CET, NEET, JEE(MAINS), AIIMS, GBPUAT, UUHF

Physics 11th and 12th Book

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Topic 0 Basic Mathematics for Physics

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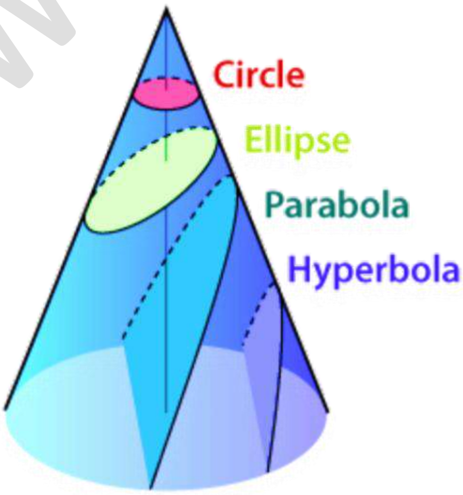


Circle
Ellipse
Parabola
Hyperbola

0.01 Logarithms

0.01.01 Indices

When a number is wrote in the form 2^4 , here 2 is known as base and 4 is known as power, index or exponent.

Rules of exponent

Consider we want to multiply 4 and 8 which is equal to 32

$4 \times 8 = 32$

Now $4 = 2^2$ and $8 = 2^3$.

As $4 \times 8 = 32$

$2^2 \times 2^3 = 32$

$(2\times2) \times (2\times2\times2) = 32$

$2^5 = 32$

From above we can conclude that if two number in exponential form, if their base is same then power or index or exponent gets added or

$a^m \times a^n = a^{(m+n)}$

Similarly it can be proved that

$a^m \div a^n = a^{(m-n)}$

Consider $(2^2)^3$

$(2^2)^3 = (2\times2)^3$

$(2^2)^3 = (2\times2) \times (2\times2) \times (2\times2)$

$(2^2)^3 = 2^6$

In general

$(a^m)^n = a^{(m\times n)}$

0.01.02 Logarithm

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$.

This is stated as 'log of 16 to base 2 equals 4'.

We see that the logarithm is the same as the power or index in the original expression.

In general we can write

$x = a^m$ then $\log_a x = m$

From above

$10 = 10^1$ thus $\log_{10} 10 = 1$

Or $2 = 2^1$ thus $\log_2 2 = 1$

In general

$\log_a a = 1$

Exercises 0.1.01

1. Write the following using logarithms instead of powers

a) $8^2 = 64$ b) $3^5 = 243$ c) $2^{10} = 1024$ d) $5^3 = 125$

e) $10^6 = 1000000$ f) $10^{-3} = 0.001$ g) $3^{-2} = \frac{1}{9}$ h) $6^0 = 1$

i) $5^{-1} = \frac{1}{5}$ j) $\sqrt{49} = 7$ k) $27^{2/3} = 9$ l) $32^{-2/5} = 1/4$

2. Determine the value of the following logarithms

a) $\log_3 9$ b) $\log_2 32$ c) $\log_5 125$ d) $\log_{10} 10000$

e) $\log_4 64$ f) $\log_{25} 5$ g) $\log_8 2$ h) $\log_{81} 3$

i) $log_3\left(\frac{1}{27}\right)$ j) $\log_7 1$ k) $log_8\left(\frac{1}{8}\right)$ l) $\log_4 8$

m) $\log_a a^5$ n) $\log_c \sqrt{c}$ o) $\log_s s$ p) $log_e\left(\frac{1}{e^3}\right)$

0.01.03 Laws of logarithms

1) The first law of logarithms

Suppose

$x = a^n$ and $y = a^m$

then the equivalent logarithmic forms are

$\log_a x = n$ and $\log_a y = m$ …….(1)

Using the first rule of indices

$xy = a^{(n+m)}$

$\log_a xy = n+m$ and

from (1) and so putting these results together we have

$\log_a xy = \log_a x + \log_a y$

2) The second law of logarithms

Suppose $x = a^n$, or equivalently $\log_a x = n$. suppose we raise both sides of $x = a^n$ to the power m:

$X^m = (a^n)^m$

Using the rules of indices we can write this as

$x^m = a^{nm}$

Thinking of the quantity x^m as a single term, the logarithmic form is

$\log_a x^m = nm = m\log_a x$

This is the second law. It states that when finding the logarithm of a power of a number, this can be evaluated by multiplying the logarithm of the number by that power.

3) The third law of logarithms

As before, suppose

$x = a^n$ and $y = a^m$

with equivalent logarithmic forms

$\log_a x = n$ and $\log_a y = m$ (2)

Consider $x \div y$.

$$\frac{x}{y} = \frac{a^n}{a^m} = a^{(n-m)}$$

using the rules of indices.

In logarithmic form

$$log_a\left(\frac{x}{y}\right) = log_a a^{(n-m)}$$

$$log_a\left(\frac{x}{y}\right) = n - m$$

which from (2) can be written

$$log_a\left(\frac{x}{y}\right) = log_a x - log_a x$$

This is the third law.

The logarithm of 1

Recall that any number raised to the power zero is 1: $a^0 = 1$. The logarithmic form of this is $\log_a 1 = 0$

Change of base

Example

Suppose we wish to find $\log_5 25$.

This is the same as being asked 'what is 25 expressed as a power of 5 ?'

Now $5^2 = 25$ and so

$\log_5 25 = 2$.

Example

Suppose we wish to find $\log_{25} 5$.

This is the same as being asked 'what is 5 expressed as a power of 25 ?'

We know that 5 is a square root of 25, that is $5 = \sqrt{25}$. So $25^{1/2} = 5$

$$log_{25}5 = \frac{1}{2}$$

Notice from the last two examples that by interchanging the base and the number

$$log_5 25 = \frac{1}{log_{25}5}$$

In general

$$log_a b = \frac{1}{log_b a}$$

$$\frac{\log_e b}{\log_e a} = \frac{\log_c b}{\log_c a} = log_a b$$

Exercise 0.1.02

Each of the following expressions can be simplified to logN. Determine the value of N in each case. We have not explicitly written down the base. You can assume the base is 10, but the results are identical whichever base is used.

a) log 3 + log 5 b) log 16 − log 2 c) 3 log 4

d) 2 log 3 − 3 log 2 e) log 236 + log 1

f) log 236 − log 1 g) 5 log 2 + 2 log 5

h) log 128 − 7 log 2 i) log 2 + log 3 + log

j) log 12 − 2 log 2 + log 3

k) 5 log 2 + 4 log 3 − 3 log 4

l) log 10 + 2 log 3 − log 2

Common bases:

log means $\log_{10}$

ln means $\log_e$ where e is the exponential constant.

We can convert ln to log as follows

ln a = 2.303 log a

Exercises 0.1.03

Use logarithms to solve the following equations

a) $10^x = 5$ b) $e^x = 8$ c) $10^x = ½$ d) $e^x = 0.1$ e) $4^x = 12$ f) $3^x = 2$ g) $7^x = 1$

h) $\left(\frac{1}{2}\right)^x = \frac{1}{100}$

0.01.04 Using log table

Four figure logarithms

Logarithms can be used to calculate lengthy multiplication and division numerical

We can use log tables , for four figure logarithms.

Logarithm of number consists of two parts

Characteristic : Integral part of log

Mantissa : Fractional or decimal part of the log

Characteristic

If number is >1, then count number of digits before decimal, then reduce one from the number of digits

For example

6.234 : Number of digits before decimal is 1 ,

thus Characteristic number = 1-1 =0

62.34 : Number of digits before decimal are 2,

thus Characteristic number =2-1=1

623.4 : Number of digits before decimal are 3,

thus Characteristic number =3-1=2

6234.0 : Number of digits before decimal are 4,

thus Characteristic number =4-1=3

If number is <1, then count number of zero after decimal, then add one from the number of digits , and Characteristic number is negative represented as bar

For example

0.6234 : Number of zero's after decimal is zero ,

thus Characteristic number =-(0+1) = $\bar{1}$

0.0623: Number of zero's after decimal is 1 ,

thus Characteristic number = -(1+1) =$\bar{2}$

0.00623 : Number of zero's after decimal is 2 ,

thus Characteristic number = -(2+1) = $\bar{3}$

Exercises 0.1.04

Find characteristic number of following

a)523.045 b) 0.02569 c) 569325 d) 0.0023 e) 2.37×10^3 f) 0.876

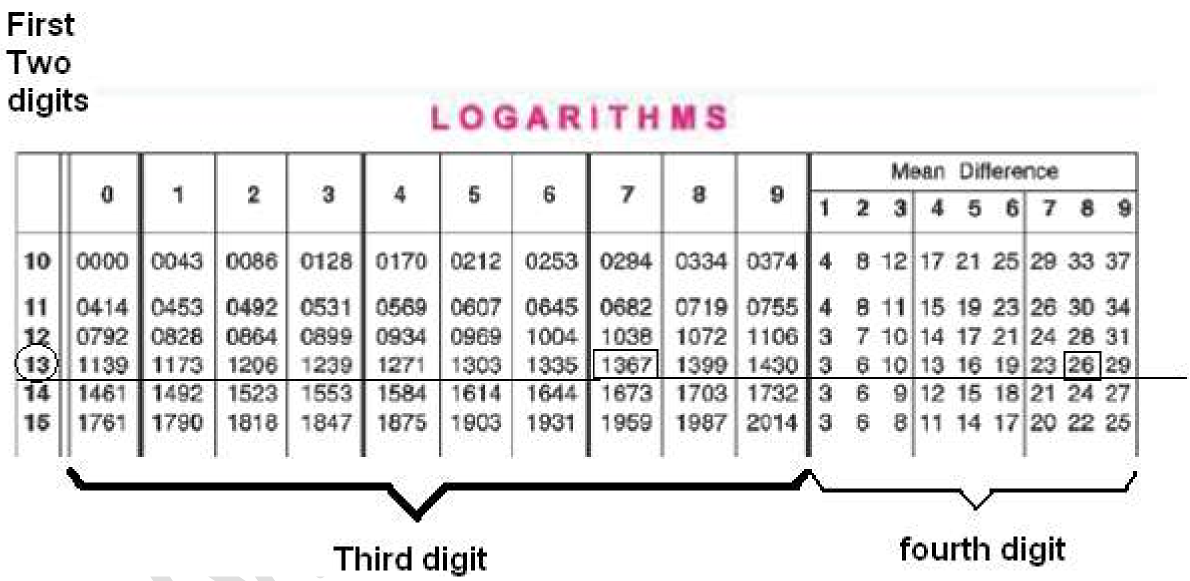
g) 2.569 h) 24.567 i) 0.00006 j) 1.236×10^{-3} k) 26.30×10^{-6} l) $.002\times10^4$

Ans

a) 2 b) $\bar{2}$ c) 5 d) $\bar{3}$ e) 3 f) $\bar{1}$ g) 0 h) 1 i) $\bar{5}$ j) $\bar{3}$ k) $\bar{5}$ l)1

Finding log of number using log table

Suppose we want log of number 1378 . characteristic number is 3



	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25

As shown in figure first two digits from left column, third digit in middle and fourth digit in right column

Now from table log of 137 refers to 1367 now add mean difference 26

We get 1367 +26 = 1393

Thus log 1378 = 3.1393

Log of 137.8 = 2.1393 (note only characteristic number changed to 2)

Log 13.78 = 1.1393 (note only characteristic number changed to 1)

Log 1.378 = 0.1393 (note only characteristic number changed to 0)

Log 0.1378 = $\bar{1}$.1393 (note only characteristic number changed to -1)

Log 0.01378 = $\bar{2}.1393$ (Note that zeros after decimal is omitted while finding log and characteristic number changed)

log5 = 0.6990 [note in table look for 50 = 6990, but characteristic is 0]

log50 = 1.6990

Finding Antilog of number

First Two digits

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	$\boxed{1161}$	1164	1167	1169	1172	0	1	1	1	1	2	2	$\boxed{2}$	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3

Third digit

fourth digit

For Antilog of any number, digits after decimal is considered

For example

Antilog (3.0658)

Here 3 is characteristic number hence should not be considered. Antilog of 0.0658 will be 1161 +2 = 1163 as shown in figure

Now put decimal point after one digit from left = 1.163

Characteristic number 3 will be now power of 10 thus final antilog will be

Antilog (3.0658) =1.163×10^{3}

Antilog($\bar{1}.0658$)

As stated earlier Antilog of 0.0658 will be 1161 +2 = 1163 as shown in figure

Now put decimal point after one digit from left = 1.163

Characteristic number $\bar{1}$ will be now power of 10 thus final antilog will be

Antilog ($\bar{1}$.0658) =1.163×10^{-1}

Similarly antilog of $\bar{5}.0658$ = 1.163×10^{-5}

To find antilog of 6.4632

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	$\boxed{2904}$	2911	2917	2924	2931	2938	2944	1	$\boxed{1}$	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

As stated earlier 6 is characteristic number , which is power of 10 and will be dropped

From table above antilog of 4632 = 2904+1 =2905 =2.905

Thus antilog(6.4632)=2.905 ×10^{6}

Similarly antilog ($\bar{3}.4632$) = 2.905 ×10^{-3}

Using log for calculation

To calculate value of y

$$y = \frac{4568 \times 3258}{0.02568}$$

Take log on both sides

$$\log y = \log\left(\frac{4568 \times 3258}{0.02568}\right)$$

Using log rules we get

Log y = log 4568 + log 3258 – log 0.02568 ----(1)

Now log 4568 , characteristic number 3

From log table 6590+8 =6598

Thus log(4568) = 3.6598

Similarly log(3258) = 3.5130

From equation (1) we have to add log of 4568 and 3258 thus

$$\begin{array}{rl} & \log(4568) = 3.6598 \\ + & \log(3258) = 3.5130 \\ \hline & 7.1728 \end{array}$$

Now log (0.025686) = $\bar{2}.4097$ (note we have round-off number as 0.02569)

From equation (i) we have to subtract log from previous log value 7.1728

$$\begin{array}{r} 7.1728 \\ - \bar{2}.4097 \\ \hline 8.7631 \end{array}$$

Note subtraction of characteristic number 7-1 =6 because of carry forward. Then 6 –(2) = 8.

Now we will find antilog of 8.7631. Digit 8 which is before decimal point refers to characteristic number and will be power of ten. From antilog table we get,

Antilog (8.7631) = 5.795×10^8 which the value of y.

∴ y = 5.795×10^8

Example

$$y = 125^{\frac{1}{6}}$$

Take log on both sides,

$$\log y = \log(125)^{\frac{1}{6}}$$

By applying log rule,

$$\log y = \frac{1}{6}\log(125)$$

From log table log125 = 2.0969

Now divide 2.0969 by 6 we get 0.3495 (after round-off)

Now take antilog of 0.3495 from antilog log table we get

Antilog(0.3495) =2.237×10^0

Thus $125^{\frac{1}{6}} = 2.237$

Exercises 0.1.05

Solve

a) $39^{3/4}$ b) $25^{1/3}$ c) $5^{1/2}$

d) $\frac{0.369\times0.0569}{0.00235}$ e) $\frac{(2.569\times10^7)\times(3.421\times10^{-4})}{45689}$

Answer

a) 15.61 b) 2.924 c)2.237 d) 8.933 e) 0.1923

Antilog of negative number

such as -7.5231

First convert the negative number in to two parts one negative (Characteristic) and the decimal part(Mantissa) into positive.

by adding +8 and subtracting -8 as follows :

$-7.5231 + 8 - 8 = -8 + (8-7.5231) = -8 + 0.4769$.

Find the actual digits using 0.4769 in the anti-log table. Multiply this by 10^{-8} to account for the -8 characteristic

Ans : 2.998×10^{-8}

Example 2

-12.7777

$-12.7777 + 13 - 13 = -13 + (13-12.7777) = -13 + 0.2223$

Find the actual digits using 0.2223 in the anti-log table. Multiply this by 10^{-13} to account for the -13 characteristic

Ans 1.668×10^{-13}

Exercises 0.1.06

Find antilog of following

a) -2.5689 b) -6. 9945 c) -3. 1129

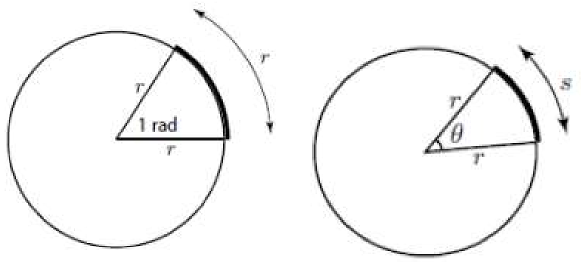
Answers

a) 2.699×10^{-3} b) 1.013×10^{-7} c) 7.711×10^{-4}

0.02 Trigonometry

0.02.01 Definition of a radian

Consider a circle of radius *r* as shown, In Figure we have highlighted part of the circumference of the circle chosen to have the same length as the radius. The angle at the centre, so formed, is 1 radian. Length of arc s = rθ Here θ is in radians



Exercises 0.2.01

Determine the angle (in radians) subtended at the centre of a circle of radius 3cm by each of the following arcs:

a) Arc of length 6 cm b) arc of length 3π cm

c) Arc of length 1.5 cm d) arc of length 6π cm

Answers

a)2 b) π c) 0.5 d) 2π

Equivalent angles in degrees and in radians

We know that the arc length for a full circle is the same as its circumference, $2\pi r$.

We also know that the arc length = $r\theta$.

So for a full circle

$2\pi r = r\theta$

$\theta = 2\pi$

In other words, when we are working in radians, the angle in a full circle is 2π radians, in other words

$360^0 = 2$ *radians*

This enables us to have a set of equivalences between degrees and radians

Exercises 0.2.02

1. Convert angle in radians

a)90^O b) 360^O c) 60^O d) 45^O e) 120^O f) 15^o g) 30^o h) 270^o

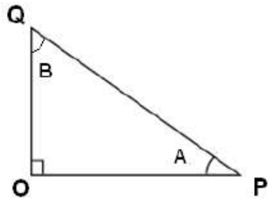
2. Convert radians to degrees

a) π/2 radians b) 3 π /4 radians c) π radians d) π /6 radians

e) 5 π radians f) 4 π /5 radians g) 7 π /4 radians h) π /10 radians

0.02.02 Trigonometric ratios for angles in a right-angled triangle

Refer to the triangle in Figure 1.



The side opposite the right-angle is called the hypotenuse

Recall the following important definitions:

$sinA = \frac{OQ}{QP}$ but $cosB = \frac{OQ}{QP}$

$\therefore$ sin A = cos B

We know that ∠ A + ∠ B = 90 $\therefore$ ∠ B = 90 - ∠ A

$\therefore$ sin A = cos (90-A)

$cosA = \frac{OP}{QP}$ but $sinB = \frac{OP}{QP}$

$\therefore$ cos A = sin B

OR cos A = sin (90-A)

$tanA = \frac{OQ}{OP}$ but $cotB = \frac{OQ}{OP}$

$\therefore$ tan A = cot B OR tan A = cot (90-A)

Angles

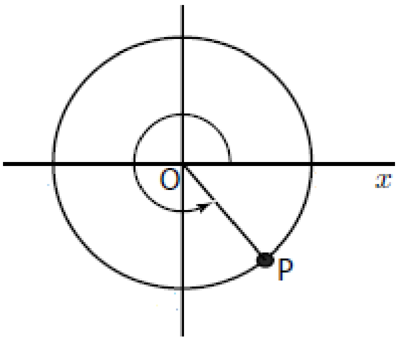


figure (a)

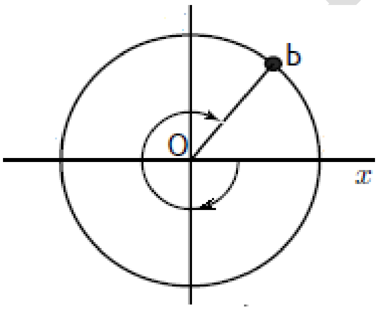


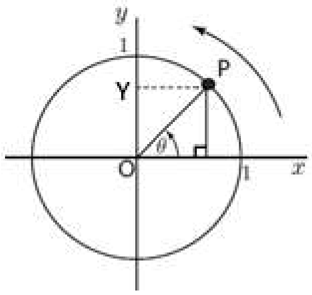
figure (b)

If angle is measured in anticlockwise direction from positive x-axis as shown in figure a. is positive If angle is measured in clockwise direction from positive x-axis as shown in figure b is negative

0.02.03 The sign of an angle in any quadrant

Sin of an angle in first quadrant

Consider Figure which shows a circle of radius 1 unit.



The side opposite θ has the same length as the projection of OP onto the y axis OY.

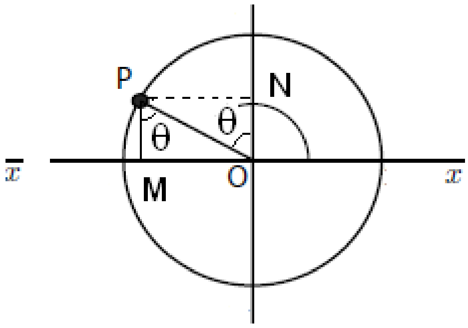
The arm OP is in the first quadrant and we have dropped a perpendicular line drawn from P to the x axis in order to form the right-angled triangle shown.

Consider angle θ. The side opposite this angle has the same length as the projection of OP onto the y axis. So we define

$$sin\theta = \frac{Projection\ of\ OP\ on\ y - axis}{OP}$$

Sinθ = Projection of OP on y axis

Sin of an angle in second quadrant



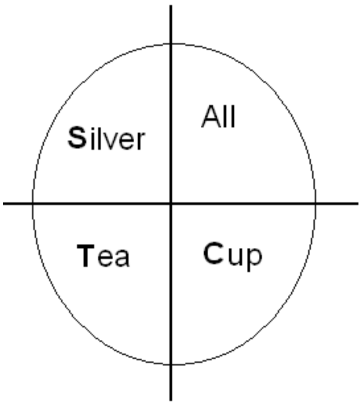
Consider adjacent figure here OP makes angle is 90 + θ with positive x-axis

Now as stated earlier sin(90+θ) = Projection of OP on y-axis = ON

From the geometry of figure we can find that $cos\theta = \frac{PM}{OP}$

Thus sin(90+θ) = cos θ

By using above we can obtain various relations, which can be quickly remembered by following way



Quadrant I: All

All ratios sin, cos, tan, cosec, sec, cot have POSITIVE value

Quadrant II: Silver

Only sine or cosec have POSITIVE value

Remaining have negative value

Quadrant III: Tea

Only tan and cot have POSITIVE value .Remaining have negative value

Quadrant IV: Cup

Only cos and sec have POSITIVE value

Remaining have negative values

Angles π + θ function do not change

For example sin(π +θ) = -sinθ

Here π +θ is in Third quadrant where sin is NEGATIVE thus negative sign appears

For angle $\left(\frac{\pi}{2}+\theta\right)$ and $\left(\frac{3\pi}{2}+\theta\right)$function changes from

sin → cos ; sec → cosec ; cosec → sec;

tan → cot; cos →sin ; cot → tan

And the sign of resultant depends on the quadrant of the first function

Sin(-θ)	Angle in IV quadrant Sin is negative	-sinθ
Cos (-θ)	Angle in IV quadrant cos is positive	cosθ
tan (-θ)	Angle in IV quadrant tan is negative	-tanθ
Cot (-θ)	Angle in IV quadrant cot is negative	-cotθ
$sin\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant sin positive and change function	cosθ
$cos\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant cos negative and change function	-sinθ
$tan\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant tan negative and change function	-cotθ
$cot\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant cot negative and change function	-tanθ
sin(π+θ)	Angle is in III quadrant sin negative and same function	-sinθ
cos (π+θ)	Angle is in III quadrant cos negative and same function	-cosθ
tan(π+θ)	Angle is in III quadrant tan positive and same function	tan θ
cot(π+θ)	Angle is in III quadrant cot positive and same function	cotθ

By using same technique you may find formula for

(π − θ) ,

$\left(\frac{3\pi}{2}+\theta\right)$ and

$\left(\frac{3\pi}{2}-\theta\right)$

0.02.04 Trigonometric identities

$$Sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = cosec^2\theta$$

$$sec^2\theta - tan^2\theta = 1$$

$$cosec^2\theta - cot^2\theta = 1$$

$$sin2\theta = 2sin\theta\, cos\theta = \frac{2\tan\theta}{1 + tan^2\theta}$$

$$cos2\theta = cos^2\theta - sin^2\theta = 2\,cos^2\theta - 1 = 1 - 2sin^2\theta = \frac{1 - tan^2\theta}{1 + tan^2\theta}$$

$$sin(\alpha \pm \beta) = sin\alpha \cos\beta \pm \cos\alpha\, sin\beta$$

$$cos\,(\alpha \pm \beta) = cos\,\alpha\, cos\,\beta \mp sin\alpha\, sin\beta \text{ (note sign changed)}$$

$$sin\alpha + sin\beta = 2sin\left(\frac{\alpha + \beta}{2}\right) cos\left(\frac{\alpha - \beta}{2}\right)$$

$$sin\alpha - sin\beta = 2cos\left(\frac{\alpha + \beta}{2}\right) sin\left(\frac{\alpha - \beta}{2}\right)$$

$$cos\alpha + cos\beta = 2\,cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$cos\alpha - cos\beta = -2\,sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$2sin\,\alpha\, cos\,\beta = sin\,(\alpha + \beta) + sin\,(\alpha - \beta)$$

$$2\,cos\,\alpha\, sin\,\beta = sin\,(\alpha + \beta) - sin\,(\alpha - \beta)$$

$$2\,cos\,\alpha\, cos\,\beta = cos\,(\alpha + \beta) + cos\,(\alpha - \beta)$$

$$-2\,sin\,\alpha\, sin\,\beta = cos\,(\alpha + \beta) - cos\,(\alpha - \beta)$$

$$Sin\,3\theta = 3sin\,\theta - 4\,sin^3\theta$$

$$Cos\,3\theta = 4\,cos^3\theta - 3\,cos\,\theta$$

$$tan\,(\,\alpha \pm \beta\,) = \frac{tan\,\alpha \pm tan\,\beta}{1 \mp tan\,\alpha\, tan\,\beta}$$

$$tan\,2\theta = \frac{2\tan\theta}{1 - tan^2\theta}$$

0.03 Introduction to vectors

0.03.01 Scalar quantity

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.

For example if I get 5 numbers of apple from east direction and 3 numbers of apple from west direction number of apple I will have is 5+3 = 8 numbers of apples. Thus total number of apples does not depends on the directions.

Now if I say I want 5 water. It does not make any sense, as I have not mentioned any unit. Is it 5 litre or 5 ml or 5 kL of water

Thus scalar quantities have unit and magnitude (number) to give full description and is independent of direction

Some examples of scalars include the mass, charge, volume, time, speed, temperature, electric potential at a point inside a medium, energy .

The distance between two points in three-dimensional space is a scalar, but the direction from one of those points to the other is not, since describing a direction requires two physical quantities such as the angle on the horizontal plane and the angle away from that plane. Force cannot be described using a scalar, since force is composed of direction and magnitude, however, the magnitude of a force alone can be described with a scalar, for instance the gravitational force acting on a particle is not a scalar, but its magnitude is. The speed of an object is a scalar (e.g. 180 km/h), while its velocity is not (i.e. 180 km/h *north*). Other examples of scalar quantities in Newtonian mechanics include electric charge and charge density.

0.03.02 Vector quantities

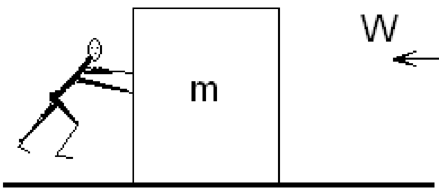
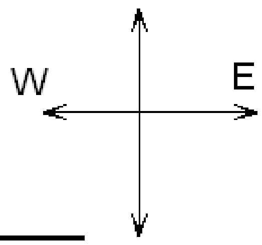


figure a



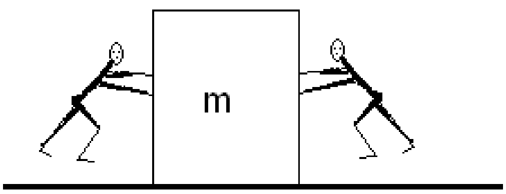


figure b

Vectors are quantities that are fully described by both a magnitude and a direction.

As shown in figure a, person is pushing a box of mass m towards East. Box will move in direction towards East. If F is the force applied by a man then acceleration of box is F/m and towards East

Now as shown in figure b, first person is pushing a box towards East while another person pushing box towards West. Now if both persons apply equal force then box does not move. As force towards East is cancelled or nullified by the force towards West.

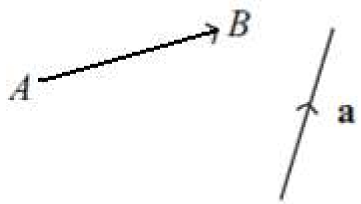
Thus to know the effective force on the box we should know the direction of force. If in above example if both person push the box in same direction(say towards East) then their forces would have got added and box would have started to move towards East.

Thus to understand vector quantity and its effect on object or at a particular point, we not only require magnitude but also direction such quantity are call vector quantity.

Some examples of vectors include weight as it is a gravitational force on object, velocity, acceleration, Electric filed, Magnetic field, Area,

0.03.03 Geometric Representation of vector

We can represent a vector by a line segment. This diagram shows two vectors.

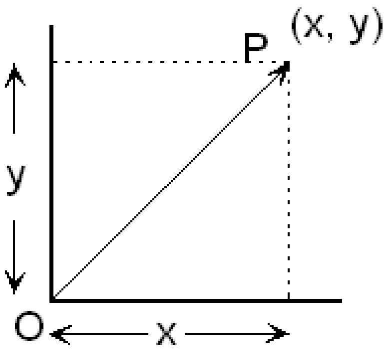


We have used a small arrow to indicate that the first vector is pointing from A to B. A vector pointing from B to A would be going in the opposite direction. Sometimes we represent a vector with a small letter such as a, in a bold typeface. This is common in textbooks, but it is inconvenient in handwriting. In writing, we normally put a bar underneath, or sometimes on top of, the letter: $\bar{a}$ or $\vec{a}$. In speech, we call this the vector "a-bar".

Arrow at B shows the direction called as head of vector, Length of arrow length AB is magnitude of vector let it be a , while point A is called as tail of vector.

Vector may be represented as $\overrightarrow{AB}$ it indicates tail of vector is at A and head is at B. If two vector p and q are equal of $\vec{p} = \vec{q}$ it means both the vectors have same magnitude and same direction.

0.03.04 Position vector

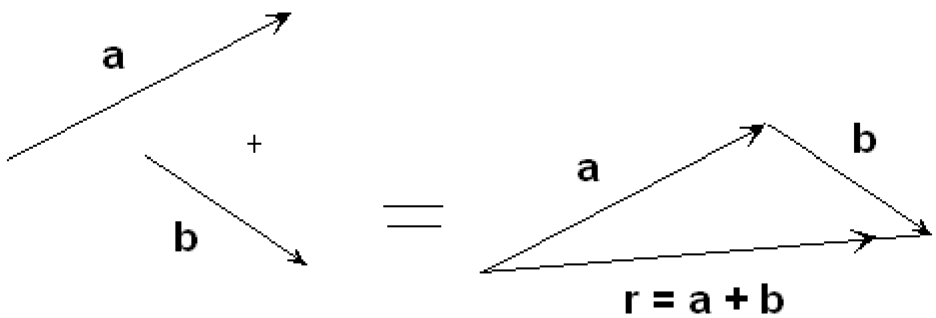


Vector OP or $\overrightarrow{OP}$ is called a position vector as it shows position of point P in co-ordinate frame. Its tail is at origin. It may be also represented as (x, y)

0.03.05 Adding two vectors

One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on.

In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So the second displacement must start where the first one finishes.

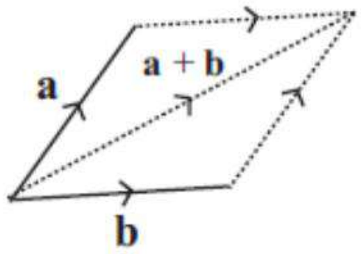


To add vector b in vector a, we have drown vector b from the head of a, keeping direction and magnitude same as of b. Then drawn vector from tail of vector a to head of vector b

The sum of the vectors, a + b (or the *resultant*, as it is sometimes called) is what we get when we join up the triangle. This is called the *triangle law* for adding vectors.

There is another way of adding two vectors. Instead of making the second vector start where the first one finishes, we make them both start at the same place, and complete a parallelogram.

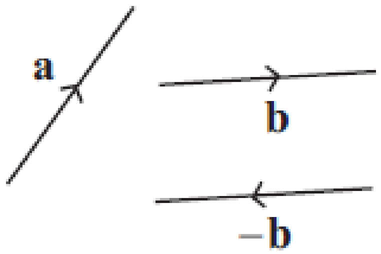
This is called the *parallelogram law* for adding vectors. It gives the same



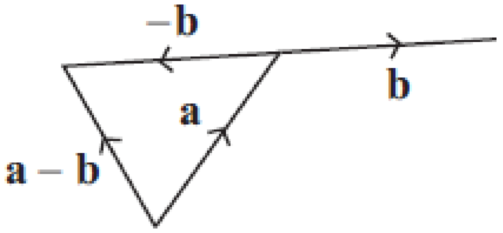
result as the triangle law, because one of the properties of a parallelogram is that opposite sides are equal and in the same direction, so that b is repeated at the top of the parallelogram.

0.03.06 Subtracting two vectors

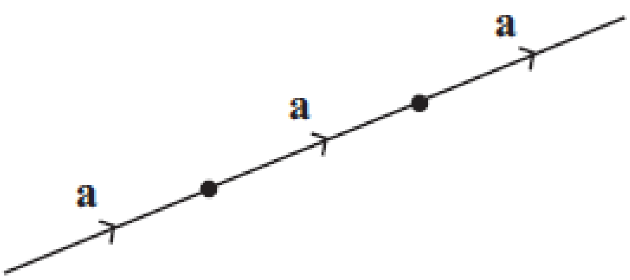
What is a −b? We think of this as a +(−b), and then we ask what −b might mean. This will
be a vector equal in magnitude to b, but in the reverse direction.



Now we can subtract two vectors. Subtracting b from a will be the same as *adding* −b to a.



Adding a vector to itself



What happens when you add a vector to itself, perhaps several times? We write, for example,

a + a + a = 3a.

In same we can write na = a + a+ a ….. n times

Or we can multiply any vector by constant (say n) and result is again a vector having magnitude n times of the previous.

Exercises 0.3.01

In Δ*OAB*, *OA* = a and *OB* = b. In terms of a and b,

(a) What is *AB*?

(b) What is *BA*?

(c) What is *OP*, where *P* is the midpoint of *AB*?

(d) What is *AP*?

(e) What is *BP*?

(f) What is *OQ*, where *Q* divides *AB* in the ratio 2:3?

Ans:

(a) b − a (b) a − b (c) $\frac{1}{2}$ (a + b)

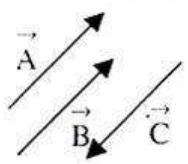
(d) $\frac{1}{2}$ (b − a) (e) $\frac{1}{2}$ (a − b) (f) $\frac{3}{5}$ a + $\frac{2}{5}$ b

0.03.07 Different types of vectors

1) Collinear vectors:- Vectors having a common line of action are called collinear vectors. There are two types of collinear vectors. One is parallel vector and another is anti-parallel vector.

2) Parallel Vectors:- Two or more vectors (which may have different magnitudes) are said to be parallel (θ = 0°) when they are parallel to the same line. In the figure below, the vectors $\vec{A}$ and $\vec{B}$are parallel.

3) Anti Parallel Vectors:-



Two or more vectors (which may have different magnitudes) acting along opposite direction are called anti-parallel vectors. In the figure below, the vectors $\vec{B}$ and $\vec{C}$ are anti-parallel vectors.

4) Equal Vectors: - Two or more, vectors are equal if they have the same magnitude (length) and direction, whatever their initial points. In the figure above, the vectors A and B are equal vectors.

5) Negative Vectors: - Two vectors which have same magnitude (length) but their direction is opposite to each, other called the negative vectors of each other. In figure above vectors A and C or B and C are negative vectors.

6) Null Vectors: - A vector having zero magnitude an arbitrary direction is called zero vector or 'null vector' and is written as = O vector. The initial point and the end point of such a vector coincide so that its direction is indeterminate. The concept of null vector is hypothetical but we introduce it only to explain some mathematical results.

Properties of a null vector:-

(a) It has zero magnitude.

(b) It has arbitrary direction

(c) It is represented by a point.

(d) When a null vector is added or subtracted from a given vector the resultant vector is same as the given vector.

(e) Dot product of a null vector with any vector is always zero.

(f) Cross product of a null vector with any vector is also a null vector.

Co-planar Vector: - Vectors situated in one plane, irrespective of their directions, are known as co-planar vectors.

0.03.08 Unit Vector or vector of unit length

If a is vector and |a| represents the magnitude of vector then $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Represent unit vector. Unit vector $\hat{a}$ is called as a-cap or a-hat.

Thus if vector a has magnitude of p then $\vec{a} = p\,\hat{a}$

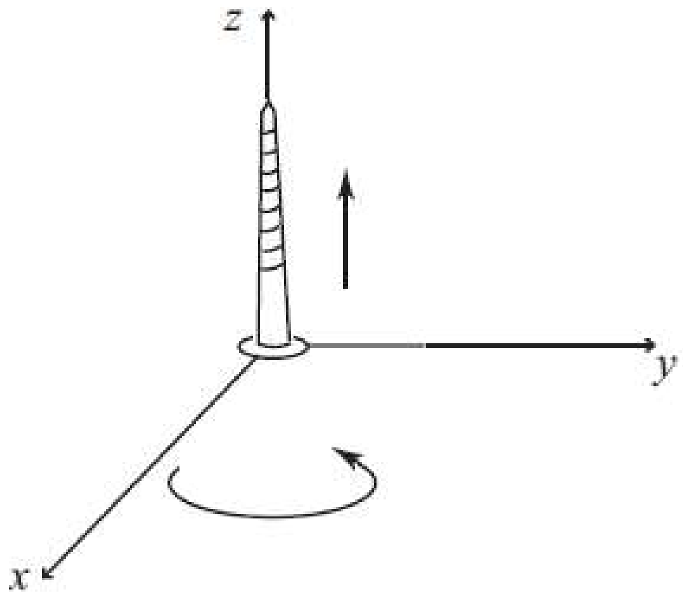
Unit vector along x-axis is represent by $\hat{\imath}$

Unit vector along y-axis is represent by $\hat{\jmath}$

Unit vector along z-axis is represent by $\hat{k}$

0.03.09 Cartesian frame of reference

In three dimensions we have three axes, traditionally labeled x, y and z, all at right angles to each other. Any point P can now be described by three numbers, the coordinates with respect to the three axes.



Now there might be other ways of labeling the axes. For instance we might interchange x and y, or interchange y and z. But the labeling in the diagram is a standard one, and it is called a right-handed system. Imagine a right-handed screw, pointing along the z-axis. If you tighten the screw, by turning it from the positive x-axis towards the positive y-axis, then the screw will move along the z-axis. The standard system of labeling is that the direction of movement of the screw should be the positive z direction.

This works whichever axis we choose to start with, so long as we go round the cycle x, y, z, and then back to x again. For instance, if we start with the positive y-axis, then turn the screw towards the positive z-axis, then we'll tighten the screw in the direction of the positive x-axis.

Algebraic representation of vector

0.03.10 Vectors in two dimensions.

The natural way to describe the position of any point is to use Cartesian coordinates. In two dimensions, we have a diagram like this, with an x-axis and a y-axis, and an origin O. To include vectors in this diagram, we have unit vector along x-axis is denoted by $\hat{\imath}$ and a unit vector along y-axis is denoted by $\hat{\jmath}$. In figure (a) vector along positive x-axis having magnitude of 6 is represented as $6\hat{\imath}$, while in figure (b) vector along positive y axis having magnitude is represented as $5\hat{\jmath}$.

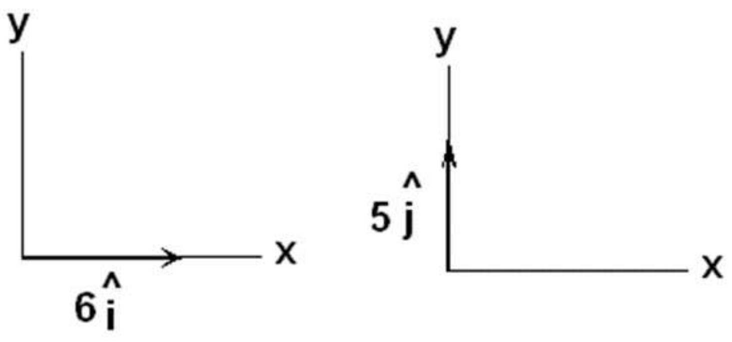
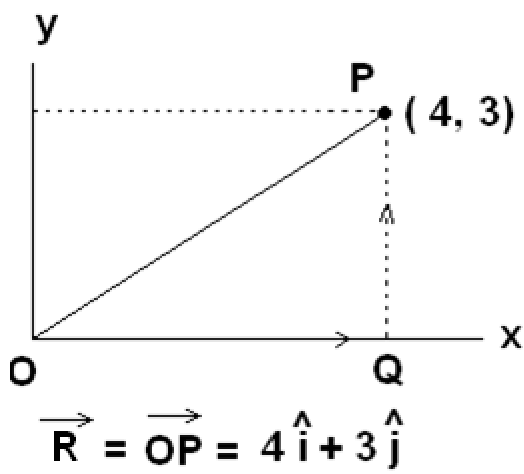


figure (a) figure (b)



In adjacent figure point P has co-ordinates (4, 3). If we want to reach from point O to P. We can walk 4 units along positive x-axis and 3 units after taking 90° turn.

Thus vector $\overrightarrow{OP}$ is result of the addition of two vectors $\overrightarrow{OQ}$ and $\overrightarrow{QP}$. Mathematically it can be represented as

$$\overrightarrow{OP} = 4\hat{\imath} + 3\hat{\jmath}$$

$4\hat{\imath}$ Can be said to be x-component of vector OP. or effectivity of vector OP along x-axis is 4 unit.

$3\hat{\jmath}$ Can be said to be y-component of vector OP or effectivity of vector OP along y-axis is 3 unit.

Angle made by the vector with x-axis

$$tan\theta = \frac{3}{4}$$

In general

$$tan\theta = \frac{y - component}{x - component}$$

Example : If force of $\vec{R} = (6\hat{\imath} + 8\hat{\jmath})$ N then force 6N force will cause motion along positive x axis and 8 N force will cause motion along y –axis.

If mass of object is 2kg. Then acceleration along +ve x-axis will be 6/2 = 3 m/s^2 and acceleration along +ve y-axis will be 8/2 = 4 m/s^2.

Magnitude of vector R can be calculated using Pythagoras equation

$$\vec{R} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

And unit vector is $\hat{R} = \frac{\vec{R}}{|\vec{R}|}$

$$\hat{R} = \frac{6\hat{\imath} + 8\hat{\jmath}}{10} = \left(\frac{6}{10}\hat{\imath} + \frac{8}{10}\hat{\jmath}\right)\ unit$$

Vector R can be represented in terms of unit vector as magnitude × unit vector

$$\vec{R} = 10\left(\frac{6}{10}\hat{\imath} + \frac{8}{10}\hat{\jmath}\right)\ unit$$

In general vector in two dimension is represented as

$$\vec{A} = x\hat{\imath} + y\hat{\jmath}$$

Magnitude as

$$|\vec{A}| = \sqrt{x^2 + y^2}$$

Unit vector

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Angle made by force vector

$$tan\theta = \frac{y - component}{x - component} = \frac{8}{6} = 1.333$$

$$\theta = 53^o\ 8'$$

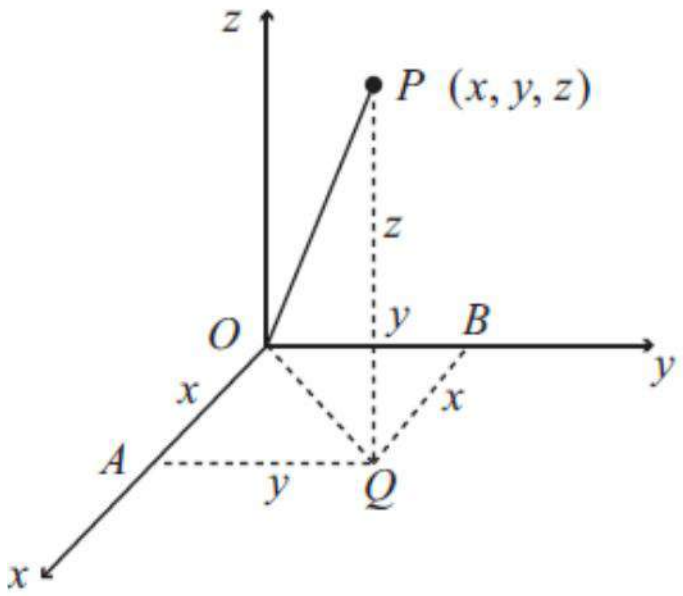
0.03.11 Vectors in three dimensions.

Vector OP in adjoining figure represent vectors in three dimensions and can be represented as

$$\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{\jmath}$$

And magnitude as

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$



Unit vector is

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$$

x, y and z are component of vector along x-axis , y axis and z-axis.

Algebraic addition of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Both vectors can be added to get resultant vector

$$\vec{R} = \vec{A} + \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$\vec{R} = (A_x + B_x)\hat{\imath} + \left(A_y + B_y\right)\hat{\jmath} + (A_z + B_z)\hat{k}$$

If $\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}$

From above we get

$R_x = A_x + B_x$, $R_y = A_x + B_x$, $R_z = A_z + B_z$

Algebraic subtraction of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Both vectors can be added to get resultant vector

$$\vec{R} = \vec{A} - \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) - \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$\vec{R} = (A_x - B_x)\hat{\imath} + \left(A_y - B_y\right)\hat{\jmath} + (A_z - B_z)\hat{k}$$

If

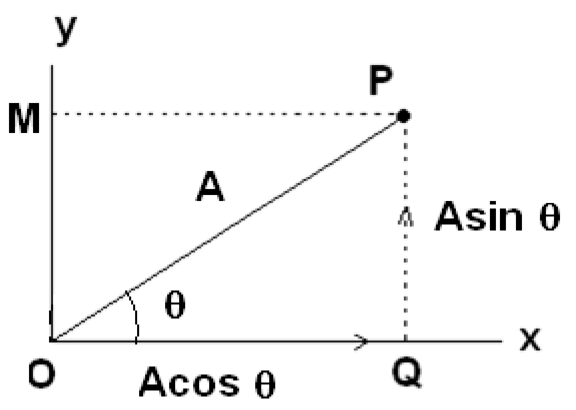
$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}$$

From above we get

$$R_x = A_x - B_x,\ R_y = A_x - B_x,\ R_z = A_z - B_z$$

Note: when two vectors are added or subtracted, their components get add or subtracted to give new vector

0.03.12 Polar representation of vector



Let vector OP makes an angle of θ with positive x-axis. Then draw a perpendicular from point P on x-axis intercept at Q and perpendicular on y-axis intercept at point M. Then OQ is called projection of vector OP on x-axis

From trigonometry OQ = A cos θ or effectively of OP along x-axis or a component of Vector OP along x-axis

Thus Vector OQ can be represented as $A\,cos\theta\ \hat{\imath}$

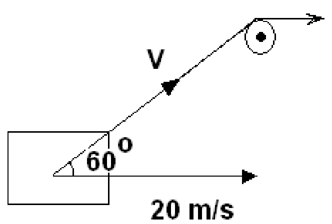
Similarly OM is projection of vector OP on y-axis. From trigonometry MO = PQ = A sin θ or effectively of OP along y-axis or a component of vector along –y-axis. Thus vector QP can be represented as $Asin\theta\hat{\jmath}$

As vector OP is made up of two mutually perpendicular vectors we can get

$$\overrightarrow{OP} = Acos\theta\ \hat{\imath} + Asin\theta\hat{\jmath}$$

Note that is the magnitude of component along OQ or x-axis is x then magnitude of vector will be |A| = x/cos θ

Example :

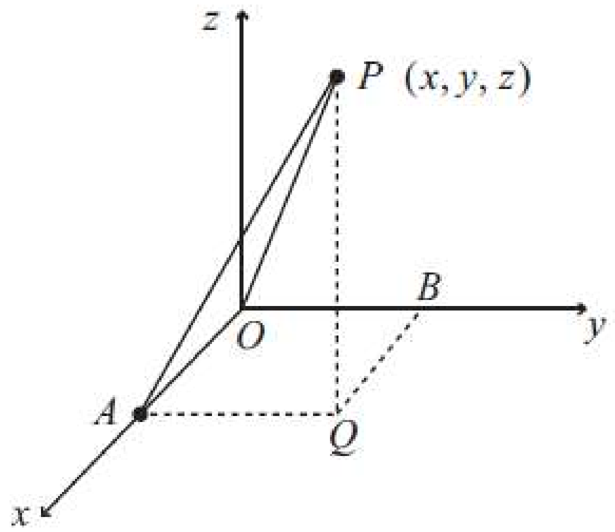


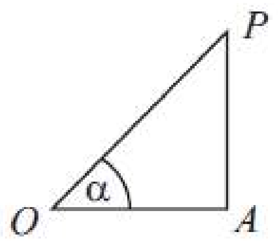
An object as shown in figure is moved with velocity along x-axis is 20m/s. By a thread making an angle of 60^{O} passing over a pulley. What is the speed of thread V

Solution

Given Vcos60 = 20 ∴ V = 20/Cos60 = 40m/s

Cosine rule





Let vector OP makes an angle of $\alpha$with x –axis, β with y-axis, γ with z-axis.

Since coordinates of 'P' are (x,y,z) and is position vector thus magnitude is

$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

AP is perpendicular on x axis thus OA = x

But the triangle *POA* is a right-angled triangle, so we can write down the cosine of the angle *POA*. If we call this angle α, then

As shown in figure on right

$$cos\alpha = \frac{OA}{OP} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

The quantity cos α is known as a *direction cosine*, because it is the cosine of an angle which helps to specify the direction of *P*; α is the angle that the position vector *OP* makes with the *x-axis.*

Similarly

$$cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

We can also be proved that

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

Exercises 0.3.02

1. Find the lengths of each of the following vectors

(a) $2\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ (b) $5\hat{\imath} - 2\hat{\jmath} + \hat{k}$ (c) $2\hat{\imath} - \hat{k}$

(d) $5\hat{\imath}$ (e) $3\hat{\imath} - 2\hat{\jmath} - \hat{k}$ (f) $\hat{\imath} + \hat{\jmath} + \hat{k}$

2. Find the angles giving the direction cosines of the vectors in Question

3. Determine the vector *AB* for each of the following pairs of points

(a) A (3,7,2) and B (9,12,5) (b) A (4,1,0) and B (3,4,-2)

(c) A (9,3,-2) and B (1,3,4) (d) A (0,1,2) and B (-2,1,2)

(e) A (4,3,2) and B (10,9,8)

4. For each of the vectors found in Question 3, determine a unit vector in the direction of) $\overrightarrow{AB}$

Ans:

1. (a)√29 (b)√30 (c)√5 (d) 5 (e)√14 (f)√3

2. (a) 68.2°, 42 °10′ , 56 ° *15′* (b) 24.1 °, 111.4 °, 79.5 °

(c), 26.6 °, 90 ° ,116.6 ° (d) 0 °, 90 ° , 90 °

(e) 36.7 °, 122.3 °, 105.5 ° (f) 54.7 °, 54.7 °, 54.7 °

3. (a) $6\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$ (b) $-\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ (c) $-8\hat{\imath} + 6\hat{k}$

(d) $-2\hat{\imath}$ (e) $6\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$

4. (a) $\frac{1}{\sqrt{70}}(6\hat{\imath} + 5\hat{\jmath} + 3\hat{k})$ (b) $\frac{1}{\sqrt{14}}(-\hat{\imath} + 3\hat{\jmath} - 2\hat{k})$

(c) $\frac{1}{10}(-8\hat{\imath} + 6\hat{k})$ (d) $-\hat{\imath}$ (e) $\frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$

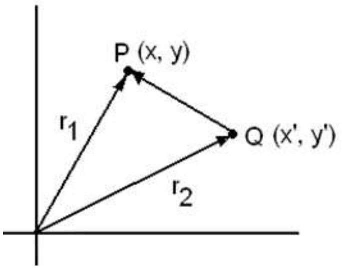
0.03.13 Displacement vector

As shown in figure point P coordinates are (x ,y) while point Q coordinates are (x′, y′)

$$\vec{r_1} = x\hat{\imath} + y\hat{\jmath}$$

And

$$\vec{r_2} = x'\hat{\imath} + y'\hat{\jmath}$$



Now Vector QP = $r_1 - r_2$ (*triangle law*)

$$\overrightarrow{QP} = x\hat{\imath} + y\hat{\jmath} - x'\hat{\imath} - y'\hat{\jmath}$$

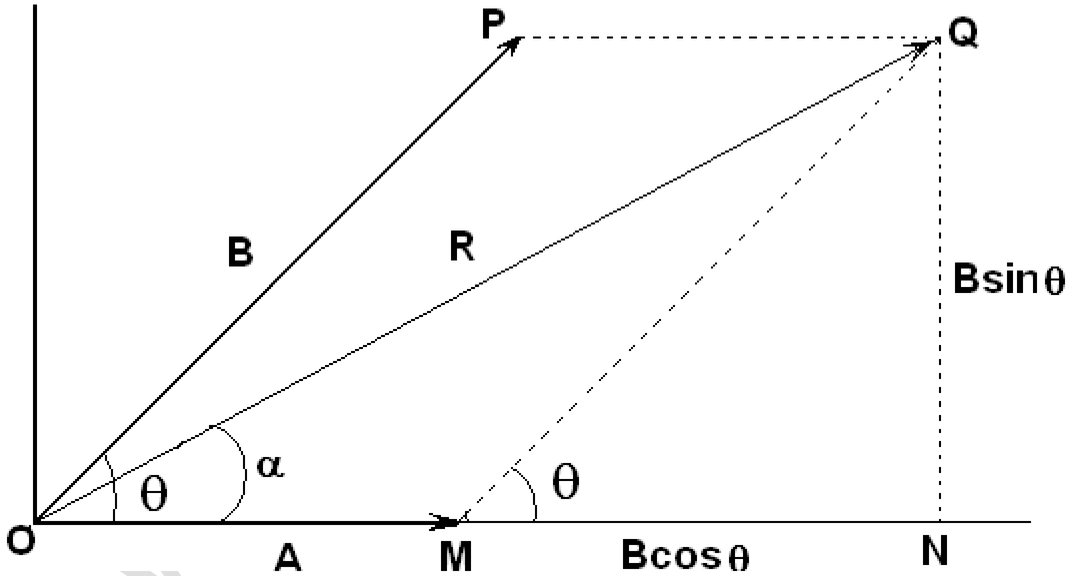
$\overrightarrow{QP} = x\hat{\imath} - x'\hat{\imath} + y\hat{\jmath} - y'\hat{\jmath}$

$$\overrightarrow{QP} = (x - x')\hat{\imath} + (y - y')\hat{\jmath}$$

0.03.14 Important result

If A and B are the two vector and θ is the angle between them then we can find a formula to get magnitude and orientation or direction of resultant vector.

We can obtain formula as follows



Let vector A be along x-axis and Vector B is making angle of θ with x-axis. Vector R is the resultant vector according to parallelogram law.

From figure Component of R along x-axis is ON and along y-axis is NQ

$$\vec{R} = \overrightarrow{ON} + \overrightarrow{NQ}$$

But

$$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$$

From the figure OM = A and MN = Bcosθ and MN =Bsinθ

$$\vec{R} = A\hat{\imath} + B cos\theta \hat{\imath} + B sin\theta \hat{\jmath}$$

$$\vec{R} = (A + Bcos\theta)\hat{\imath} + Bsin\theta \hat{\jmath}$$

$$\left|\vec{R}\right|^2 = (A + Bcos\theta)^2 + (Bsin\theta)^2$$

$$\left|\vec{R}\right|^2 = A^2 + 2ABcos\theta + B^2cos^2\theta + B^2sin^2\theta$$

$$\left|\vec{R}\right|^2 = A^2 + 2ABcos\theta + B^2$$

$$\left|\vec{R}\right| = \sqrt{A^2 + B^2 + 2ABcos\theta}$$

Orientation or direction of resultant vector from triangle OQN

$$tan\alpha = \frac{QN}{ON} = \frac{QN}{OM + MN}$$

$$tan\alpha = \frac{Bsin\theta}{A + Bcos\theta}$$

Exercises 0.3.03

Find magnitude of resultant vector and orientation

a) |A| = 6 and |B| = 8 θ = 30^{O} b) |A| = 15 and |B| = 8 θ = 60^{O}

c) |A| = 12 and |B| = 8 θ = 90^{O}

Answers

13.53 units , tanα = 0.3095 angle is with vector A

20.22 units , tanα =0.3646 angle is with vector A

14.42 units, tanα = 0.666 angle is with vector A

0.03.15 Scalar product of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Scalar product or dot product is defined as

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|cos\theta$$

Not that right hand side of the equation is scalar quantity while right hand side is vector quantity.

θ is the angle between vector A and B . | A | and | B | are the magnitude of vector A and B.

We may right above equation in different form to get more information

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left(\left|\vec{B}\right|cos\theta\right)$$

Bcosθ is projection of vector B along vector A

$$\vec{A}\cdot\vec{B} = |\vec{A}|(\text{projection of vector B along vector A})$$

Or

$$\vec{A}\cdot\vec{B} = |\vec{B}|(\text{projection of vector A along vector B})$$

If we take dot product of unit vectors

$\hat{\imath}\cdot\hat{\imath} = 1\times1cos0 = 1$ as magnitude of unit vector is 1 and angle between two $\hat{\imath}$ is 0,

Similarly $\hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$

$\hat{\imath}\cdot\hat{\jmath} = 1\times1cos90 = 0$ as angle between x-axis and y-axis is 90°

Similarly $\hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$

Thus we can say that if unit vectors are parallel their dot product is 1. If unit vectors are perpendicular their dot product is zero.

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$\begin{aligned}\vec{A}\cdot\vec{B} = &\left(A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{k}\right)\\ &+ \left(A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k}\right)\\ &+ \left(A_zB_x\hat{k}\cdot\hat{\imath} + A_zB_y\hat{k}\cdot\hat{\jmath} + A_zB_z\hat{k}\cdot\hat{k}\right)\end{aligned}$$

$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and remaining all unit vector dot products give zero value thus

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Uses of dot products

Dot product is used to check vectors are perpendicular or not

To find angle between two vectors

Projection of one vector along the another vector

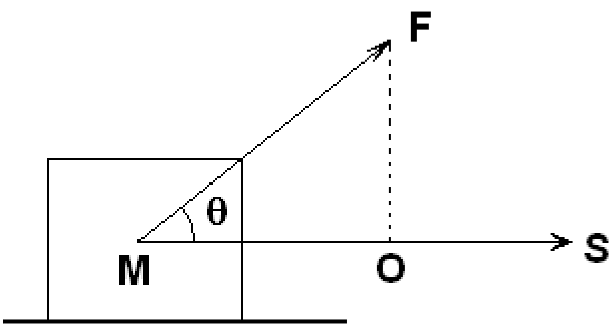
Explanation for why dot product gives scalar

According to definition of work,

work = displacement × force in the direction of displacement

Note work is scalar quantity

Consider following diagram in which force is vectors making angle of θ



with displacement which is a vector.

MO is the effective force along MS which represents displacement.

MO = Fcosθ

Now according to definition of work

W= S × Fcosθ

Which can be represented as $W = \vec{S} \cdot \vec{F}$ Thus dot product gives scalar product

Exercises 0.3.04

1. If a = 4i + 9j and b = 3i + 2j find (a) a · b (b) b · a (c) a · a (d) b · b.
2. Find the scalar product of the vectors 5i and 8j.
3. If p = 4i + 3j + 2k and q = 2i − j + 11k find
(a) p · q, (b) q · p, (c) p · p, (d) q · q.
4. If r = 3i + 2j + 8k show that r · r = $|r|^2$.

5 Determine whether or not the vectors 2i + 4j and −i + 0.5 j are perpendicular.

6. Evaluate p · i where p = 4i + 8j. Hence find the angle that p makes with the x axis.
7. Obtain the component of a vector A = 3i+4j in the direction of 2i+2j

Answers

1. (a) 30, (b) 30, (c) 97, (d) 13.
2. 0.
3. (a) 27, (b) 27, (c) 29, (d) 126.
4. Both equal 77.

5 Their scalar product is zero. They are non-zero vectors. We deduce that they must be perpendicular.

6. p · i = 4. The required angle is 63.4°. 7. $\frac{7}{\sqrt{2}}$

0.03.16 Vector product or cross product

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Vector product or cross product is defined as

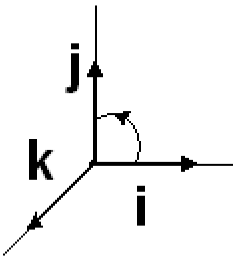
$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|sin\theta\hat{n}$$

Note that cross product gives again vector, direction of resultant vector is determined by right hand screw rule.

If we take cross product of unit vectors

$\hat{\imath}\times\hat{\imath} = 1\times1sin0 = 0$ As magnitude of unit vector is 1 and angle between two $\hat{\imath}$ is 0,

Similarly $\hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$



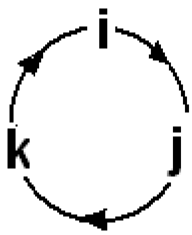
$$\hat{\imath}\times\hat{\jmath} = 1\times1sin90 = \hat{k}$$

As angle between x-axis and y-axis is 90°. Direction of resultant can be obtained by rotating right hand screw in the direction as shown in figure which gives the direction in +z axis

Similarly $\hat{\jmath}\times\hat{k} = \hat{\imath}$ and $\hat{k}\times\hat{\imath} = \hat{\jmath}$

But $\hat{\jmath}\times\hat{\imath} = -\hat{k}$

$\hat{k}\times\hat{\jmath} = -\hat{\imath}$ and $\hat{\imath}\times\hat{k} = -\hat{\jmath}$



This sequence can be remembered by following the arrows for positive resultant vector in adjacent figure and if followed in opposite to direction we get Negative vector

Thus we can say that if unit vectors are parallel their cross product is 0.

$$\vec{A}\cdot\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$\vec{A}\times\vec{B} = (A_xB_x\hat{\imath}\times\hat{\imath} + A_xB_y\,\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k}) \quad + (A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_y\hat{\jmath}\times\hat{\jmath} + A_yB_z\hat{\jmath}\times\hat{k})$$
$$+(A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath} + A_zB_z\hat{k}\times\hat{k})$$

Since $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$ above equation reduce to

$$\vec{A}\times\vec{B} = \left(A_xB_y\,\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k}\right) + \left(A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_z\hat{\jmath}\times\hat{k}\right) + \left(A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath}\right)$$

Now $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\imath}\times\hat{k} = -\hat{\jmath}$, $\hat{\jmath}\times\hat{\imath} = -\hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$, $\hat{k}\times\hat{\jmath} = -\hat{\imath}$

Substituting above values we get

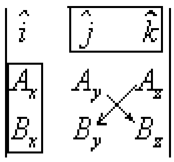
$$\vec{A}\times\vec{B} = \left(A_xB_y\,\hat{k} - A_xB_z\hat{\jmath}\right) + \left(-A_yB_x\hat{k} + A_yB_z\hat{\imath}\right) + \left(A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}\right)$$

By taking common $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ we get

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

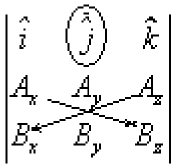
Above equation can be obtained by solving determinant

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



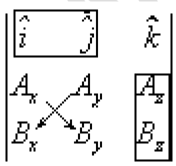
First select $\hat{\imath}$ then follow the arrow as shown in adjacent figure

To get term $\left(A_yB_z - A_zB_y\right)\hat{\imath}$



Similarly select $\hat{\jmath}$ and then follow the arrow as shown in adjacent figure

To get term $(A_xB_z - A_zB_x)\hat{\jmath}$ give negative sign



select $\hat{k}$ and then follow the arrow as shown in adjacent figure

To get term$\left(A_xB_y - A_yB_x\right)\hat{k}$

Now add all these terms to get equation as

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

Note that A × B = - B × A negative sign indicate directions are opposite

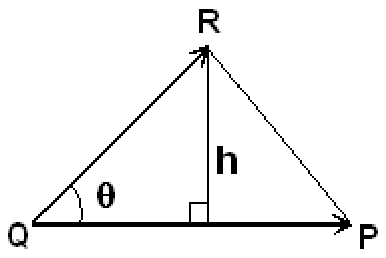
The vector product is distributive over addition. This means

a × (b + c) = a × b + a × c

Equivalently,

(b + c) × a = b × a + c × a

Important results

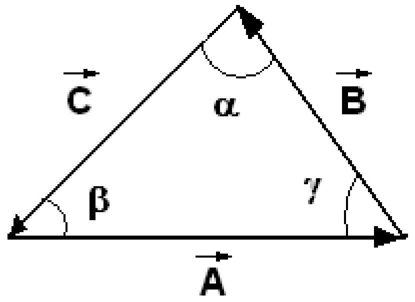


From figure area of triangle QPR =

$$A = \frac{1}{2}|\overrightarrow{QP}|h$$

$$A = \frac{1}{2}|\overrightarrow{QP}||\overrightarrow{QR}|sin\theta$$

$$A = \frac{1}{2}(\overrightarrow{QP}\times\overrightarrow{QR})$$



From above derivation for area we get

$$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{C}| = |\vec{C}\times\vec{A}|$$

AB sin(180 −γ) = BC sin(180-α) = CA sin(180-β)

Dividing each term by ABC, we have

$$\frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

Exercises 0.3.05

1. Find the cross products of following vectors

(a) p = i + 4j + 9k, q = 2i − k.

(b) p = 3i + j + k, q = i − 2j − 3k.

2. For the vectors p = i + j + k, q = −i − j − k show that, in this special case, p × q = q × p.

3.For the vectors a = i + 2j + 3k, b = 2i + 3j + k, c = 7i + 2j + k, show that

a × (b + c) = (a × b) + (a × c)

4. Find a unit vector which is perpendicular to both a = i + 2j − 3k and b = 2i + 3j + k.

5. Calculate the triple scalar product (a × b) · c when a = 2i − 2j + k, b = 2i + j and c = 3i + 2j + k.

Answers

1.(a) −4i + 19j − 8k, (b) −i + 10j − 7k.

2.Both cross products equal zero, and so, in this special case p × q = q × p. The two given vectors are anti-parallel.

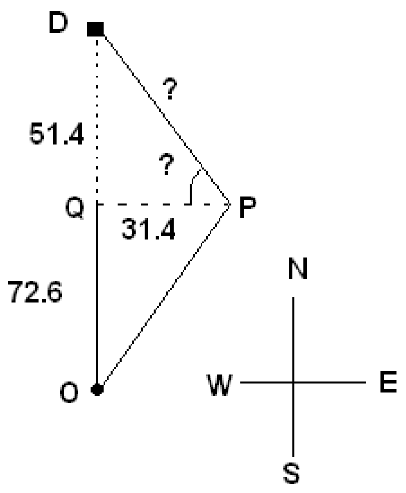
3. Both equal −11i + 25j − 13k

4. $\frac{1}{\sqrt{171}}(11\hat{\imath}-7\hat{\jmath}-\hat{k})$ 5. 7

Solved numerical

A ship sets out to sail a point 124 km due north. An unexpected storm blows the ship to a point 72.6 km to the north and 31.4 km to the east of its starting point. How far, and in what direction, must it now sail to reach its original destination?

Solution



As shown in figure O is starting point , reached to point P due to wind. , OQ = 72.6 km, Thus QD = 51.4. given QP = 31.4 km.

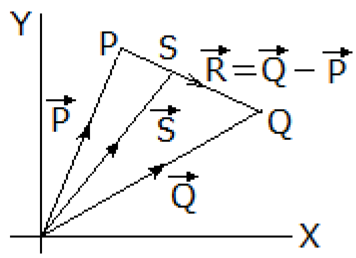
From the triangle DQP $tan\theta = \frac{51.4}{31.4}$

θ = 58.5°

Therefore the ship must go in the direction 58.5° north of west to reach its destination

Using Pythagoras we get PD = 62.2 km

Q) Three vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are shown in the figure. Let S be any point on the vector $\vec{R}$ The distance between the points P and S is $b\,|\vec{R}|$. The general relation among vectors$\vec{P}$, $\vec{Q}$ and $\vec{S}$ is ... [IIT advance 2017]



$a)\ \vec{S} = (1-b)\vec{P} + b^2\vec{Q}$

$b)\ \vec{S} = (b-1)\vec{P} + b\vec{Q}$

$c)\ \vec{S} = (1-b)\vec{P} + b\vec{Q}$

$d)\ \vec{S} = (1 - b^2)\vec{P} + b\vec{Q}$

Solution:

From figure

$\vec{S} = \vec{P} + b\vec{R}$

But $\vec{R} = \vec{Q} - \vec{P}$

$\vec{S} = \vec{P} + b(\vec{Q} - \vec{P})$

On rearranging terms

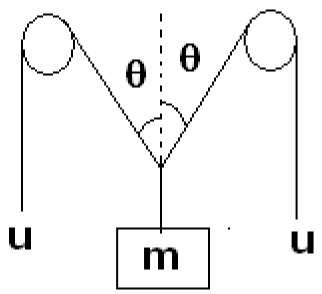
$\vec{S} = (1 - b)\vec{P} + b\vec{Q}$

Correct option c

Exercises 0.3.06

1. If the magnitude of vector A, B and C are 12, 5 and 13 units respectively and A+B = C, then what is the angle between A and B
2. The vectors 3i -2j + k and 2i + 6j + ck are perpendicular then find value of c.
3. If A = 2i+3j+k and B = 3i – 2j then what will be dot product.
4. If A= Acosθ i – Asinθj find the vector which is perpendicular to A
5. A vector A of magnitude 10 units and another vector of magnitude 6 units are acting at 60^{O} to each other. What is the magnitude of vector product of two vectors

6. A river is flowing at the rate of 6km/hr. A swimmer across with a velocity of 9 km/hr. What will be the resultant velocity of the man in(km/hr)
7. At what angle two forces 2F and √2 F act, so that the resultant force is F√10
8. In the arrangement shown in figure rope is pulled with velocity u in down



Then at what speed mass m will move up

9. If A · B = AB then what is angle between A and B
10. If A = 2i+3j and B=i+4j+k, then what will be the unit vector along (A+B)
10. A vector x is added to two vectors A = 3i -5j+7k and B = 2i + 4j -3k
11. 0.4 i +0.8 j + ck represents a unit vector when c is
12. A boat moves 10 km due west, 5 km due north, and then 10 km due east. The displacement of the boat from its initial position is
13. Unit vector along i+j is

Answers: 1)180° 2) C = 6

3) 0	4) Bsinθ i + Bcosθ j
5) 30√3 unit	6) √117
7)45°	8) u/cosθ
9) 0	10)$\frac{1}{\sqrt{59}}(3\hat{\imath}+7\hat{\jmath}+\hat{k})$
11) -5i + 2j -4k	12) √(0.2)
13) 5 km ,North	14) $\frac{\hat{\imath}+\hat{\jmath}}{\sqrt{2}}$

0.04 Differentiation

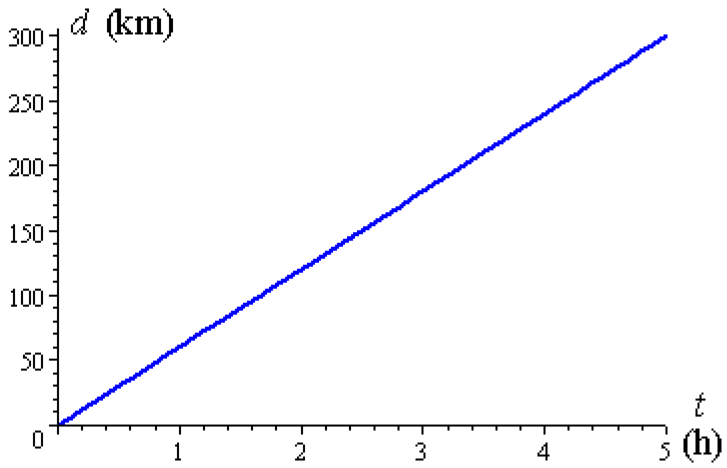
What is Differentiation?

Differentiation is all about finding rates of change of one quantity compared to another. We need differentiation when the rate of change is not constant.

What does this mean?

0.04.01 Constant Rate of Change

First, let's take an example of a car travelling at a constant 60 km/h. The distance-time graph would look like this:



We notice that the distance from the starting point increases at a constant rate of 60 km each hour, so after 5 hours we have travelled 300 km. We notice that the slope (gradient) is always 300/5 =60 for the whole graph. There is a constant rate of change of the distance compared to the time. The slope is positive all the way (the graph goes up as you go left to right along the graph.)

Equation of line is d = 60t, here d is distance and t is time and 60 is slope of the equation and is constant

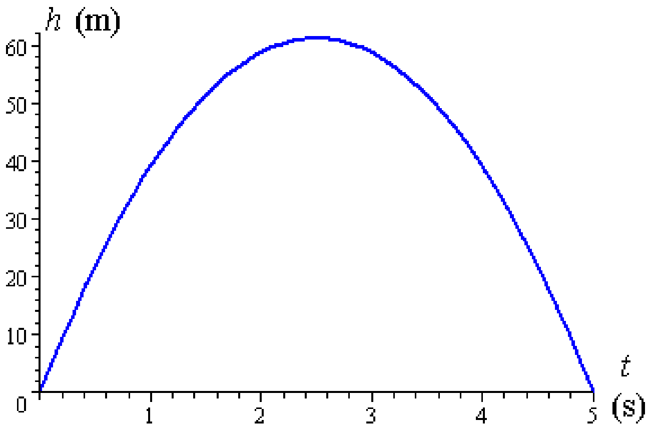
Now if we take a very small period of time then for that very small time displacement will be also very small but ratio of displacement and time will be again 60 km/hr. Such time period which is tending to zero time period but not zero is denoted by dt. Note it is not d and t but one word dt = very small time period tending to zero. And ds denote very small displacement for such very small period.

Now the ratio of ds and dt is known as instantaneous velocity or velocity at that particular time. Similar to velocity noted by us when we look into the Speedo meter. If Speedo meter 60 km/hr it means speed of the car is 60km/hr at that particular moment. $v = \frac{ds}{dt}$

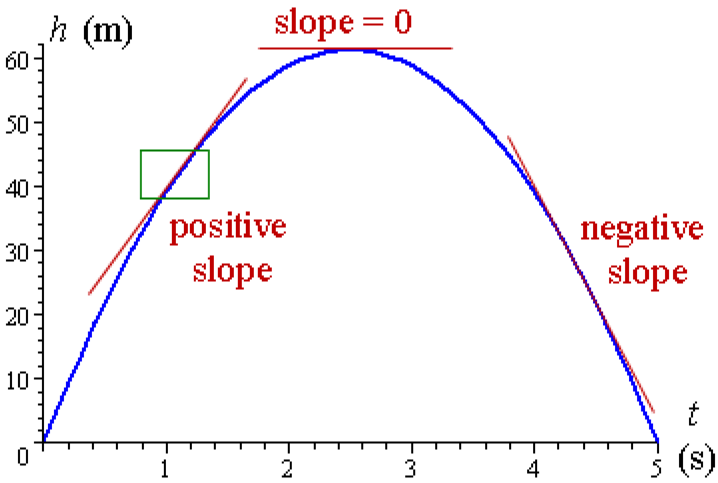
0.04.02 Rate of Change that is Not Constant

Now let's throw a ball straight up in the air. Because gravity acts on the ball it slows down, then it reverses direction and starts to fall. All the time during this motion the velocity is changing. It goes from positive (when the ball is going up), slows down to zero, then becomes negative (as the ball is coming down). During the "up" phase, the ball has negative acceleration and as it falls, the acceleration is positive.

Now let's look at the graph of height (in metres) against time (in seconds).



Notice this time that the slope of the graph is changing throughout the motion. At the beginning, it has a steep positive slope (indicating the large velocity we give it when we throw it). Then, as it slows, the slope get less and less until it becomes 0 (when the ball is at the highest point and the velocity is zero). Then the ball starts to fall and the slope becomes negative (corresponding to the negative velocity) and the slope becomes steeper (as the velocity increases).



TIP :

The slope of a curve at a point tells us the rate of change of the quantity at that point

We can differentiate function, which relates two interdependent variables. For example if position of object is changed from one point to another, time also change with it thus we can have mathematical equation for displacement in terms of time. As shown in first graph equation was S = 60t

For object going up against gravitational force is given by equation of motion is

$$S = ut - \frac{1}{2}gt^2$$

Note here that u is constant, g is also constant. Thus we can easily differentiate above equation to get equation for velocity which can indicate velocity of the object at that particular time. To differentiate above equation we have learn common derivatives of polynomial which are stated below without proof

0.04.03 Derivatives of Polynomials

Common derivatives

a) Derivative of a Constant

$$\frac{dc}{dx} = 0$$

This is basic. In English, it means that if a quantity has a constant value, then the rate of change is zero.

Example

$$\frac{d5}{dx} = 0$$

b) Derivative of n-th power of x

$$\frac{d}{dx}x^n = nx^{n-1}$$

But

$$\frac{d}{dt}x^n = nx^{n-1}\frac{dx}{dt}$$

Example

$$\frac{d}{dx}x^5 = 5x^4$$

c) Derivative of Constant product

$$\frac{d}{dx}(cy) = c\frac{dy}{dx}$$

Exercises 0.4.01

Find the derivative of each of the following with respect to x:

a) x^{6} b) p^{10} c) $\frac{3}{2}x^2$ d) x^{-2} e) $5x^{-5}$ f) $x^{13/2}$ g) $6q^{3/5}$ h) $\frac{2}{3}x^{\frac{3}{2}}$ i) $\frac{1}{x^4}$ j) $\sqrt{x}$ k) $x^{-3/2}$ l) $y^{-1/5}$

Answers : a) $6x^5$ b) $10p^9\frac{dp}{dx}$ c) 3x d) $-2x^{-3}$ e) $-25x^{-6}$ f) $\frac{13}{2}x^{\frac{11}{2}}$

g) $\frac{18}{5}q^{\frac{-2}{5}}\frac{dq}{dx}$ h) $x^{1/2}$ i) $-4x^{-5}$ j) $\frac{1}{2}x^{\frac{-1}{2}}$ k) $\frac{-3}{2}x^{\frac{-5}{2}}$ l) $\frac{-1}{5}y^{\frac{-6}{5}}\frac{dy}{dx}$

0.04.04 Linearity rules

We frequently express physical quantities in terms of variables. Then, functions are used to describe the ways in which these variables change. We now look at some more examples which assume that you already know the following rules: if

y =v ± u , here v and u are functions of 'x'

$$\frac{dy}{dx} = \frac{dv}{dx} \pm \frac{du}{dx}$$

Example 1

Example Suppose we want to differentiate y = $6x^{3} - 12x^{4} + 5$.

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3 - 12x^4 + 5)$$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3) - \frac{d}{dx}(12x^4) + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 6\frac{d}{dx}(x^3) - 12\frac{d}{dx}(x^4) + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 6\times 3x^2 - 12\times 4x^3 + 0$$

$$\frac{dy}{dx} = 18x^2 - 48x^3$$

Example 2: Equation for displacement is given by, find equation for velocity and acceleration

$$S = ut - \frac{1}{2}gt^2$$

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(ut - \frac{1}{2}gt^2\right)$$

$$v = \frac{d}{dt}(ut) - \frac{d}{dt}\left(\frac{1}{2}gt^2\right)$$

Since u and g and ½ are constant

$$v = u\frac{d}{dt}(t) - \frac{1}{2}g\frac{d}{dt}(t^2)$$

$$v = u - \frac{1}{2}g(2t) \ \dots(i)$$

v = u – gt is the equation for velocity

By taking further derivative or (i) with respect to time we get acceleration

$$a = \frac{dv}{dt} = -g$$

Similarly power is rate of work done per unit time, can be written as

$$P = \frac{dw}{dt}$$

Example: $y = (5x^3+2x)^2$

When we to take derivative of function consists of power of bracket, the first take derivative of bracket, then take derivative of function in the bracket

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 2x)^2$$

$$\frac{dy}{dx} = 2(5x^3 + 2x)^{2-1}\frac{d}{dx}(5x^3 + 2x)$$

$$\frac{dy}{dx} = 2(5x^3 + 2x)^1(15x^2 + 2)$$

Exercises 0.4.02

Find the derivative of each of the following:

a) $X^2 + 12$ b) x^5+x^3+2x

c) $(2x^5+x^2)^3$ d) $\left(\frac{2}{5}x^{\frac{3}{6}} + 7x^{-2}\right)^2$

e) $7p^2 - 5q^3 + 12x + 5$ f) $y = x + \frac{1}{x}$

Answers a) 2x b) $5x^4 + 3x^2 + 2$

c) $3(2x^5+x^2)^2\,(10x^4+2x)$ d) $2\left(\frac{2}{5}x^{\frac{3}{6}} + 7x^{-2}\right)\left(\frac{1}{5\sqrt{x}} - 14x^{-3}\right)$

e) $14p\frac{dp}{dx} - 15q^2\frac{dq}{dx} + 12$ f) $1 - x^{-2}$

Solved numerical

Q1) The displacement x of a particle moving along a straight line at time t is given by $x=a_0+a_1t+a_2t^2$ The find formula for velocity and acceleration.

Solution: as displacement x is the function of time then derivative with respect to time (t) will give us velocity and derivative of velocity will give us acceleration.

$$v = \frac{dx}{dt} = \frac{d}{dt}(a_0 + a_1t + a_2t^2)$$

$$v = \frac{d}{dt}a_0 + \frac{d}{dt}(a_1t) + \frac{d}{dt}(a_2t^2)$$

$$v = \frac{d}{dt}a_0 + a_1\frac{d}{dt}(t) + a_2\frac{d}{dt}(t^2)$$

Since a_0, a_1, a_2 all are constant we get equation for velocity as

$$v = 0 + a_1 + a_2(2t)$$

$$v = a_1 + 2a_2t$$

By again taking derivative of velocity we get equation for acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}(a_1 + 2a_2t)$$

$$a = \frac{d}{dt}(a_1) + 2a_2\frac{d}{dt}(t)$$

$$a = 2a_2$$

Q2) A particle moves along a straight line OX. At a time t (in seconds) the distance x(in meters) of the particle from O is given by $x=40+12t-t^3$

How long would the particle travel before coming to rest?

Solution: Given equation for displacement is a function of time. When particle comes to rest it means its velocity will become zero. Thus by taking first derivative of displacement equation with respect to time we get formula for velocity

$$v = \frac{dx}{dt} = \frac{d}{dt}(40 + 12t - t^3)$$

$$v = \frac{d}{dt}(40) + \frac{d}{dt}(12t) - \frac{d}{dt}(t^3)$$

$$v = 0 + 12 - 3t^2$$

v= $12-3t^2$

Since object comes to rest v = 0

∴ 0 = $12-3t^2$

∴ t = 2 sec

Thus particle will come to rest after 2 seconds, by substituting t=2 sec in equation for displacement we will displacement of particle before it comes to rest

X = 40 +12 (2) - 2^3

X = 40 + 24 - 8 = 56m

But its initial position t= 0 was

X= 40 +12 (0) –(0)3 = 40 cm

Thus displacement in 2 second = final position – initial position = 56 – 40 = 16 cm

Thus particle will travel 16m before it comes to rest.

Q 3) The relation between time and distance x is $t=\alpha x^2+\beta x$, where α and β are constant. Obtain formula for retardation.

Solution:

Since t depends on displacement we can take derivative of t with respect to x

$$\frac{dt}{dx} = 2\alpha x + \beta$$

We know that $\frac{dx}{dt} = v$

$$\frac{1}{v} = 2\alpha x + \beta$$

$$v = (2\alpha x + \beta)^{-1}$$

Now $a = \frac{dv}{dt}$

$$a = \frac{dv}{dt} = (-1)(2\alpha x + \beta)^{-2}\frac{d}{dt}(2\alpha x + \beta)$$

$$a = (-1)(2\alpha x + \beta)^{-2}\left(2\alpha\frac{dx}{dt}\right)$$

$$a = -2\alpha v^2 v \quad [as\ v = (2\alpha x + \beta)^{-1}]$$
$$a = -2\alpha v^3$$

Exercises 0.4.03

1. A particle moves along a straight line such that its displacement at any time 't' is given by s=(t^3-6t^2+3t+4) meters Find the velocity when the acceleration is zero

Ans V =-9 m/s

2. The displacement of a particle is represented by the following equation s=3t^3 + 7t^2 + 5t + 8 , s is in meters and t in second. What will be the acceleration of particle at t=1 s .

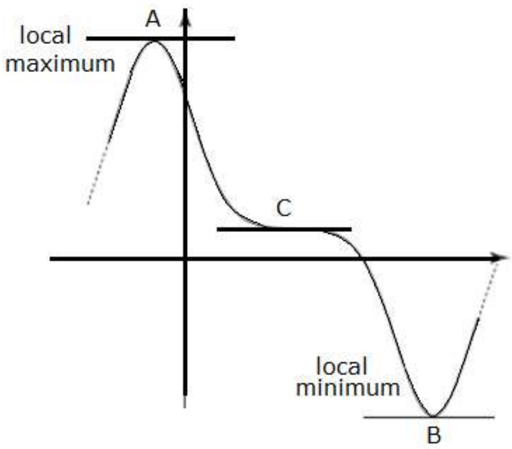
Ans 32 m/s^2

3. The displacement of a particle varies with time(t) as s=at^2 - bt^3.

At what time acceleration of the particle will become zero

Ans $t = \frac{a}{3b}$

0.04.05 Maxima and Minima of function



Refer adjacent graph. Notice that at points A and B the curve actually turns. These two stationary points are referred to as turning points. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points.

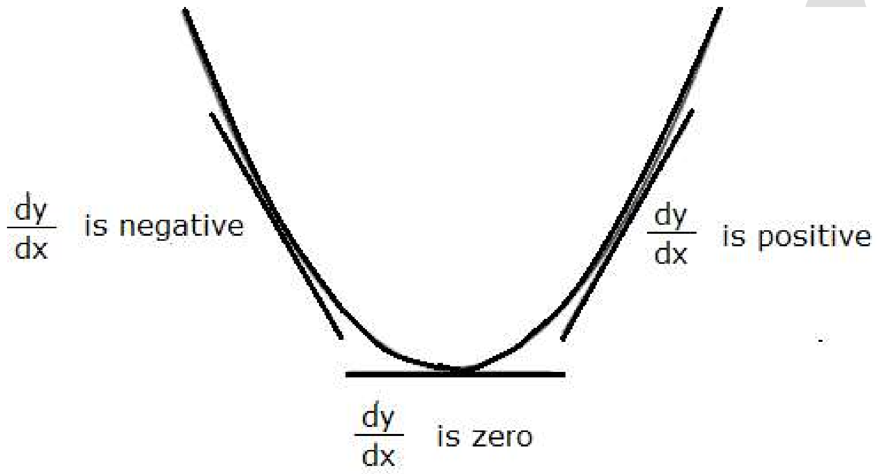
But not all stationary points are turning points (e.g. point C).

In other words, there are points for which $\frac{dy}{dx} = 0$ are stationary point but not necessarily be turning points.

Point A in graph is called a maximum and Point B is called minima

0.04.06 Distinguishing maximum points from minimum points

1) Minimum points



Notice that to the left of the minimum point dy/dx is negative. Because the tangent has negative gradient. At the minimum point, dy/dx is zero. To the right of the minimum point dy/dx is positive because here the tangent has a positive gradient. So, dy/dx goes from negative, to zero, to positive as x increases. In other words, dy/dx must be increasing as x increases.

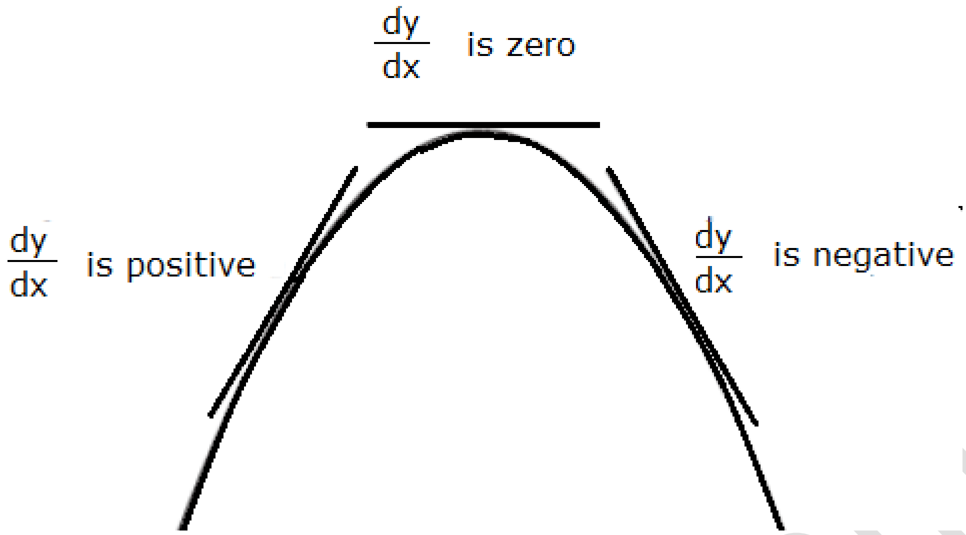
Or rate of change of slope with respect to x is increasing or becomes positive. Thus derivative slope with respect to x is decreasing ,can be represented as

$$\frac{d}{dx}(slope) = \frac{d}{dx}\left(\frac{d}{dx}y\right) = \frac{d^2y}{dx^2}$$

$\frac{d^2y}{dx^2}$ is known as second order derivative .

Note if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at a point then that point is minima

2) Maximum points



Notice that to the left of the minimum point, $\frac{dy}{dx}$ is positive because the tangent has positive gradient. At the minimum point, $\frac{dy}{dx} = 0$

To the right of the minimum point $\frac{dy}{dx}$ is negative because here the tangent has a negative gradient. So, $\frac{dy}{dx}$ goes from positive, to zero, to negative

as x increases. In other words, $\frac{dy}{dx}$ must be decreasing as x increases. Or rate of change of slope with respect to x is increasing or becomes negative. Thus derivative slope with respect to x is decreasing

Note if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at a point then that point is maxima

Example : 1) $y = \frac{1}{2}x^2 - 2x$

$$\frac{dy}{dx} = x - 2 \;\ldots\ldots eq(1)$$

For finding stationary point equate $\frac{dy}{dx} = 0$

X – 2 = 0 Thus x = 2

Now we will take second order derivative of equation to check maxima or minima

$$\frac{d^2y}{dx^2} = -2$$

Since second-order derivative is less than zero, At point x = 2 equation attends maxima and to find the maximum value substitute x = 2 in equation

$$y = \frac{1}{2}x^2 - 2x$$

$$y = \frac{1}{2}2^2 - 2\times2 = -2$$

Answer is : maximum at (2,−2)

Q. 2) $y = 2x^3 - 9x^2 + 12x$

First order derivative

$$\frac{dy}{dx} = 6x^2 - 18x + 12 \dots\dots(1)$$

Equate $6x^2 - 18x + 12 = 0$

Or $x^2 - 3x + 2 = 0$

Root x = +2 and x = +1

To check maxima and minima take derivative of equation 1

$$\frac{d^2y}{dx^2} = 12x - 18 \dots\dots.(2)$$

Substituting x = +2

We get

$$\frac{d^2y}{dx^2} = 12(2) - 18 = 6 > 0$$

Since value of second order derivative at x = +2 is greater than zero, at point x =+2 function has minima by substituting x = 2 in given equation we get

$y = 2x^3 - 9x^2 + 12x$

$y = 2(2)^3 - 9(2)^2 + 12(2) = 4$

Minimum at (2,4)

Now by Substituting x = +1

We get

$$\frac{d^2y}{dx^2} = 12(1) - 18 = -6 < 0$$

Since value of second order derivative at x = +1 is less than zero, at point x =+1 function has maxima by substituting x = 1 in equation

$y = 2x^3 - 9x^2 + 12x$

$y = 2(1)^3 - 9(1)^2 + 12(1) = 5$

Maximum at (1,5)

Exercises 0.4.04

Locate the position and nature of any turning points of the following functions.

a) $y = x^2 + 4x + 1$ b) $y = 12x - 2x^2$

c) $y = -3x^2 + 3x + 1$ d) $y = x^4 + 2$

e) $y = 7 - 2x^4$ f) $y = 4x^3 - 6x^2 - 72x + 1$

g) $y = -4x^3 + 30x^2 - 48x - 1$,

Ans a) Minimum at $(-2,-3)$

b) Maximum at $(3,18)$

c) Maximum at (1/ 2 ,7/4),

d) not Minimum not maximum at (0, 2),

e) not Minimum not maximum at (0, 7),

f) Maximum at $(-2, 89)$, minimum at $(3,-161)$,

g) Maximum at $(4, 31)$, minimum at $(1,-23)$,

Application of maxima and minima to real life problems

Q1) The daily profit, P of an oil refinery is given by $P = 8x - 0.02x^2$, here x is the number of barrels production per day. Find the value of x for which profit become maximum

Solution:

Take derivative and equate it with zero to find value of x

$$\frac{dP}{dx} = 8 - 0.04x$$

8 -0.04x = 0

X = 200

Now verify value of x is for maxima or not by taking second order derivative of P

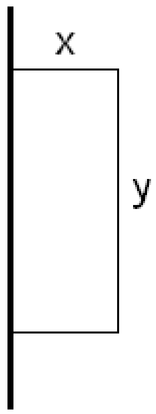
$$\frac{d}{dx}\left(\frac{d}{dx}P\right) = -0.04 < 0$$

As second order derivative gives negative value thus function have maximum value at x = 200

Thus maximum profit $P = 8(200) - 0.02(200)^2 = 1600 - 800 = 800$ \$ per day

Q2) A rectangular storage area is to be constructed alongside of a tall building. A security fence is required along the three remaining side of the area. What is the maximum area that can be enclosed with 800 m of fence wire?

Ans:



Length of fence wire = 2x+ y = 800

Y = 800 – 2x

Area enclosed = xy

Area A = x (800 -2x) = 800x – $2x^2$

To find maxima take first order derivative w.r.t x and equate to zero

$$\frac{dA}{dx} = 800 - 4x$$

800-4x = 0

X =200 m and y = 800 – 2x = 800 – 400 = 400 m

Verification of maxima

$$\frac{d^2A}{dx^2} = -4 < 0$$

Since second order derivative is less than zero for x=200 area is maximum

Area = xy = 200 ×(400) = 80,000 m^2

Maximum area that can be enclosed = 80,000 m^2

Try your self

Q3) A box with a square bas has no top. If 64√3 cm^2 of material is used, what is the maximum possible volume for the box

Some more important rules

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{(v)^2}$$

Some more formula of Differentiation

$$\frac{d}{dx} sinx = cosx$$

$$\frac{d}{dx} cosx = -sinx$$

$$\frac{d}{dx} tanx = sec^2 x$$

$$\frac{d}{dx} cotx = -cosec^2 x$$

$$\frac{d}{dx} secx = secx\ tanx$$

$$\frac{d}{dx} cosecx = -cosecx\ cotx$$

$$\frac{d}{dx} e^x = e^x$$

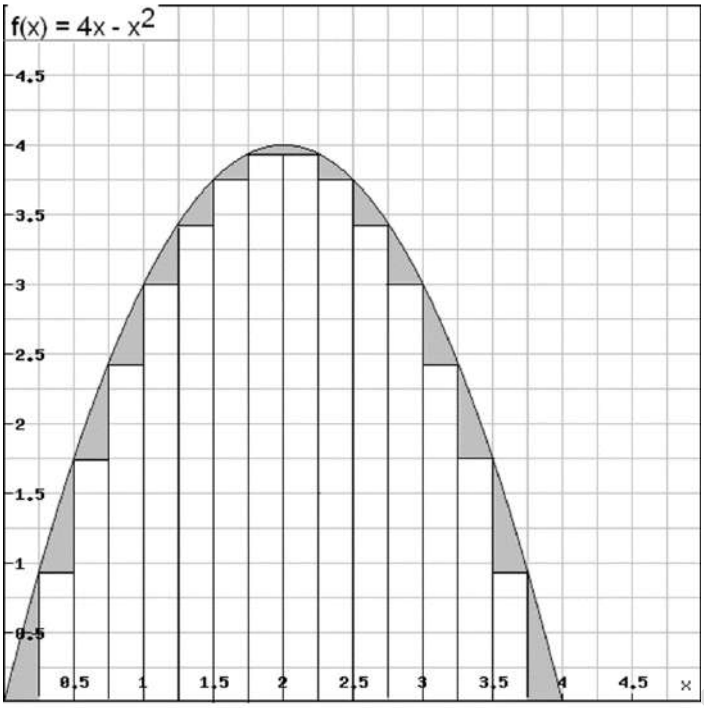
$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}\ a^x = a^x\ log_e a$$

0.05 Integration

0.05.01 Definition

In mathematics, an integral assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data. Integration is one of the two main operations of calculus, with its inverse, differentiation, being the other.
If we want to calculate the area covered under the graph we have to draw rectangle of fixed width and variable height in the curve.

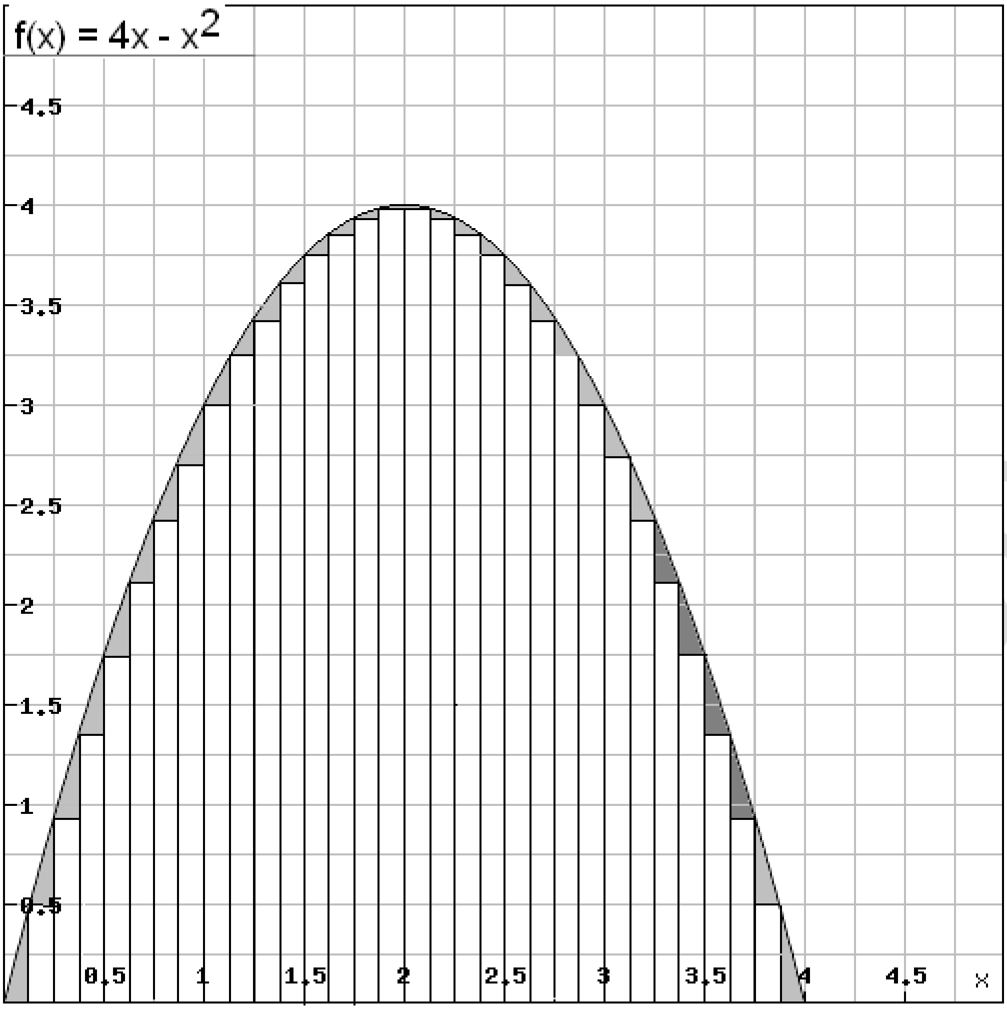


As shown in the graph, we have drawn rectangle of fix with about 0.25. By calculating area of each rectangle and then adding we will get area enclosed by graph much smaller than actual. Area not calculated is shown by shaded portion. Thus to have better approximation of area under curve. We have to take smaller fixed width

If a is the width of each rectangle, and y_1 , y_2, y_3 , … …. are the height of the rectangle than Area under curve A

$A = ay_1 + ay_2 + ay_3 + \ldots.. ay_n$

In this case n = 14



As show in figure above now width is taken as 0.125 on x-axis. Then shaded region which we could not calculate by adding area of rectangle is reduced compare to previous 0.25 width

If we continue to take smaller width, shaded region will go on reducing.

If b is smaller width than 'a' then

$A_1 = by_1 + by_2 + by_3 + \ldots.. by_n$

In this case n = 30

Clearly $A_1 > A$, and A_1 is more accurate than A

If we make very small with tending to zero, then we will get accurate area under curve

Let dx be very small width. Such that $dx \to 0$ then are under curve

$A = y_1dx + y_2dx+y_3dx + \ldots\ldots+y_ndx$

Above equation can be expressed mathematically as summation

$$A = \sum_{i=1}^{\infty} y_i \, dx$$

It is not possible for anyone to add millions and trillion of rectangles. Thus summation sign is replaced by ∫ , singe of Summation of function

Now the given graph equation is f(x) = 4x – x^2. Thus our equation for integration reduced to

$$A = \int (4x - x^2) \, dx$$

Above type of integration is called indefinite integral and to not have staring or end point

But in our case starting point is x = 0 called as lower limit and x = 4 called as upper limit our integration equation change to

$$A = \int_0^4 (4x - x^2) dx$$

Without any proof we will use basic formula for integration

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

Here c is called constant of integration

Note in integration power increases by 1 and increased power divides

We cannot integrate above with respect to dt or any other, if function is in terms of y then dy must appear not dx or dt.

Thus by using above formula

$$A = \int_0^4 (4x - x^2) dx$$

$$A = 4\int_0^4 (x) dx - \int_0^7 (x^2) \, dx$$

$$A = 4\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4$$

Substitute x= 4 as upper limit and x=0 as lower limit in above equation

$$A = \frac{4}{2}(4^2 - 0^2) - \frac{1}{3}(4^3 - 0^2)$$

$$A = 2(16) - \frac{1}{3}(64)$$

$$A = 32 - 21.33$$

Thus area under the given curve = 10.67 units (accurate up to two decimal)

Exercises 0.5.01

Integrate following

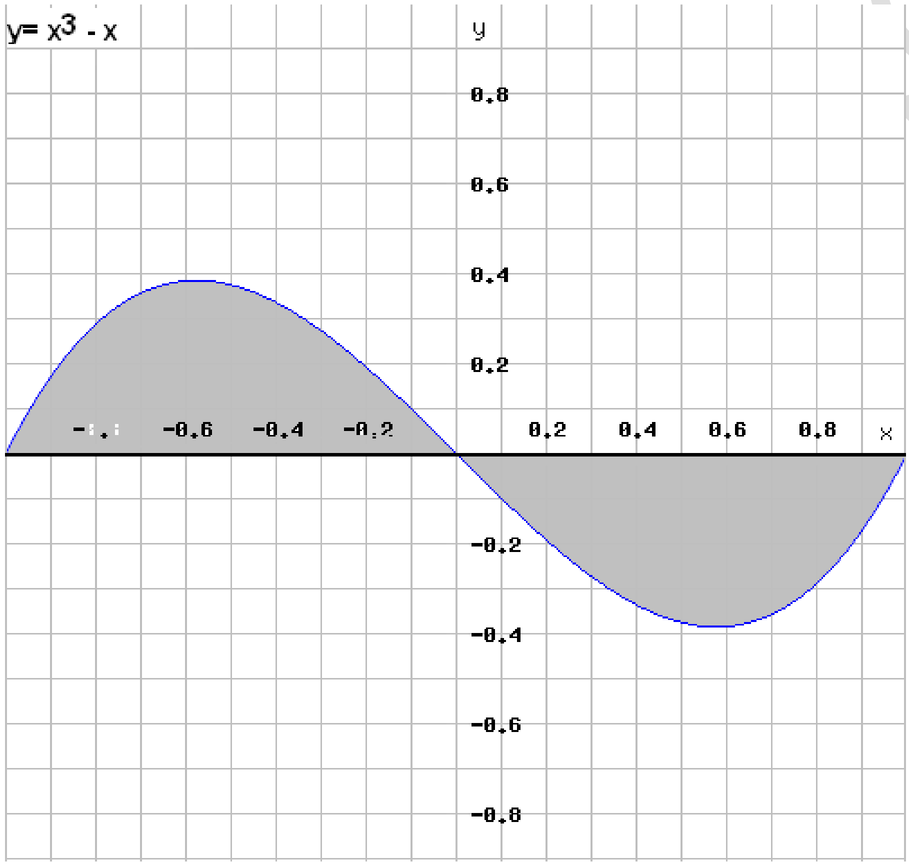
a) $X^{-3/5}$ b) $x^{5/3}$ c) x^{7} d) $X^6 - 10x^3 + 5$

Answers: a) $\frac{5}{2}x^{\frac{2}{5}}$ b) $\frac{3}{8}x^{8/3}$ c) $\frac{x^8}{8}$ d) $\frac{1}{7}x^7 - \frac{5}{2}x^4 + 6x$

Example

Find the area contained by the curve y = x(x − 1)(x + 1) and the x-axis.

Above function intercept x axis at x= 0



To find other intercept take

$$x(x - 1)(x + 1) = 0$$

(x-1)(x+1) = 0

Thus at x = +1 and x = -1 curve intercept x-axis

Thus we get two regions x=-1 to x = 0 and x =0 to x =1 as shown in graph.

Area enclosed by $y=x^3 - x$ and axis is shown in graph by shaded region.

Therefore we have to integrate above function for two limits

Y =x(x-1)(x+1)

$$Y = x^3 - x$$

$$A = \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} (x^3 - x)dx$$

$$A = \left\{ \int_{-1}^{0} x^3 dx - \int_{-1}^{0} x dx \right\} + \left\{ \int_{0}^{1} x^3 dx - \int_{0}^{1} x dx \right\}$$

$$A = \left\{ \left[\frac{x^4}{4}\right]_{-1}^{0} - \left[\frac{x^2}{2}\right]_{-1}^{0} \right\} + \left\{ \left[\frac{x^4}{4}\right]_{0}^{1} - \left[\frac{x^2}{2}\right]_{0}^{1} \right\}$$

$$A = \left\{ \left[0 - \frac{(-1)^4}{4}\right] - \left[0 - \frac{(-1)^2}{2}\right] \right\} + \left\{ \left[\frac{(1)^4}{4} - 0\right] - \left[\frac{(1)^2}{2} - 0\right] \right\}$$

$$A = \left\{ -\frac{1}{4} + \frac{1}{2} \right\} + \left\{ \frac{1}{4} - \frac{1}{2} \right\}$$

$$A = \frac{1}{4} + \left\{ -\frac{1}{4} \right\} = 0$$

Since the graph is symmetric about x axis have equal areas above and below x axis. Thus we get area equal to 0 which is not possible

So to solve such we solve the area under the curves separately and their absolute values are added, so we get

A = |1/4| + |-1/4| = 1/2

$$A = \frac{1}{2}\ units$$

Exercises 0.5.02

1\. a) Find the area between the curve y = x(x − 3) and the ordinates x = 0 and x = 5.

b) Find the area bounded by the curve y = $x^2 + x + 4$, the x-axis and the ordinates x = 1 and x = 3.

2. Calculate the value of $\int_{-1}^{+1} x(x-1)(x+1)dx$

3. Calculate the value of $\int_0^6 (4x - x^2)dx$

Answers

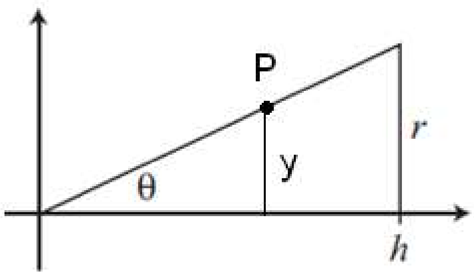
1. a) $\frac{25}{6}\ units$ b) $\frac{62}{3}\ units$

2. -1 /2

3. 0.

0.05.02 Application of integration for finding volume

Volume of Cone



Suppose we have a cone of base radius *r* and vertical height *h*. We can imagine the cone being formed by rotating a straight line through the origin by an angle of 360◦ about the *x*-axis.

If we rotate point around x-axis it will trace a circle of radius y and area πy^2

Now equation for hypo, is y =tan θ x, here tan θ is slope of hypo

Area of circle for radius y, a= π(tan θ x)^2

Distance of circles increases from 0 to h and if we add area of all circle = volume of come

$$V = \int_0^h \pi tan^2\theta\, x^2\, dx$$

$$V = \pi tan^2\theta \int_0^h x^2\, dx$$

$$V = \pi tan^2\theta \left[\frac{x^3}{3}\right]_0^h$$

$$V = \pi tan^2\theta\left(\frac{h^3}{3}\right)$$

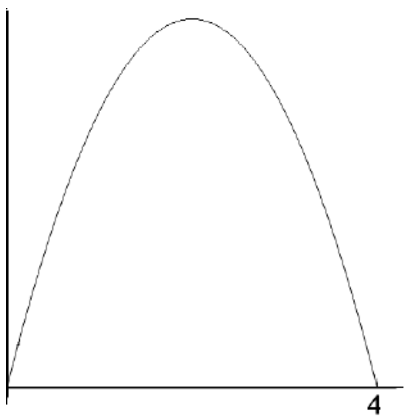
As tanθ =r/h

$$V = \pi tan^2\theta\left(\frac{h^3}{3}\right)$$

$$V = \pi\frac{y^2}{h^2}\left(\frac{h^3}{3}\right)$$

$$V = \pi\frac{hy^2}{3}$$

Example: General equation



Graph shown in figure is for y = 4x-x^2

Any point on the graph line if rotated by 360^O around x-axis will represent circle. We can find the volume of the line rotated along x can be calculated as

V= ∫πy^{2}dx.

Now upper limit point is x=4 and lower limit point is x=0

$$V = \int_0^4 \pi(4x - x^2)^2\,dx$$

$$V = \int_0^4 \pi(16x^2 - 8x^3 + x^4)\,dx = \pi\left[\left(\frac{16}{3}x^3 - \frac{8}{4}x^4 + \frac{1}{5}x^5\right)\right]_0^4 = 34.1\pi\ units$$

0.05.03 Application of integration in Physics

We can obtain equations of motion using integration for constant acceleration

We know that

1) Acceleration is rate of change of velocity

$$a = \frac{dv}{dt}$$

$$(a)dt = dv \quad \ldots(1)$$

Let initial velocity be v_0 and final velocity be v at time t then by integrating above equation (1)

$$\int_0^t (a)\, dt = \int_{v_0}^{v} dv$$

$$a[t]_0^t = [v]_{v_0}^{v}$$

$$at = v - v_0$$

2) From the definition of velocity

$$v = \frac{dx}{dt}$$

$$vdt = dx$$

Note Velocity is not constant but acceleration is velocity

$v = v_0 + at$ or $(v_0 + at)\, dt = dx$

Let initial position of object be x_i at time t=0 and final position be x_f at time t

Integrating above equation we get

$$\int_{x_i}^{x_f} dx = \int_0^t (v_0 + at)\, dt$$

On solving above equation we get

$$[x]_{x_i}^{x_f} = \left[v_0 t + a\frac{t^2}{2}\right]_0^t$$

$$x_f - x_i = v_0 t + \frac{1}{2}at^2$$

3) Velocity is changing with time thus must be changing with position

$$\frac{dv}{dx} = ?$$

We will split left side of the above equation

$$\frac{dv}{dt} \times \frac{dt}{dx} = a \times \frac{dt}{dx}$$

$$But \ \frac{dx}{dt} = v \quad \therefore \ \frac{dt}{dx} = \frac{1}{v}$$

$$\frac{dv}{dx} = \frac{a}{v}$$

$$\therefore \quad vdv = adx$$

Integrating above equation

$$\int_{v_0}^{v} (v)dv = a \int_{x_i}^{x_f} dx$$

$$\left[\frac{v^2}{2}\right]_{v_0}^{v} = a[x]_{x_i}^{x_f}$$

$$\frac{v^2 - v_0^2}{2} = a(x_f - x_i)$$

$$v^2 - v_0^2 = 2a(x_f - x_i)$$

$x_f - x_i = S$ (displacement)

Thus by using integration we have proved three basic equations for motion.

Example

Let object starts its motion which was at rest from point P at a distance b from origin under the influence of force given by equation F = $-kx^{-2}$, obtain equation for velocity.

Solution

Equation of force is position dependent

Now F = ma

Ma = $-kx^{-2}$

But

$$a = \frac{dv}{dt}$$

$$\therefore\ m\frac{dv}{dt} = -kx^{-2}$$

$$\frac{dv}{dt} = -\frac{k}{m}x^{-2}$$

$$\frac{dv}{dx}\frac{dx}{dt} = -\frac{k}{m}x^{-2}$$

$$\frac{dv}{dx}v = -\frac{k}{m}x^{-2}$$

$$vdv = -\frac{k}{m}x^{-2}dx$$

Given velocity is zero when particle is at distance b, let v_x be the velocity at distance x from origin

At time t

$$\int_0^{v_x} vdv = -\frac{k}{m}\int_b^x x^{-2}dx$$

$$\frac{v_x^2}{2} = +\frac{k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

$$v_x = \frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

0.05.04 Some important Integration formulas

$$\int sin\theta\ d\theta = \ -cos\theta$$

$$\int cos\theta\ d\theta = \ sin\theta$$

$$\int tan\theta\ d\theta = \log sec\ \theta = \ -\log cos\theta$$

$$\int cot\theta\ d\theta = \log sin\theta = \ -\log cosec\theta$$

$$\int \frac{1}{x} dx = logx$$

$$\int e^x\, dx = e^x$$

0.06 Quadratic equation

The standard quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$.

The solution of the quadratic equation is the values of variable x which satisfied the given quadratic equation.

To solve the quadratic equation factorization is the proper method.

But if polynomial cannot be factorize then to find the roots of equation we find discriminant denoted by Δ or D

$D = b^2 - 4ac$

Depending upon the value of D we have conditions

(i) If D < 0, then no real roots for given equation.

(ii) If D = 0, then equal and real roots and roots are given by $\frac{-b}{2a}$

(iii) If D > 0, then two distinct real roots . The roots are denoted by α and β.

Where

$$\alpha = \frac{-b+\sqrt{D}}{2a} \; and \; \beta = \frac{-b-\sqrt{D}}{2a}$$

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \; and \; \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots = $\alpha\beta = \frac{c}{a}$

Difference of the roots = $\alpha - \beta = \frac{\sqrt{D}}{a}$

Application of quadratic equation in physics

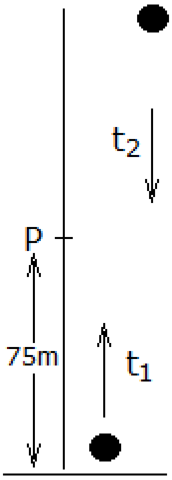
Example

Let object is thrown up with initial speed of 50m/s. At what time it will cross point (P) at height 75 m from ground. And what is the difference between the time when object cross point P while going up and coming down. Take g=10m/s^2

Solution

From equation of motion

$$s = ut + \frac{1}{2}at^2$$



As object is going up and gravitational acceleration down

a=g= -10m/s^2

$$75 = 50(t) - \frac{1}{2}\times 10\times t^2$$

$$t^2 - 10t + 15 = 0$$

Now comparing above equation with standard equation

a =1 , b = -10, c = 15

From formula D = b^2 – 4ac

D= $(-10)^2$ - 4×1×15

D = 40

As D > 0, two real roots

$$t_2 = \frac{10+\sqrt{40}}{2\times 1} \ \ and \ \ t_1 = \frac{10-\sqrt{40}}{2\times 1}$$

t_2 = 8.162 and t_1 = 1.838

Thus at time 1.838s object will cross point P while going up

And at time 8.162s object will cross point O while going down

Thus difference in timing = 8.162 – 1.838 = 6.234 s

This difference in timing can be calculated directly using formula=

$$\alpha - \beta = \frac{\sqrt{D}}{a}$$

$$t_1 - t_2 = \frac{\sqrt{40}}{1} = 6.324\, s$$

Solve:

a) $x^2 + x - 4 = 0$ b) $x^2 - 3x - 4 = 0$.

c) $6x^2 + 11x - 35 = 0$ d) $x^2 - 48 = 0$.

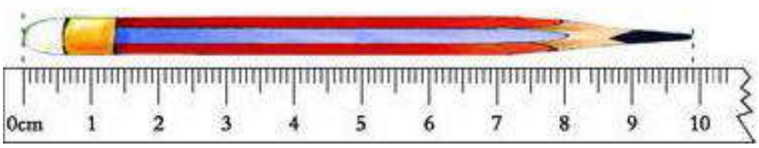
e) $x^2 - 7x = 0$.

Answer:

a) x = −1, 3 b) x = −1, 4 c) x = −7/2, 5/3 d) x = ± 4√(3) e) x = 0, 7

Topic 1: Dimension, Measurement and error

CONTENTS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1.01 Systems of units

All the quantities in terms of which laws of Physics are described and whose measurement is necessary are called physical quantities

For example length, mass, time, electric current, temperature, the amount of substance, luminous intensity etc.

The chosen standard of measurement of a quantity which has essentially the same nature as that of the quantity is called the unit of the quality

For example when we talk about length then unit of length is meter or kg represents the mass of the object.

When we say 5 kg which equals to 5 times of 1kg. Thus it is basically a comparison process

1.02 Fundamental and Derived unit

A fundamental unit is a unit adapted for measurement of a fundamental quantity. A fundamental quantity is one of a conventionally chosen subset of physical quantities, where no subset quantity can be expressed in terms of the others. In the International System of Units, there are fundamental units

1.02.01 Fundamental and Supplementary units of SI

Sr .No	Physical fundamental quantity	Fundamental units	Symbol
1	Mass	Kilogram	kg
2	Length	Metre	m

3	Time	Second	s
4	Electric current	Ampere	A
5	Temperature	Kelvin	K
6	Luminous intensity	Candela	cd
7	Quantity of matter	mole	mol
Sr.No	Supplementary physical quantity	Supplementary unit	Symbol
1	Plane angle	Radian	rad
2	Solid angle	Steradian	sr

1.02.02 Choice of a standard unit

The unit chosen for measurement of nay physical quantity should satisfy following requirements

i) It should be of suitable size

ii) It should be accurately defined

iii) It should be easily accessible

iv) Replicas of unit should be available easily

iv) It should not change with time

v) It should not change with change in physical condition like temperature, pressure etc

1.03 Various systems of units

a) The f.p.s system is the British engineering system of units, which uses the foot as the unit of length, pound as the unit of mass and second as the unit of time.

b) The c.g.s system is the Gaussian system which uses the centimeter, gram and second as the three basic units.

c) The M.K.S system is based on meter, kilogram and second as the three basic units

d) The International system of unit (SI) This system is introduced in 1960, by the General Conference of Weight and Measures. This system of units is essentially a modification over the m.k.s. system

1.03.01 Advantages of SI

1. SI system assigns only one unit to a particular quantity. For example, Joule is the unit for all types of energy, while in MKS system joule is the unit of energy and calories is the unit for heat energy.

2. SI system follows decimal system i.e. the multiples and submultiples of units are expressed as power of 10

3. SI system is based on certain fundamental units, from which all derived units are obtained by multiplying or division without introducing numerical factors

4. SI is an internationally accepted system.

5. SI is an absolute system of units. There are no gravitational units on the system. The use of factor ‘g’ is thus eliminated

1.04 Some important Practical Units

1. Astronomical unit (AU)

It is the average distance of the center of the sun from the center of the earth

1AU $=1.496\times10^{11}$ m $\cong 1.5\times10^{11}$ m

2. Light Year (ly)

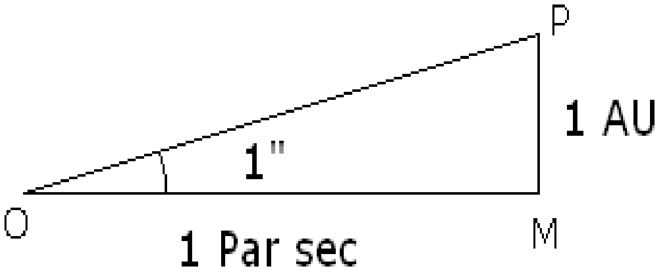
One light year is the distance traveled by light in vacuum in one year.

1 light year = $3\times10^{8}\times(365\times24\times60\times60)$ meter

1ly = 9.46×10^{15} m

3. Parsec.

One Parsec is the distance at which an arc 1AU long subtends an angle of 1" (one second)



Conversion of 1 Parsec in meter

Since angle is very small $\tan\theta = \theta$ = PM/OM

1" = 1 AU/ 1Par sec

1 Parsec = 1AU/1" -- -- eq(1)

We know that $1" = \frac{\pi}{180\times60\times60}\ rad$ and 1AU $=1.496\times10^{11}$ m

Substituting values of 1" and 1AU in eq(1) and on simplification we get

1 Parsec = 3.1×10^{16} m

4. Relation between AU, ly, and parsec

1 ly = 6.3×10^4 AU

1 parsec = 3.26 ly

1 Å (angstrom) = 10^{-12} m

1 amu = 1.66×10^{-27} kg

1.04.01 Metric prefixes

Sr.No	Power of 10	Prefix	Symbol
1	10^{-1}	deci	d
2	10^{-2}	centi	c
3	10^{-3}	milli	m
4	10^{-6}	micro	μ
5	10^{-9}	nano	n
6	10^{-12}	pico	p
7	10^{-15}	femto	F
8	10^{-18}	atto	a

Sr.No	Power of 10	Prefix	Symbol
9	10^1	deca	da
10	10^2	hecto	h
11	10^3	kilo	k
12	10^6	mega	M
13	10^9	giga	G
14	10^{12}	tera	T
15	10^{15}	peta	P
16	10^{18}	exa	e

Exercise 1.01

1. How many astronomical units are there in 1 meter?

2. Calculate the number of light years in one meter

3. How many amu make 1 kg

Answers : 1) 6.68×10^{-12} AU,

2) 1.057×10^{-16} ly

3) 0.6×10^{27} amu

1.05 SI Derived Units

Derived units are units which may be expressed in terms of base units by means of mathematical symbols of multiplication and division.

Certain derived units have been given special names and symbols, and these special names and symbols may themselves be used in combination with the SI and other derived units to express the units of other quantities.

Examples of derived unit

SrNo	Physical quantity	Relation with other quantities	SI unit
1	Area	Length × breadth	m^2
2	Density	$\frac{mass}{volume}$	kg m^{-3}
3	Speed or velocity	$\frac{displacement}{time}$	ms^{-1}
4	Linear momentum	Mass × velocity	$Kgms^{-1}$
5	Acceleration	$\frac{change\ in\ velocity}{time}$	ms^{-2}
6	Force	Mass × Acceleration	$Kgms^{-2}$ or N
7	Moment of force (torque)	Force × displacement	N-m
8	Work	Force × displacement	J (joule)

9	Power	$\frac{work}{time}$	Js^{-1} or W(watt)
10	Electric charge	Current × time	As
11	Pressure	$\frac{Force}{Area}$	Nm^{-2}
12	Frequency	$\frac{1}{T}$	s^{-1}

1.06 Significant figure

In scientific work, all numbers are assumed to be derived from measurements and therefore the last digit in each number is uncertain. All certain digits plus the first uncertain digit are significant.

For example, if we measure a distance using meter scale. Least count of meter scale is 0.1 cm. Now if we measure a length of the rod and it is between 47.6 cm and 47.7cm then we may estimate as 47.68 cm. Now this expression has 3 significant figure 4,7,6 are precisely known but the last digit 8 is only approximately known.

1.06.01 Common rules for counting significant figure

Rule 1: All nonzero digits are significant

Example: x= 2365 have four significant digits

Rule 2: All the zeros between two nonzero digits are significant no matter where the decimal point is it at all.

Example: X = 1007 has four significant digits, Whereas x = 2.0807 have five significant digit

Rule3: If the number is less than 1 then zeros on the right of decimal point but to the left of the first nonzero digit are not significant.

Example: X = 0.0057 has only two significant digits, but x = 1.0057 have five significant digits according to Rule2

Rule4: All zeros on the right of the last non-zero digit in the decimal part are significant

Example: X = 0.00020 have two significant digits

Rule5: All zeros on the right of non-zero digit are not significant

Example: X = 8000 have only one significant digit while x = 32000 have only two significant digits

Rule6: All zeros on the right of the last nonzero digit become significant when they come from a measurement. Also, note that change in the units of measurement of a quantity does not change the number of significant digits.

Example: If the measured quantity is 2030 m then the number has 4 significant digits. Same can be converted in cm as 2.030×10^5 cm here also a number of significant digits is to be four.

Illustration

7) State the number of significant figures in the following: (i) 600900 (ii) 5212.0 (iii) 6.320 (iv) 0.0631 (v) 2.64×10^{24}

Answers

i) 4 ii) 5 iii) 4 iv) 3 v) 3

1.06.02 Round off

Rule1: If the digit to be dropped is less than 5, then the preceding digit is left unchanged. Example 5.72 round-off to 5.7

Rule 2: If the digit to be dropped is more than 5, then the preceding digit is increased by one. Example 5.76 round-off to 5.8

Rule3: If the digit to be dropped is 5 followed by nonzero number, then preceding digit is increased by one Example: 13.654 round-off to 13.7

Rule4: If the digit to be dropped is 5, then preceding digit is left unchanged, if even Example 4.$\underline{2}$50 or 4.$\underline{2}$52 becomes 4.2

Rule5: If the digit to be dropped is 5, then the preceding digit is increased by one if it is odd. Example 4.$\underline{3}$50 or 4.$\underline{3}$52 becomes 4.4

Illustration

8) Round off the following numbers to three significant digits (a) 15462 (b) 14.745 (c)14.750 (d) 14.650×10^{12}

Solution

(a) The third significant digit is 4. This digit is to be rounded. The digit next to it is 6 which is greater than 5. The third digit should, therefore, be increased by 1. The digits to be dropped should be replaced by zeros because they appear to the left of the decimal. Thus, 15462 become 15500 on rounding to three significant digits.

(b) The third significant digit in 14.745 is 7. The number next to it is less than 5. So 14.745 become 14.7 on rounding to three significant digits.

(c) 14.750 will become 14.8 because the digit to be rounded is odd and the digit next to it is 5.

1.06.03 Arithmetical operations with significant figure

Addition and subtraction

In addition or subtraction, the number of decimal places in the result should the smallest number of decimal places of terms in the operation

Example1: The sum of three measurements 2.2 m, 2.22m, 2.222m is 6.642 round off is 6.6m

Example 2: If x= 2.35 and y = 2.1 the x-y = 0.25 Round off to 0.2 (as per round-off rule 2)

Multiplication and division

In multiplication and division, the number of significant figure in the product or in the quotient is the same as the smallest number of digits in any of the factor

Example 1: If x= 2.35 and y = 2.1 then xy =4.935 round off will be 4.9 as least significant digits is 2 in 2.1

Example 2: If x = 2500 and y = 123 then x/y =20.3252 round-off 20 as 2500 have only two significant numbers.

Illustration

9) Multiply 2.2 and 0.225. Give the answer correct to significant figures.

Solution: 2.2 × 0.225 = 0.495 since the least number of significant figures in the given data is 2, the result should also have only two significant figures.

∴ $2.2 \times 0.225 = 0.50$

10) Find the value of π^2 correct to significant figures, if $\pi = 3.14$

Solution

$\pi^2 = 3.14 \times 3.14 = 9.8596$

Since the least number of significant figure in the given data is 3, the result should also have only three significant figures $\pi^2 = 9.86$ (rounded off)

Exercise 1.02

Q1) 5.74 g of a substance occupies a volume of 1.2 cm^3. Calculate its density applying the principle of significant figures. [Ans: 4.8 g cm^{-3}]

Q2) The length, breadth, and thickness of a rectangular plate are 4.234 m, 1.005 m and 2.01 cm respectively. Find the total area and volume of the plate to correct significant figures. [Ans :4.255 m^2 ,0.0855 m^3]

1.07 Errors of measurement

Measurement cannot be perfect as the errors involved in the process cannot be removed completely. Difference between measured value and true value is called the error of measurement. The error in measurement are classified as Systemic errors, Random errors and Gross errors

1.07.01 Systemic error

1) Instrumental error: It may be due to a manufacturing defect of the instrument, there may be zero error.

2) Personal error: May be due to the inexperience of the observer. For example improper setting of the instrument, not following proper method of taking an observation.

3) Error due to imperfection: Arises on account of ignoring the fact. For example, if temperature should be at say 25^{O} C while taking observation, and if temperature is more or less than 25^{O} C then error will happen

4) Error due to external causes: If there is sudden radiation or temperature change which is not under your control will affect observation

1.07.02 Random error

The random error is those error, during repeated observation by the same person, the cause may be different at every time. It occurs because there are a very large number of parameters beyond the control of the experimenter that may interfere with the results of the experiment. A random error can also occur due to the measuring instrument and the way it is affected by changes in the surroundings. Such error can be minimized by taking an average of many readings. Unlike systematic errors, random errors are not predictable, which makes them difficult to detect but easier to remove since they are statistical errors and can be removed by statistical methods like averaging

1.07.03 Gross errors

These errors arise on account of the sheer carelessness of the observer. For example Reading an instrument without setting properly. Recording the observation wrongly. Using wrong values of the observations in the calculation.

1.08 Absolute Error, Relative error, and Percentage error

1.08.01 Absolute error

Absolute error in the measurement of a physical quantity is the magnitude of the difference between the true value and the measured value of the quantity

Let a physical quantity being measured n times. Let the measured values be a_1, a_2, a_3, a_4, …., a_n. the arithmetic mean is

$$a_m = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

If the true value is not known then the arithmetic mean is taken as the true value. By definition, absolute errors in the measured values are

$$\Delta a_1 = a_m - a_1$$

$$\Delta a_2 = a_m - a_2$$

$$\Delta a_3 = a_m - a_3$$

$$\Delta a_n = a_m - a_n$$

The absolute error may be positive or negative.

1.08.02 Mean absolute error

It is the arithmetic mean of the modulus absolute error. It is represented as $\Delta\bar{a}$

$$\Delta\bar{a} = \frac{|\Delta a_1| + |\Delta a_2| + \cdots + |\Delta a_n|}{n}$$

Hence final result of measurement is $a = a_m \pm \Delta\bar{a}$

Illustration

11) A student measures the time period of 100 oscillations of a simple pendulum four times. The data set is 90 s, 91 s, 95 s and 92 s. If the minimum division in the measuring clock is 1 s, then what will be the reported meantime?

Solution

Mean of reading = (90+91+95+92)/4 = 92 s

Absolute error =

$$\frac{|92-90| + |92-91| + |92-95| + |92-92|}{4} = 1.5s$$

Minimum division is 1 and absolute error is 1.5 hence reported error = 2s

Reported mean time = 92±2s

1.08.03 Relative error

The relative error is the ratio of mean absolute error to the mean value of the quantity measured

$$relative\ error = \frac{\overline{\Delta a}}{a_m}$$

$$percentage\ error = \frac{\overline{\Delta a}}{a_m} \times 100\%$$

Illustration

12) The length of a rod is measured as 25.0 cm using a scale having an accuracy of 0.1 cm. determine the percentage error in length.

Solution

Accuracy is the maximum possible error = 0.1 cm

∴ % error = (0.1/25)×100 = 0.4 %

1.09 Combination of errors

If any experiment involves many observations and involves many mathematical operations then errors in measurement get combined. For example, density is the ratio of mass and volume. Here error will be in mass and length.

1.09.01 Errors in summation

Suppose z = a + b

Let Δa and Δb be the absolute error in measurement of a and b

Then $z \pm \Delta z = (a \pm \Delta a) + (b \pm \Delta b)$

$z \pm \Delta z = (a+b) \pm (\Delta a + \Delta b)$

Thus $\Delta z = \Delta a + \Delta b$

Note error gets added

1.09.02 Error in subtraction

Suppose z = a - b

Let Δa and Δb be the absolute error in measurement of a and b

Then $z \pm \Delta z = (a \pm \Delta a) - (b \pm \Delta b)$

$z \pm \Delta z = a \pm \Delta a - b \mp \Delta b$

$z \pm \Delta z = a - b \pm \Delta a \mp \Delta b$

$\pm \Delta z = \pm \Delta a \mp \Delta b$

Thus there are four possible values of error

$(+\Delta a + \Delta b), (+\Delta a - \Delta b), (-\Delta a + \Delta b), (-\Delta a - \Delta b)$

Therefore maximum error $\Delta z = \Delta a + \Delta b$

Note error gets added

1.09.03 Error in product

Let $z = a \times b$

$z \pm \Delta z = (a \pm \Delta a) \times (b \pm \Delta b)$

$z \pm \Delta z = ab \pm a\Delta b \pm \Delta a b + \Delta a \Delta b$

as Δa and Δb are very small, $\Delta a \Delta b$ can be neglected

$z \pm \Delta z = ab \pm a\Delta b \pm \Delta a b$

$\pm \Delta z = \pm(a\Delta b + \Delta a b)$

Now dividing above equation by z

$$\frac{\Delta z}{z} = \frac{a\Delta b}{z} + \frac{b\Delta a}{z}$$

But z = ab

$$\frac{\Delta z}{z} = \frac{a\Delta b}{ab} + \frac{b\Delta a}{ab}$$

$$\frac{\Delta z}{z} = \frac{\Delta b}{b} + \frac{\Delta a}{a}$$

Note relative error gets added

1.09.04 Error in division

Let $z = \frac{a}{b} = ab^{-1}$

By taking first order derivative with respect to z we get

$$1 = -ab^{-2}\frac{db}{dz} + b^{-1}\frac{da}{dz}$$

$$dz = -ab^{-2}(db) + b^{-1}(da)$$

Dividing by z

$$\frac{dz}{z} = -\frac{ab^{-2}(db)}{z} + \frac{b^{-1}(da)}{z}$$

But z = ab^{-1} and error are either positive or negative

$$\pm\frac{dz}{z} = \mp\frac{ab^{-2}(db)}{ab^{-1}} \pm \frac{b^{-1}(da)}{ab^{-1}}$$

$$\pm\frac{dz}{z} = \mp\frac{(db)}{b} \pm \frac{(da)}{a}$$

Or

Thus four possible values of errors

$$= -\frac{(db)}{b} + \frac{(da)}{a}, -\frac{(db)}{b} - \frac{(da)}{a}, +\frac{(db)}{b} - \frac{(da)}{a}, +\frac{(db)}{b} + \frac{(da)}{a}$$

Thus maximum error in terms of fractional error cab be written as

$$\frac{\Delta z}{z} = \left(\frac{\Delta b}{b} + \frac{\Delta a}{a}\right)$$

Note relative error gets added

1.09.05 Errors in powers

$$z = \frac{a^n}{b^m}$$

by taking derivative we get

$$\frac{dz}{z} = n\frac{da}{a} - m\frac{db}{b}$$

In terms of fractional error

$$\pm\frac{\Delta z}{z} = \pm n\frac{\Delta a}{a} \mp m\frac{\Delta b}{b}$$

Therefore maximum value of error

$$\frac{\Delta z}{z} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b}$$

General formula

$$z = \frac{a^n b^m}{c^p}$$

$$\frac{\Delta z}{z} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b} + p\frac{\Delta c}{c}$$

Illustration

13) In an experiment to determine acceleration due to gravity by a simple pendulum, a student commits 1%positive error in the measurement of length

and 3% negative error in the measurement of the time period. What will be percentage error in the value of g

Solution

We know that period T

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$g = 4\pi\frac{l}{T^2}$$

$$\frac{\Delta g}{g}\% = \frac{\Delta l}{l}\% + 2\frac{\Delta T}{T}\%$$

$$\frac{\Delta g}{g}\% = 1\% + 2\times 3\% = 7\%$$

Thus error in gravitational acceleration = 7%

1.09.06 Error in reciprocal

If a given formula is in reciprocal form, we can determine error

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{c} = \frac{a+b}{ab}$$

$$c = \frac{ab}{a+b}$$

$$\frac{\Delta c}{c} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta(a+b)}{a+b}$$

Illustration

14) In an experiment, the value of resistance of two resistance is r_1 = (10±0.2)Ω ohm and r_2 = (30±0.4) Ω. Find the value of total resistance if connected in parallel with limit of error

Solution

Calculation of total resistance R using formula for parallel connection

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$R = \frac{r_1 r_2}{r_1 + r_2} = \frac{10\times 30}{10 + 30} = 7.5\Omega$$

Using formula for relative error

$$\frac{\Delta c}{c} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta a + \Delta b}{a + b}$$

$$\frac{\Delta R}{7.5} = \frac{0.2}{10} + \frac{0.4}{30} + \frac{0.2 + 0.4}{10 + 30}$$

ΔR = 0.048Ω

∴R = (7.5±0.048)Ω

15) Calculate focal length of a spherical mirror from the following observation object distance u = 10±0.03, image distance 40±0.02

Solution:

Calculation of focal length

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$f = \frac{uv}{u + v} = \frac{10\times 40}{10 + 40} = 8cm$$

Using formula for relative error

$$\frac{\Delta c}{c} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta a + \Delta b}{a + b}$$

$$\frac{\Delta f}{8} = \frac{0.03}{10} + \frac{0.02}{40} + \frac{0.03 + 0.02}{10 + 40}$$

$$\frac{\Delta f}{8} = 0.0045$$

∴Δf =0.036 cm

Thus focal length of spherical mirror = (8.0±0.036)cm

16) The time period of simple pendulum is given by $t = 2\pi\sqrt{\frac{l}{g}}$. What is the accuracy in the determination of g if 10 cm length is known to 1mm accuracy and 0.5s time period is measured from time of 100 oscillations with watch of 1 second resolution?

Solution

Total time for 100 oscillation = 100×0.5s = 50 s. Watch resolution is 1s thus Δt = 1s . 1mm = 0.1 cm

$$t = 2\pi\sqrt{\frac{l}{g}} \quad or\ g = 4\pi^2\frac{l}{t^2}$$

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2\frac{\Delta t}{t}$$

$$\frac{\Delta g}{g}\% = \frac{0.1}{10}\times 100 + \frac{2}{50}\times 100 = \pm 5\%$$

1.10 Accuracy and precision

1.10.01 Measurement’s accuracy

Accuracy refers to the closeness of a measured value to a standard or known value. Problems with accuracy are due to errors. For example, if in the lab you obtain a weight measurement of 3.2 kg for a given substance, but the actual or known weight is 10 kg, then your measurement is not accurate. In this case, your measurement is not close to the known value. We can necessary precaution and reduce different types of error discussed in section 1.13 on errors

1.10.02 Precision

Precision refers to the closeness of two or more measurements to each other. Using the example above, if you weigh a given substance five times, and get 3.2 kg each time, then your measurement is very precise. Precision is independent of accuracy. You can be very precise but inaccurate, as described above. You can also be accurate but imprecise.

Precision describes the limitation of the measuring instrument. Measurement's precision is determined by least count of the measuring instrument. Smaller the least count, greater is the precision.

A good analogy for understanding accuracy and precision is to imagine a basketball player shooting baskets. If the player shoots with accuracy, his aim will always take the ball close to or into the basket. If the player shoots with precision, his aim will always take the ball to the same location which may or may not be close to the basket. A good player will be both accurate and precise by shooting the ball the same way each time and each time making it in the basket.

1.11 Dimensional analysis

1.11.01 Dimensions of a physical quantity

The dimensions of a physical quantity are the power to which the fundamental units of mass, length and time have to be raised to represent a derived unit of quantity.

The fundamental unit of mass is represented by [M], length by [L] and time by [T].

Suppose we obtained derived unit area

As Area = length × width

Area = [L][L] = $[L^2]$

Thus to represent area we have to raise [L] to the power of 2. Thus area is said to have two dimensions. Similarly, the volume is said to have three dimensions.

Since for area and volume mass and time are not required we write

A = $[M^0L^2T^0]$ and V = $[M^0L^3T^0]$ are also called as dimensional formula or dimensional equation for velocity

For velocity, we write

$$V = \frac{dispalcement}{ime} = \frac{[L]}{[T]}$$

Velocity = $[M^0L^1T^{-1}]$ is also called as dimensional formula or dimensional equation for velocity

Hence dimension of velocity is zero in mass, +1 in length and -1 in time.

1.11.02 Types of physical quantities

1. Dimensional constant

These are the quantities whose values are constant and they posse's dimensions. For examples: velocity of light, Planck's constant

2. Dimensional Variable

These are the quantities whose values are variable and they have dimensions For example velocity, acceleration, volume etc.

3. Dimensionless constants

These are the quantity, whose values are constant but do not have dimensions. Example numbers 1, 2, 3 … or mathematical constant e and π

4. Dimensionless variables

These are the quantities, whose values changes and they do not have dimensions. For example angle, solid angle, specific gravity, refractive index

1.11.03 some important dimension formula

SrNo	Physical quantity	Dimension	SI unit
1	Force (F)	$[M^1 L^1 T^{-2}]$	newton
2	Work (W)	$[M^1 L^2 T^{-2}]$	joule
3	Power(P)	$[M^1 L^2 T^{-3}]$	watt
4	Gravitational constant(G)	$[M^{-1} L^3 T^{-2}]$	N-m^2/kg^2
5	Angular velocity (ω)	$[T^{-1}]$	radian/s
6	Angular momentum(L)	$[M^1 L^2 T^{-1}]$	kg-m^2/s
7	Moment of inertia (I)	$[M^1 L^2]$	kg-m^2

8	Torque (τ)	[$M^1 L^2 T^{-2}$]	N-m
9	Young's modulus (Y)	[$M^1 L^{-1} T^{-2}$]	N/m^2
10	Surface Tension (S)	[$M^1 T^{-2}$]	N/m
11	Coefficient of viscosity (η)	[$M^1 L^{-1} T^{-1}$]	$N\text{-}s/m^2$
12	Pressure (p)	[$M^1 L^{-1} T^{-2}$]	N/m^2(Pascal)
13	Specific heat capacity (c)	[$L^2 T^{-2} K^{-1}$]	J/kg-K
14	Stefan's constant (σ)	[$M^1 T^{-3} K^{-4}$]	watt/m^2-k^4
15	Current density (j)	[$I^1 L^{-2}$]	ampere/m^2
16	Electrical conductivity (σ)	[$I^2T^3M^{-1}L^{-3}$]	$\Omega^{-1}m^{-1}$
17	Electric dipole moment (p)	[$L^1I^1T^1$]	C-m
18	Electric field (E)	[$M^1L^1I^{-1}T^{-3}$]	V/m
19	Electrical potential (V)	[$M^1L^2I^{-1}T^{-3}$]	volt
20	Electric flux (φ)	[$M^1L^{-3}I^{-1}T^3$]	volt/m
21	Capacitance (C)	[$M^{-1}L^{-2}I^2T^4$]	farad (F)
22	Permittivity (ε)	[$M^{-1}L^{-3}I^2T^4$]	$C^2/N\text{-}m^2$
23	Permeability (μ)	[$M^1L^1I^{-2}T^{-3}$]	Newton/A^2
24	Magnetic dipole moment (M)	[L^2I^1]	N-m/T
25	Magnetic flux (ϕ)	[$M^1L^2I^{-1}T^{-2}$]	Weber(Wb)
26	Magnetic field (B)	[$M^1I^{-1}T^{-2}$]	tesla
27	Inductance (L)	[$M^1L^2I^{-2}T^{-2}$]	henry
28	Resistance (R)	[$M^1L^2I^{-2}T^{-3}$]	ohm (Ω)
29	Intensity of wave (I)	[M^1T^{-3}]	watt/m^2
30	Thermal conductivity (k)	[$M^1L^1T^{-3}K^{-1}$]	watt/m-K

1.12 Uses of dimensional equation

Dimensional analysis is used for 1. Conversion of one system of units to another 2. Checking the accuracy of various formulae. 3. Derivation of formulae

1.12.01 Conversion of one system of units to another

The magnitude of a physical quantity remains same, whatever be the system of its measurement for example length of 1 m length rod will remain same if expressed as 100 cm. If Q is the magnitude of physical quantity, u_1 and u_2 are two units of measurement, n_1 and n_2 are their respective numerical value then

$$Q = n_1u_1 = n_2u_2$$

Let M_1, L_1, T_1, and M_2, L_2, T_2 be the fundamental units for respective system, let a, b, c are the respective dimensions of the quantity of mass, length and time of the both systems

$u_1 = [M_1^a\ L_1^b\ T_1^c]$ and $u_2 = [M_2^a\ L_2^b\ T_2^c]$

$$\text{Then}\quad n_1[M_1^a\ L_1^b\ T_1^c] = n_2[M_2^a\ L_2^b\ T_2^c]$$

By knowing $M_1\ L_1\ T_1$, $M_2\ L_2\ T_2$, (a,b,c) and n_1, we can find the value of n_2

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^b$$

Illustration

17) Convert 76 cm of mercury pressure into N m^{-2} using the method of dimensions. (Density of mercury is 13.6 g/cm^3)

Use formula for pressure P = hρg , here h is height of mercury column, ρ is the density of mercury, g is gravitational acceleration

Solution :

In cgs system, 76 cm of mercury pressure = 76 × 13.6 × 980 dyne cm^{-2}

Let this be P_1. Therefore P_1 = 76 × 13.6 × 980 dyne cm^{-2}

Dimension of pressure is [$M^1L^{-1}T^{-2}$]

By using formula

$$n_1[M_1^a\ L_1^b\ T_1^c] = n_2[M_2^a\ L_2^b\ T_2^c]$$

$$= 76 \times 13.6 \times 980\ [M_1^1\ L_1^{-1}\ T_1^{-2}] = n_2[M_2^1\ L_2^{-1}\ T_2^{-2}]$$

$$n_2 = 76 \times 13.6 \times 980 \left[\frac{M_1}{M_2}\right]^1 \left[\frac{L_1}{L_2}\right]^{-1} \left[\frac{T_1}{T_2}\right]^{-2}$$

$$n_2 = 76 \times 13.6 \times 980 \left[\frac{M_1}{M_2}\right]^1 \left[\frac{L_1}{L_2}\right]^{-1} \left[\frac{T_1}{T_2}\right]^{-2}$$

Since we want to convert cgs to SI unit we have to convert all units of CGS in terms of SI So mass unit g = 10^{-3}kg, cm = 10^{-2}m time will not change as both units have second as unit

$$n_2 = 76 \times 13.6 \times 980 \left[\frac{10^{-3}\text{kg}}{1\text{kg}}\right]^1 \left[\frac{10^{-2}}{1\text{m}}\right]^{-1} \left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$$n_2 = 76 \times 13.6 \times 980 \times 10^{-3} \times 10^{2} \times 1$$

$$n_2 = 101292.8 \text{ N m}^{-2}$$

$$n_2 = 1.01 \times 10^5 \text{ Nm}^{-2}$$

18) If the units of force, energy, and velocity in a new system be 10 N, 5J and 0.5 ms^{-1} respectively, find the units of mass, length and time in that system.

Solution

New system values will be denoted by n_1

From formula

$$n_2 = n_1 \left[\frac{M_1}{M_2}\right]^a \left[\frac{L_1}{L_2}\right]^b \left[\frac{T_1}{T_2}\right]^b$$

Force

Here n_1 = 10 N , n_2 =1 N and a = 1, b= 1, c = -2 for energy as dimension formula is $[M^1L^1T^{-2}]$

$$1 = 10 \left[\frac{M_1}{M_2}\right]^1 \left[\frac{L_1}{L_2}\right]^1 \left[\frac{T_1}{T_2}\right]^{-2} \ldots\ldots eq(1)$$

Energy

n_1 = 5J , n_2 = 1J and a = 1, b=2, c = -2 , as for energy dimension formula is $[M^1L^2T^{-2}]$

$$1 = 5 \left[\frac{M_1}{M_2}\right]^1 \left[\frac{L_1}{L_2}\right]^2 \left[\frac{T_1}{T_2}\right]^{-2} \ldots\ldots eq(2)$$

N_1 =0.5 m/s , n_2 = 1.0 m/s and a = 0, b = 1, c = -1 , as for velocity dimension formula is $[M^0L^1T^{-2}]$

$$1 = 0.5 \left[\frac{M_1}{M_2}\right]^0 \left[\frac{L_1}{L_2}\right]^1 \left[\frac{T_1}{T_2}\right]^{-1} \ldots\ldots eq(3)$$

Dividing eq(2) by eq(1) we get

$$1 = 0.5 \left[\frac{L_1}{L_2}\right]^1 \; or \; \left[\frac{L_1}{L_2}\right]^1 = 2$$

∴ $L_2 = 0.5\ L_1$

As L_1 =1m L_2 = 0.5m

Substituting value of $\left[\frac{L_1}{L_2}\right]^1 = 2$ in equation 3 we get

$$1 = 0.5\times 2\times \left[\frac{T_1}{T_2}\right]^{-1}$$

$$\left[\frac{T_1}{T_2}\right]^1 = 1$$

As T1 = 1 s , T_2 = 1s

Substituting values of $\left[\frac{L_1}{L_2}\right]^1 = 2\ and\ \left[\frac{T_1}{T_2}\right]^1 = 1$ in eq(1) we get

$1 = 10\left[\frac{M_1}{M_2}\right]^1 \times 2\times 1$ $\therefore M_2 = 20\ M_1$ as $M_1 = 1$ $M_2 = 20$ kg

Hence units of mass, length and time are 20 kg, 0.5 m and 1 sec respectively

19) The density of a substance is 8 g/cm^3. Now we have a new system in which unit of length is 5cm and unit of mass 20g. Find the density in this new system

Solution

Let the symbol of a unit of length be Ln and mass be Mn.

Since Ln = 5 cm Therefore 1cm = 1/5 Ln, Mn= 20g Therefore 1g = Mn/20

Substituting in formula for density

$$\rho = \frac{m}{V} = \frac{m}{l^3}$$

If volume is 1cm^3 mass is 8gm

$$\rho = \frac{8gm}{1cm^3} = \frac{8\frac{Mn}{20}}{\left(\frac{1}{5}Ln\right)^3}$$

$$\rho = 50\frac{Mn}{(Ln)^3}$$

Thus as per new system density is 50Mn/Ln^3

20) The moment of inertia of body rotating about a given axis is 3.0 kgm^2 in the SI system. What will be the value of the moment of inertia in a system of a unit of length 5cm and the unit mass is 20g?

Dimension formula for moment of inertia = [ML^2] , thus a = 1 and b = 2

n_1 = 6, M_1 = 1kg, L_1 = 1m, M_2 = 20g and L_2 = 5cm

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^b$$

$$n_2 = 6\left[\frac{1000g}{20g}\right]^1\left[\frac{100}{5}\right]^2$$

N_2 = 6×50×400 = 120000=1.2×10^5 units

Exercise 1.03

Convert using dimensional analysis (i) $\frac{18}{5}$ kmph into m s^{-1} (ii) $\frac{5}{18}$ ms^{-1} into kmph (iii) 13.6 g cm^{-3} into kg m^{-3}

Answers:

i) 1 m s^{-1}, ii) 1 kmph, iii) 1.36 × 10^4 kg m^{-3}

1.12.02 Checking the accuracy of various formulae

We can check the correctness of the formula based on the principle of homogeneity of dimension. According to this principle, dimensions of various terms of left side of the equation should be same as that of right side

Note that we cannot add or subtract different dimension, for example, we cannot add L with M. or we cannot add 5metre with 2 kilograms. But we can multiply or divide.

To check the correctness of the given relation we shall write the dimensions of the quantities on both sides of the relation. If the principle of homogeneity is obeyed, the formula is correct.

Note that If the equation is dimensionally correct, it is not necessarily that equation describing the relation is correct. However, if the equation is dimensionally incorrect then equation describing relation is also incorrect.

Illustration

20) Write the dimensions of a and b in the relation, P is pressure, x is distance ad t is time

$$P = \frac{b - x^2}{at}$$

Solution

X^2 is subtracted from b thus b and x^2 must have the same dimension, the dimension of $x^2 = L^2$. Therefore dimension of b = $[L^2]$

Expressing given equation in dimensions we get

$$[M^1L^{-1}T^{-2}] = \frac{[L^2]}{a[T]}$$

$$a = \frac{[L^2]}{[M^1L^{-1}T^{-2}][T]}$$

$$a = [M^{-1}L^3T^1]$$

21) Check whether the equation $\lambda = \frac{h}{mv}$ is dimensionally correct

(λ - wavelength, h - Planck's constant, m - mass, v - velocity).

Solution :

Dimension of Planck's constant h is $[ML^2\,T^{-1}]$

Dimension of λ is [L]

Dimension of m is [M]

Dimension of v is $[LT^{-1}]$

Representing given equation in dimensions we get

$$[L] = \frac{[ML^1T^{-1}]}{[M][LT^{-1}]}$$

[L] = [L]

As the dimensions on both sides of the equation are same, the given equation is dimensionally correct

Exercise 1.04

1) Check the correctness of the following equation by dimensional analysis

(i) $F = \frac{mv^2}{r^2}$ where F is force, m is mass, v is velocity and r is radius

(ii) $n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ where n is frequency, g is acceleration due to gravity and l is length.

(iii) $\frac{1}{2}mv^2 = mgh^2$ where m is mass, v is velocity, g is acceleration due to gravity and h is height.

2) The number of particle crossing per unit time, normal to cross section is given by

$$N = q\left(\frac{n}{v}\right)$$

Here n is the number of particles per unit volume, v is velocity of particle, find dimension of q

3) Check following formula dimensionally correct or wrong

$$v^2 = \frac{\pi P r^2}{8\eta t}$$

Here v is velocity, P is pressure, η is coefficient of viscosity, t is time

Answers

1 i) wrong, ii) correct, iii) wrong 2) $[L^{-2}\ T^{-2}]$ 3) Correct

1.12.03 Derivation of formulae

Using the principle of homogeneity of dimensions, we can derive the formula of a physical quantity, provided we know the factors on which the physical quantity depends

Using principle of homogeneity of dimensions, equating the powers of M, L, T on both sides of the dimensional equation. Three equations are obtained, on solving the equations we get powers of dimensions. On substituting the values of powers in dimensional equation, we obtain the preliminary form of the relation

Illustration

22) Derive an expression for time period (t) of a simple pendulum, which may depend on mass of Bob (m), length of pendulum(l) and acceleration due to gravity(g)

Solution

Since powers of mass, length, and acceleration are not described let us consider a be the power of mass (m), b be the power of l and c be the power of acceleration (g)

Thus we get equation

$$t \propto m^a l^b g^c$$

here a, b and c are the dimensions of m, l, and g, now let k be dimensionless constant equation becomes $t = k\, m^a l^b g^c$ eq(1)

Writing the dimensions in terms of M, L, T on either side of eq(1) we get

$[M^0L^0T^1] = M^a L^b (LT^{-2})^c$ { dimension of acceleration = $[M^0L^1T^{-2}]$ }

$[M^0L^0T^1] = M^a L^{b+c} T^{-2c}$

Applying the principle of homogeneity of dimensions, we get

a = 0, b+c =0 and -2c = 1

Thus b = ½ Substituting the values in eq(1) we get

$$t = k\, m^0 l^{1/2} g^{-1/2}$$

$$t = k\sqrt{\frac{l}{g}}$$

Value of k a dimensionless constant is calculated to be 2π

$$t = 2\pi\sqrt{\frac{l}{g}}$$

23) Obtain by dimensional analysis an expression for the surface tension of a liquid rising in a capillary tube. Assume that the surface tension T depends on mass m of the liquid, pressure P of the liquid and radius r of the capillary tube (Take the constant k = 1 2).

Solution

Let power of m be a, power of pressure be b and radius be c

$T \propto m^a p^b r^c$

$T = k\, m^a p^b r^c$

Dimension of Surface tension is $T = [M^1 L^0 T^{-2}]$

Dimension of mass $m = [M^1L^0T^{-1}]$

Dimension of Pressure $P = [M^1L^{-1}T^{-2}]$

Dimension of radius $r = [M^0L^1T^0]$

Expressing eq(1) in terms of dimension formula

$[M^1 L^0 T^{-2}] = [M^1L^0T^{-1}]^a [M^1L^{-1}T^{-2}]^b [M^0L^1T^0]^c$

On simplification we get

$[M^1 L^0 T^{-2}] = M^{a+b} L^{-b+c} T^{-a-2b}$

Comparing powers we get equations a+b = 1, -b+c = 0 and -a-2b = -2

$\therefore$ b = 1, c =1 and a =0

$T = k\,P^1\,r^1$

As k = ½ given

$$T = \frac{Pr}{2}$$

24) Rotational kinetic energy of the body depends on angular velocity and moment of inertia of the object obtain the formula for Rotational kinetic energy

Where K = Kinetic energy of rotating body and k = dimensionless constant =1

Solution:

$K = kI^a\,\omega^b$ Dimensions of left side are, $K = [ML^2\,T^{-2}]$ Dimensions of right side are, $I^a = [ML^2\,]^a$, $\omega = [T^{-1}]^b$,

According to principle of homogeneity of dimension, $[ML^2\,T^{-2}] = [M^a\,L^{2a}]\,[T^{-b}]$ Equating the dimension of both sides, a = 1 and b = 2⇒2 = 2a and –2 = –b

On substitution, we get $K = kI\omega^2$, as k =1

$K = I\omega^2$

Exercise 1.05

1) The force F acting on a body moving in a circular path depends on mass m of the body, velocity v and radius r of the circular path. Obtain an expression for the force by dimensional analysis (Take the value of k = 1).

2) When a small sphere moves at low speed through a fluid, the viscous force F opposing the motion is found experimentally to depend on the radius 'r', the velocity v of the sphere and the viscosity η of the fluid.

Find the force F (take k =6π)

3) To derive the Einstein mass - energy relation, Energy depends on velocity of light and mass, take k =1

4) A gas bubble from an explosion under water oscillates with a period T proportional to $p^a d^b E^c$ where p is the static pressure, d is the density of water and E is the total energy of the explosion. Find the values of a,b, and c.

5) The depth (x) to which bullet penetrate in a body depends on the coefficient of elasticity (η) and kinetic energy E. Establish the relation between these quantities using the method of dimensions

Answers

1) $F = \frac{mv^2}{r}$ 2) F = 6πηvr 3) $E = mc^2$ 4) a = -5/6 , b = ½, c = 1/3

5) $x \propto (E/\eta)^{1/3}$

1.13 Limitation of Dimensional Analysis

1. This method does not give the value of dimensionless constants

2. If a quantity is dependent on more than three factors, having dimension, the formula cannot be derived

3. We cannot obtain formulae containing trigonometrical function, exponential function, log functions etc

4. It does not give information about physical quantity scalar or vector.

5. This method can be used to get exact form of the relation. Which consists more than one part. For example, $s = ut + 1/2\ at^2$

1.14 Vernier Caliper

The vernier caliper consists of the main scale and a vernier scale (sliding scale) and enables readings with a precision of 1/200 cm. Figure 1 shows that the main scale is fitted with Jaws C and D on either side, with the straight edges connecting C and D vertically to the main scale forming a right angle. Simultaneously, Jaws E and F are fitted on the vernier scale, which moves over the main scale. When the jaws of the main and the vernier scales contact each other, the zeros of both scales should coincide. If the zeros don't coincide, a zero point calibration must be performed instantly. The distance between C and E or between D and F is the length of the object that is being measured.

We first use an example to demonstrate how to read the vernier caliper, followed by simple equation readings

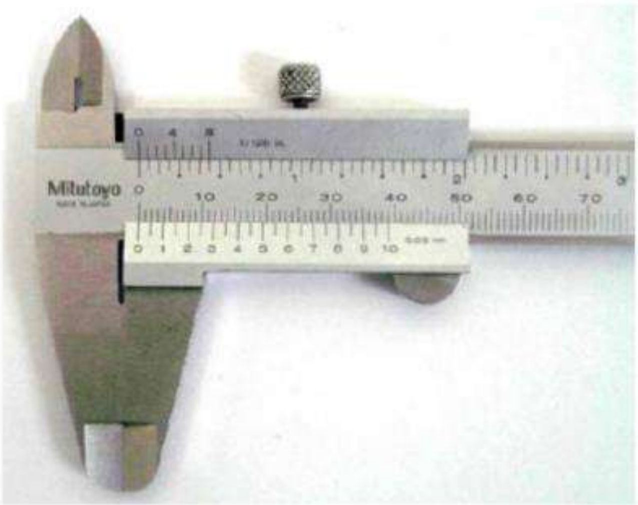


Fig 1.11

The vernier scale (sliding scale) in Figure 1.11 is graduated into 20 divisions sliding scale, which coincide with the 39 smallest divisions on the main scale (i.e., 39 mm). Assuming the length of one division on the vernier scale is S, then S can be obtained as follows:

20S = 39

S = 1.95 mm. (1)

Therefore one division of sliding scale = 2.00 - 1.95 = 0.05 mm

Or least count of vernier = 0.05mm = 0.005 cm

Least count is also known as vernier constant

Illustration

Example 25: The 20 divisions on the vernier scale (sliding scale, VSD) coincide with the 39 smallest markings on the main scale (mm). Find least count

Solution

20 S =39

S = 39/20 = 1.95

Least count = 2 – 1.95 = 0.05 mm

Example 26: Ten divisions on the vernier scale (sliding scale) coincide with 9 smallest divisions on the main scale (mm). Find least count

Solution

10S =9

S= 0.9.

Therefore, least count is equal to 1 - 0.9 = 0.1 mm.

Example 27

In a Vernier 1 cm of the main scale is divided into 20 equal parts. 19 divisions of the main scale coincide with 20 divisions on the vernier scale. Find the least count of the instrument.

Solution

Smallest division on vernier scale =1/20 = 0.05cm = 0.5 mm

19 main scale = 20 Sliding scale(S)

19×0.5 mm = 20 S

S=0.475

Least count= smallest division on vernier scale - vernier scale dive

Least count = 0.5-0.475 = 0.025mm = 0.0025 cm

Example 28) 1 main scale division=0.3 cm . 29 division of the main scale coincides with 30 division of vernier scale. Then what will be the least count of vernier

30V.S.D=29M.S.D

The least count of this vernier caliper is......

Smallest division on vernier scale:- 0.3mm

29 main scale divisions = 0.3 * 29 cm

= 3 * 29 mm

= 87mm

= 30 vernier scale divisions

Therefore 1 vernier scale division = 87/30 mm

= 2.9 mm

Least count = 1 main scale div - 1 vernier scale div

= 3mm - 2.9mm

= 0.1mm

Q39) The number of divisions on a vernier scale is 10 and are equal to 9 mm on the main scale. In the measurement of the length of a cylinder, the main scale reading is found to be 3.6 cm and the vernier coinciding division was found to be 6. Then length of cylinder is ….?

Solution

10S = 9 mm

S = 0.9 mm

Least count = 1-0.9 = 0.1m

36 +6×0.1 = 36.6 mm = 3.66 cm

Illustration

Q40) A screw gauge of pitch 1mm and 100 division on the circular scale what is its least count

Solution

$$least\ count = \frac{pitch}{no.of\ divison\ on\ circular\ scale}$$

$$least\ count = \frac{1mm}{100} = 0.01mm = 0.001\ cm$$

1.15 Exercise for practice

Q1) The mass of solid cube is 567 g and each edge has a length of 5 cm. determine the density ρ of the cubic in SI units [Ans 4.536×10^{3} kg/m^{3}]

Q2) Show that $v^2 = u^2 + 2as$ is dimensionally correct, here v is final velocity, u is initial velocity a is acceleration and s displacement.

Q3) Experiments shows that frequency (n) of tuning fork depends on length (l) of the prong, density(ρ) and Young's modulus (Y) of its material. On the basis of dimensional analysis obtain a formula for frequency. Ans $n = \frac{k}{l}\sqrt{\frac{Y}{\rho}}$

Q4) Round of the following numbers to three significant digits

a) 12.7468 b) 12.75 c) 12.652×10^{12} [Ans: 12.7 , 12.8 , 12.6×10^{12}]

Q5) The original length of the wire is (125.6±0.4)cm stretched to (128.8 ±0.2) cm. Calculate the elongation in the wire with error [Ans (3.2±0.6)]

Q6) The measurement of length of rectangles are l = (3.00±0.01)cm and breadth b= (2.00±0.02)cm. What is the area of the rectangle? [Ans (6.00±0.08)cm^2]

Q7) The change in the velocity of a body is (15.6 ±0.2) m/s in a time (2.2±0.1)s. Find the average acceleration of the body with error limits

[Ans (7.09 ±0.4) cm/s^2]

Q8) The period of oscillation of a simple pendulum is $= 2\pi\sqrt{\frac{l}{g}}$. Measured value of L is 20.0 cm known to 1 mm accuracy and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1s resolution. What will be the accuracy in the determination of g . [Ans 2.7%]

Q9) Assuming that the critical velocity of flow of liquid through a narrow tube depends on the radius of the tube, density of the liquid and viscosity of the liquid, then find the expression for critical velocity [Ans $v = \frac{k\eta}{r\rho}$]

Q10) Round off to three significant digit

a) 253674 b) 0.03937 c) 4.065×10^5

[Ans: a)254000, b) 0.0394 c) 4.06×105]

Q11) The radius of the earth is 6.37×10^6m and its mass is 5.975×10^{24} kg. Find the earth's average density to appropriate significant figure

[Ans 5.52×10^3 kg/m^3]

Q12) It is estimated that per minute, each cm^2 of the earth receives about 2 calories of heat energy from the sun. This constant is called solar constant. Express solar constant SI units. [Ans 1.kW/m^2]

Hint solar constant $S = \frac{2\ cal}{\min cm^2}$ Convert calories to joule , minutes to second and cm^2 m^2]

Q13) The time period t of the oscillation of a large star, oscillating under its own gravitational attraction, may depend on its mean radius R, its mean density ρ and the gravitational constant. Using dimensional analysis show that t is independent of R and find the formula for t [Ans $t = \frac{k}{\sqrt{\rho G}}$]

Q14) The rotational kinetic energy of a body is given by $= \frac{1}{2} I\omega^2$, where I is the moment of inertia of the body about its axis of rotation and ω is the angular velocity. Using this relation, obtain the dimensional formula for I. Ans [ML^2]

Q15) Calculate the percentage of error in specific resistance of cylindrical wire, measurements are r = radius of wire = (0.25 ±0.02)cm, Length of wire l = 225±0.1) cm, Resistance of wire = (48±3)Ω Use formula $R = \rho \frac{l}{A}$ [Ans 22.29%]

Q16) A box container of ice-cream having volume 27m^3 is to be made from a cube. What should be the length of a side in cm? [Ans 3.0×10^2 cm]

Q17) In two different system of units acceleration is represented by numbers in ratio 1:2, whilst a velocity is represented by numbers in ratio 1:3, compare units of length and time [9/2, 2/3]

Q18) In a certain system of absolute units the acceleration produced by gravity in a body falling freely is denoted by 3, the kinetic energy of 100 kg shot moving with 400 meters per second is denoted by 100 and its momentum by 10, find the units of length, time and mass: [Ans mass 200 kg, 40.82 m, 6.12 s]

Solution:

Dimension for gravitational acceleration = L^1T^2

$$3 = 9.8\left[\frac{L_1}{L_2}\right]\left[\frac{T_1}{T_2}\right]^{-2} \quad \ldots . eq(1)$$

Dimension for kinetic energy = $M^1L^2T^{-2}$

$$100 = \left(\frac{1}{2}100\times400^2\right)\left[\frac{M_1}{M_2}\right]\left[\frac{L_1}{L_2}\right]^2\left[\frac{T_1}{T_2}\right]^{-2} \quad \ldots . eq(2)$$

Dimension for momentum = $M^1L^1T^{-1}$

$$10 = (100\times400)\left[\frac{M_1}{M_2}\right]\left[\frac{L_1}{L_2}\right]^1\left[\frac{T_1}{T_2}\right]^{-1} \quad \ldots . eq(3)$$

Squaring eq(3) and dividing by eq(2) we get

$$\frac{10^2}{100} = \frac{(100\times400)^2\left[\frac{M_1}{M_2}\right]^2\left[\frac{L_1}{L_2}\right]^2\left[\frac{T_1}{T_2}\right]^{-2}}{\left(\frac{1}{2}100\times400^2\right)\left[\frac{M_1}{M_2}\right]\left[\frac{L_1}{L_2}\right]^2\left[\frac{T_1}{T_2}\right]^{-2}}$$

$$1 = 200\left[\frac{M_1}{M_2}\right]$$

$M_2 = 200\ M_1$

Substituting value of $M_1/M_2 = 1/200$ in eq(2) we get

$$100 = \left(\frac{1}{2}100\times400^2\right)\frac{1}{200}\left[\frac{L_1}{L_2}\right]^2\left[\frac{T_1}{T_2}\right]^{-2} \quad \ldots . eq(4)$$

Dividing eq(4) by eq(1) we get

$$\frac{100}{3}=\frac{\left(\frac{1}{2}100\times 400^2\right)\times\frac{1}{200}\left[\frac{L_1}{L_2}\right]^2\left[\frac{T_1}{T_2}\right]^{-2}}{9.8\left[\frac{L_1}{L_2}\right]\left[\frac{T_1}{T_2}\right]^{-2}}$$

$$\frac{100}{3}=\frac{(100\times 400^2)\left[\frac{L_1}{L_2}\right]^1}{2\times 9.8\times 200}$$

$$\frac{100\times 9.8\times 2\times 200}{(100\times 400^2)}=\left[\frac{L_1}{L_2}\right]^1$$

$L_2 = 40.82L_1$

Dividing eq(3) by eq(1) we get

$$\frac{10}{3}=\frac{(100\times 400)\left[\frac{M_1}{M_2}\right]\left[\frac{L_1}{L_2}\right]^1\left[\frac{T_1}{T_2}\right]^{-1}}{9.8\left[\frac{L_1}{L_2}\right]\left[\frac{T_1}{T_2}\right]^{-2}}$$

$$\frac{10}{3}=\frac{(100\times 400)\left[\frac{M_1}{M_2}\right]\left[\frac{T_1}{T_2}\right]^1}{9.8}$$

But $M_1/M_2 = 1/200$

$$\frac{10}{3}=\frac{(100\times 400)\left[\frac{T_1}{T_2}\right]^1}{9.8\times 200}$$

$$\left[\frac{T_1}{T_2}\right]^1=\frac{10\times 9.8\times 200}{3\times 100\times 400}$$

$$\frac{T_1}{T_2} = \frac{9.8}{60}$$

$T_2 = (60/9.8)T_1$

$T_2 = 6.12$ s

Q19) If the unit of force be the weight of one kilogram, what must be the unit of mass so that the equation F = ma may still be true? [Ans g kg]

Q20) The Young's modulus of steel in CGS system is 19.0×10^{11} dyne cm^{-2}. Express it in the SI system [Ans 19×10^{10} N/m^2]

Q21) We measure the period of oscillation of a simple pendulum. In successive measurements, the reading turns out to be 264 s, 256 s, 242 s, 270s and 280 sec for 100 oscillation. If the minimum division in the measuring clock is 1 s, then what will be the reported meantime Calculate the a) absolute error b) relative error c) percentage error d) reported mean time

Ans a) 2, 6,20,12,18 b) 0.04 c) 4 % d) 262±11 s

Q22) A physical quantity z is related to four variable a,b,c d, and e are as follows

$$z = \frac{a^2 b^{\frac{1}{3}}}{c^4\, d}$$

Percentage error of measurement in a, b, c, d are 1%, 3%, 3%, 4% respectively. Find the percentage error in z [Ans 19%]

Q23) In the formula x = 2Y Z^2, X and Z have the dimensions of capacitance and magnetic induction respectively. What are the dimensions of Y in SI system

[Ans $M^{-3}\, L^{-2}\, T^8\, A^4$]

Q24) Finding the dimensions of resistance R and inductance L, speculate what physical quantities (L/R) and $(1/2)LI^2$ represent, where I is current. [Ans time, magnetic energy]

Q25) The parallax of a heavenly body measured from two points diametrically opposite on equator of the earth is 30". If the radius of Earth is 6400km, find the distance of the heavenly body from the center of Earth in AU, taking 1AU = 1.5×10^{11} m [Ans 0.586 AU]

Q26) In an experiment, the refractive index of glass was observed to be 1.45, 1.5, 1.54, 1.44, 1.54, and 1.53. Calculate (i) mean value of refractive index (ii) mean absolute error (iii) fractional error (iv) Percentage error. Express the result in terms of absolute error and percentage error. [Ans i) 1.51 ii) ±0.04 iii) ±0.03 iv) ±3% ; μ = 1.51±0.04 ; μ = 1.51±3%]

Q27) in a submarine equipped with SONAR, the time delay between generation of probe wave and the reception of its echo after reflection from an enemy submarine is found to be 773.0s. What is the distance of the enemy submarine? [Speed of sound in water = 1450 m/s) [Ans. 55825 m]

Motion In One Dimension

Particle

A particle is ideally just a piece or a quantity of matter, having practically no linear dimensions but only a position.
In practice it is difficult to get such particle, but in certain circumstances an object can be treated as particle.
Such circumstances are
(i) All the particles of solid body performing linear motion cover the same distance in the same time. Hence motion of such a body can be described in terms of the motion of its constituent particle
(ii) If the distance between two objects is very large as compared to their dimensions, these objects can be treated as particles. For example, while calculating the gravitational force between Sun and Earth, both of them can be considered as particles.

Frame of reference

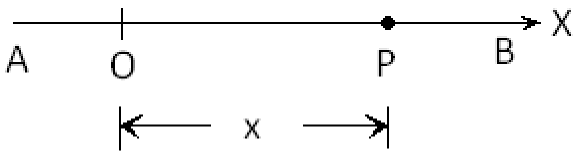
A "frame of reference" is just a set of coordinates: something you use to measure the things that matter in Newtonian problems, that is to say, positions and velocities, so we also need a clock.
Or A place and situation from where an observer takes his observation is called frame of reference.
A point in space is specified by its three coordinates (x, y, z) and an "event" like, say, a little explosion, by a place and time: (x, y, z, t).

An inertial frame is defined as one in which Newton's law of inertia holds—that is, anybody which isn't being acted on by an outside force stays at rest if it is initially at rest, or continues to move at a constant velocity if that's what it was doing to begin with. Example of inertial frame of reference is observer on Earth for all motion on surface of earth. Car moving with constant velocity
An example of a non-inertial frame is a rotating frame, such as a accelerating car,

Rest and Motion

When a body does not change its position with respect to time with respect to frame of reference, then it is said to be at rest. Revolving earth *Motion is the change of position of an object with respect to time.*
To study the motion of the object, one has to study the change in position (*x,y,z* coordinates) of the object with respect to the surroundings. It may be noted that the position of the object changes even due to the change in one, two or all the three

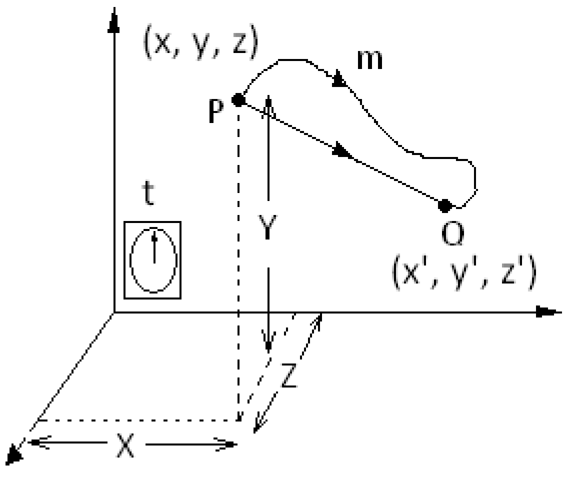
coordinates of the position of the objects with respect to time. Thus motion can be classified into three types:
(i) Motion in one dimension
Motion of an object is said to be one dimensional, if only one of the three coordinates specifying the position of the object changes with respect to time.
Example : An ant moving in a straight line, running athlete, etc.
Consider a particle moving on a straight line AB. For the analysis of motion we take origin. O at any point on the line and x-axis along the line. Generally we take origin at the point from where particle starts its motion and rightward direction as positive x-direction. At any moment if article is at P then its position is given by OP = x



(ii) Motion in two dimensions
In this type, the motion is represented by any two of the three coordinates. Example: a body moving in a plane.
(iii) Motion in three dimensions
Motion of a body is said to be three dimensional, if all the three coordinates of the position of the body change with respect to time.
Examples : motion of a flying bird, motion of a kite in the sky, motion of a molecule, etc

Position, Path-length and Displacement

POSITION
Choose a rectangular coordinate system consisting of three mutually perpendicular axes, labeled X-, Y-, and Z- axes. The point of intersection of these three axes is called origin (O) and serves as the reference point, the coordinates (x,y,x) of a particle at point P describe the position of the object with respect to this frame of reference. To measure the time we put clock in this system



If all the coordinate of particle remains unchanged with time then particle is considered at rest with respect to this frame of reference.
If position of particle at point P given by coordinates (x, y, z) at time t and particles position coordinates are (x', y', z') at time t' , that is at least one coordinates of the particle is changed with time then particle is said to be in motion with respect to this frame of reference

PATH LENGTH

The path length of an object in motion in a given time is the length of actual path traversed by the object in the given time. As shown in figure actual path travelled by the particle is PmO . Path length is always positive

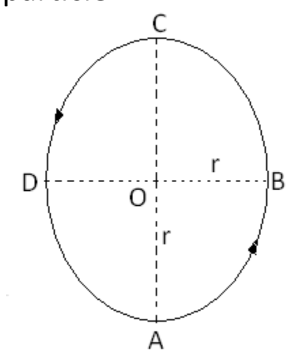
DISPLACEMENT

The displacement of an object in motion in a given time is defined as the change in a position of the object, i.e., the difference between the final and initial positions of the object in a given time. It is the shortest distance between the two positions of the object and its directions is from initial to final position of the object, during the given interval of time. It is represented by the vector drawn from the initial position to its final position. As shown in figure. Since displacement is vector it may be zero, or negative also

Solved numerical

Q) A particle moves along a circle of radius r. It starts from A and moves in anticlockwise direction as shown in figure. Calculate the distance travelled by the particle and magnitude of displacement from each of following cases

(i) from A to B (ii) from A to C (iii) from A to D (iv) one complete revolution of the particle



Solution

(i)Distance travelled by particle from A to B is One fourth of circumference thus

$$path\ length = \frac{2\pi r}{4} = \frac{\pi r}{2}$$

Displacement

$$|AB| = \sqrt{(OA)^2 + (OB)^2} = \sqrt{r^2 + r^2} = \sqrt{2}\,r$$

(ii) Distance travelled by the particle from A to C is half of the circumference

$$path\ length = \frac{2\pi r}{2} = \pi r$$

Displacement

|AC| = r+r = 2r

(iii) Distance travelled by the particle from A to D is three fourth of the circumference

$$path\ length = 2\pi r\ \frac{3}{4} = \frac{3}{2}\pi r$$

Displacement AD

$$|AD| = \sqrt{(OA)^2 + (OD)^2} = \sqrt{r^2 + r^2} = \sqrt{2}\, r$$

(iv) For one complete revolution total distance is equal to circumference of circle

Path length = 2πr

Since initial position and final position is same displacement is zero

Speed and velocity

Speed

It is the distance travelled in unit time. It is a scalar quantity.

$$speed = \frac{path\ length}{time}$$

Solved numerical

Q) A motorcyclist covers 1/3rd of a given distance with speed 10 kmh^{-1}, the next 1/3rd at 20 kmh^{-1} and the last 1/3rd at of 30kmh^{-1}. What is the average speed of the motorcycle for the entire journey

<u>Solution:</u>

Let total distance or path length be 3x

Time taken for first 1/3rd path length

$$t_1\ = \frac{path\ length}{speed} = \frac{x}{10}\ hr$$

Time taken for second 1/3rd path length

$$t_1\ = \frac{path\ length}{speed} = \frac{x}{20}\ hr$$

Time taken for third 1/3rd path length

$$t_1\ = \frac{patn\ length}{speed} = \frac{x}{30}\ hr$$

Total time taken to travel path length of 3x is, $t = t_1+t_2+t_3$

Substituting values of t_1 t_2 and t_3 in above equation we get

$$t = \frac{x}{10} + \frac{x}{20} + \frac{x}{30} = \frac{11x}{60}\ hr$$

Form the formula for speed

$$speed = \frac{path\ length}{time}$$

$$speed = \frac{path\ length}{time}$$

$$speed = \frac{3x}{\frac{11x}{60}} = \frac{180}{11} = 16.36\ kmh^{-1}$$

Velocity

The velocity of a particle is defined as the rate of change of displacement of the particle. It is also defined as the speed of the particle in a given direction. The velocity is a vector quantity. It has both magnitude and direction.

$$velocity = \frac{displacement}{time}$$

Units for velocity and speed is m s^{-1} and its dimensional formula is LT^{-1}.

Uniform velocity

A particle is said to move with uniform velocity if it moves along a fixed direction and covers equal displacements in equal intervals of time, however small these intervals of time maybe.

Non uniform or variable velocity

The velocity is variable (non-uniform), if it covers unequal displacements in equal intervals of time or if the direction of motion changes or if both the rate of motion and the direction change.

Average velocity

Let s_1 be the position of a body in time t_1 and s_2 be its position in time t_2 The average velocity during the time interval ($t_2 - t_1$) is defined as

$$\mathrm{v} = \frac{\mathrm{s_2 - s_1}}{\mathrm{t_2 - t_1}} = \frac{\Delta \mathrm{s}}{\Delta \mathrm{t}}$$

Average speed of an object can be zero ,positive or zero. It depends on sign of displacement.

In general average speed of an object can be equal to or greater than the magnitude of the average velocity

Instantaneous velocity

It is the velocity at any given instant of time or at any given point of its path. The instantaneous velocity *v* is given by

$$v = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{s}}{\Delta \mathrm{t}} = \frac{\mathrm{ds}}{\mathrm{dt}}$$

Acceleration

If the magnitude or the direction or both of the velocity changes with respect to time, the particle is said to be under acceleration. Acceleration of a particle is defined as the rate of change of velocity.

If object is performing circular motion with constant speed then also it is accelerated motion as direction of velocity is changing

Acceleration is a vector quantity.

$$acceleration = \frac{change\ in\ velocity}{time}$$

If *u* is the initial velocity and *v,* the final velocity of the particle after a time *t*, then the acceleration,

$$a = \frac{v - u}{t}$$

Its unit is m s^{-2} and its dimensional formula is LT^{-2}

The instantaneous acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

If the velocity decreases with time, the acceleration is negative. The negative acceleration is called **retardation or deceleration**

Equations of motion

Motion in straight line with uniform velocity

If motion takes place with uniform velocity v on straight line the

Displacement in time t , S = vt - - - -eq(1)

Acceleration of particle is zero

Motion in a straight line with uniform acceleration – equations of motion

Let particle moving in a straight line with velocity u (velocity at time t = 00 and with uniform acceleration a. Let its velocity be v at the end of the interval of time t (final velocity at time t). Let S be the displacement at the instant t acceleration a is

$$a = \frac{v - u}{t} \quad or$$

$$v = u + at \quad - - - eq(2)$$

If u and a are in same direction 'a" is positive and hence final velocity v will be more than initial velocity u, velocity increases

If u and a are in opposite direction final velocity v will be less than initial velocity u. Velocity is decreasing. And acceleration is negative

Displacement during time interval t = average velocity ×t

$$S = \frac{v + u}{2} \times t - -eq(3)$$

Eliminating v from equation 3 and equation 2 we get

$$S = \frac{u + at + u}{2} \times t$$

$$S = ut + \frac{1}{2}at^2 - - - eq(4)$$

Another equation can be obtained by eliminating t from equation 2 and equation3

$$v = u + at$$

$$t = \frac{v - u}{a}$$

$$S = \frac{v + u}{2} \times \frac{v - u}{a}$$

$$S = \frac{v^2 - u^2}{2a}$$

$$v^2 = u^2 + 2aS \ ---eq(5)$$

Distance transverse by the particle in n^{th} second of its motion
The velocity at the beginning of the nth second = u +a (n-1)
The velocity at the end of n^{th} second = u+an
Average velocity during n^{th} second v_{ave}

$$v_{ave} = \frac{\mathrm{u} + \mathrm{a}\,(\mathrm{n}-1) + \mathrm{u} + \mathrm{an}}{2}$$

$$v_{ave} = u + \frac{1}{2}a(2n-1)$$

Distance during this one second
S_n = average velocity × time

$$S_n = u + \frac{1}{2}a(2n-1) \times 1$$

$$S_n = u + \frac{1}{2}a(2n-1) --- eq(6)$$

The six equations derived above are very important and are very useful in solving problems in straight-line motion

Calculus method of deriving equation of motion
The acceleration of a body is defined as

$$a = \frac{dv}{dt}$$

$$dv = adt$$

Integrating we get v = at+A
Where A is constant of integration . For initial condition t = 0, v = u (initial velocity) we get A = u
∴ v = u +at

We know that instantaneous velocity v

$$v = \frac{ds}{dt}$$

ds = adt
displacement ds = vdt = (u+at)dt
integrating above equation

$$S = ut + \frac{1}{2}at^2 + B$$

B is integration constant
At t = 0, S = 0 yields B = 0

$$\therefore S = ut + \frac{1}{2}at^2$$

Acceleration a

$$a = \frac{dv}{dt} = \frac{dv}{dS} \cdot \frac{dS}{dt} = v\frac{dv}{dS}$$

$$\therefore a = v\frac{dv}{dS}$$

$$adS = v \cdot dv$$

Integrating we get

$$aS = \frac{v^2}{2} + C$$

Where C is integration constant

Applying initial condition , where S = 0, v = u we get

$$0 = \frac{u^2}{2} + C$$

$$Or\ \ C = -\frac{u^2}{2}$$

$$\therefore aS = \frac{v^2}{2} - \frac{u^2}{2}$$

$$v^2 = u^2 + 2aS$$

If S_1 and S_2 are the distances traversed during n seconds and (n-1) seconds

$$S_1 = un + \frac{1}{2}an^2$$

$$S_2 = u(n-1) + \frac{1}{2}a(n-1)^2$$

Displacement in nth second

$$S_n = S_1 - S_2$$

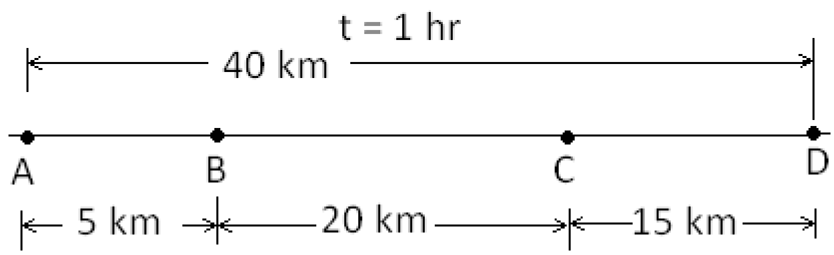
$$S_n = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$

$$S_n = u + \frac{1}{2}a(2n-1)$$

Solved numerical

Q) The distance between two stations is 40 km. A train takes 1 hour to travel this distance. The train, after starting from the first station, moves with constant acceleration for 5km, then it moves with constant velocity for 20 km and finally its velocity keeps on decreasing continuously for 15 km and it stops at the other station. Find the maximum velocity of the train.

Solution:



Motion is divided in three parts
Motion between point A and B is with constant acceleration
Here initial velocity u = 0 and final velocity at point B = v_{max}
Let time interval be t_1
From equation

$$S = \frac{v+u}{2} \times t$$
$$5 = \frac{v_{max}+0}{2} \times t_1$$
$$t_1 = \frac{10}{v_{max}}$$

Motion between point B and C is with constant velocity v_{max}
Let time period t_2
Form formula S = vt
20 = v_{max} t_2

$$t_2 = \frac{20}{v_{max}}$$

Motion between point C and D is with retardation
Initial velocity is v_{max} and final velocity v= 0 let time interval t_3
From formula

$$S = \frac{v+u}{2} \times t$$
$$15 = \frac{0+v_{max}}{2} \times t_3$$
$$t_3 = \frac{30}{v_{max}}$$

Total time taken is 1 hr
T = t_1 + t_2 + t_3

$$1 = \frac{10}{v_{max}} + \frac{20}{v_{max}} + \frac{30}{v_{max}}$$

$\therefore V_{max}$=60 km h^{-1}

Q) A certain automobile manufacturer claims that its sports car will accelerate from rest to a speed of 42.0 m/s in 8.0 s. under the important assumption that the acceleration is constant
(i)Determine the acceleration
(ii)Find the distance the car travels in 8s
(iii)Find the distance travelled in 8^{th} s
Solution
(a) Here initial velocity u = 0 and final velocity v = 42 m/s

From formula

$$a = \frac{v-u}{t}$$

$$a = \frac{42-0}{8} = 5.25\, ms^{-2}$$

(b) Distance travelled in 8.0s

From formula

$$S = ut + \frac{1}{2}at^2$$

$$S = (0)(t) + \frac{1}{2}(5.25)(8)^2 = 168\, m$$

(c) distance travelled in 8th second.

From formula

$$S_n = u + \frac{1}{2}a(2n-1)$$

$$S_n = 0 + \frac{1}{2}(5.25)(2 \times 8 - 1) = 39.375\, m$$

Q) Motion of a body along a straight line is described by the equation

x = $t^3 + 4t^2 - 2t + 5$ where x is in meter and t in seconds

(a) Find the velocity and acceleration of the body at t = 4s

(b) Find the average velocity and average acceleration during the time interval from t = 0 to t = 4 s

Solution

(a)We have to find instantaneous velocity at t = 4s

$$v = \frac{dx}{dt} = \frac{d}{dt}(t^3 + 4t^2 - 2t + 5)$$

$$v = \frac{d}{dt}t^3 + 4\frac{d}{dt}t^2 - 2\frac{d}{dt}t + \frac{d}{dt}5$$

$$v = 3t^2 + 4 \times 2t - 2$$

$$v = 3t^2 + 8t - 2$$

Thus we get equation for velocity, by substituting t = 4 in above equation we get instantaneous velocity at t = 4

v=$3(4)^2 + 8(4) - 2$

v = 78 m/s

To find instantaneous acceleration at t = 4s

$$a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 8t - 2)$$

a =6t +8

Thus we get equation for acceleration, by substituting t=4 in equation for acceleration we get instantaneous acceleration t=4

a = 6(4) + 8

a=32 m s^{-2}

(b)Average velocity

Final position of object at time t = 4 s

$X_4 = (4)^3 + 4(4)^2 – 2(4) +5 = 125$

Initial position of object at time t = 0 s

$X_0= (0)^3 + 4(0)^2 – 2(0) +5 = 5$

Displacement = 125 – 5 = 120 m, time interval t =4 seconds

Average velocity = Displacement / time = 120/4 = 30 ms^{-1}

Average acceleration

Initial velocity t = 0 from equation for velocity

$$v = 3t^2 + 8t - 2$$

$$v = 3(0)^2 + 8(0) - 2 = -2\ ms^{-1}$$

∴ Initial velocity u = -2 ms^{-1}

Final velocity is calculated as 78 ms^{-1}

From formula for average acceleration

$$a = \frac{v-u}{t} = \frac{78-(-2)}{4} = 20ms^{-2}$$

Q) A particle moving in a straight line has an acceleration of (3t – 4) ms^{-2} at time t seconds. The particle is initially 1m from O, a fixed point on the line, with a velocity of 2ms^{-1} . Find the time when the velocity is zero. Find the displacement of particle from O when t = 3

Solution:

$$a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = 3t - 4$$

$$\Rightarrow \int_2^v dv = \int_0^t (3t-4)dt$$

$$\Rightarrow v - 2 = \frac{3t^2}{2} - 4t$$

$$\Rightarrow v = \frac{3t^2}{2} - 4t + 2$$

The velocity will be zero when

$$\frac{3t^2}{2} - 4t + 2 = 0$$

i.e when

$$(3t-2)(t-2) = 0$$

$$t = \frac{2}{3}\ or\ 2$$

Using

$$\frac{ds}{dt} = v$$

We have

$$\frac{ds}{dt} = \frac{3t^2}{2} - 4t + 2$$

$$\Rightarrow \int_1^s ds = \int_0^3 \left(\frac{3t^2}{2} - 4t + 2\right) dt$$

$$\Rightarrow s - 1 = \left[\frac{3t^2}{2} - 4t + 2\right]_0^2 = 1.5$$

$$\Rightarrow s = 2.5 \ m$$

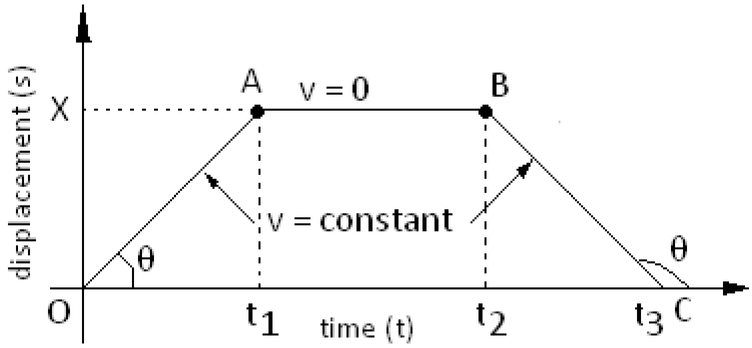
Therefore the particle is 2.5 m from O when t = 3s

Graphical representation of motion

(1) Displacement – time graph:

If displacement of a body is plotted on Y-axis and time on X-axis, the curve obtained is called displacement-time graph.

The instantaneous velocity at any given instant can be obtained from the graph by finding the slop of the tangent at the point corresponding to the time

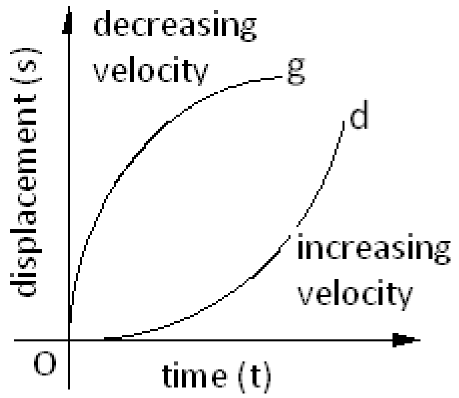


graph(A)

In graph(A) object started to move with constant velocity (a = 0) at time t = 0 from origin. Object is going away represented by OA, at time t_1 object reach position X ,note slope of graph AO is positive and constant .

For time period t_1 to t_2 object have not changed its position thus velocity is zero. Slope of graph is zero

For time period t_2 to t_3 object started to move towards its original position at time t_2 and reaches original position at time t_3 . Here velocity is constant (a=0) as slope of graph is constant. And reaching original position as slope is negative



graph (B)

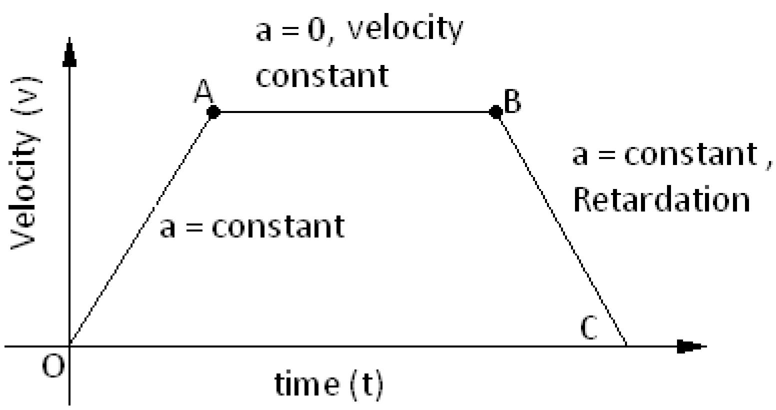
In graph(B) motion represented by Og is decelerated motion as slope is decreasing with time, hence velocity is decreasing. However object is moving away from origin

Motion represented by Od is accelerated as slope is continuously increasing with time, it indicates that velocity is increasing or acceleration is positive , object is moving away from origin

(2)Velocity-time graph

If Velocity of a body is plotted on Y-axis and time on X-axis, the curve obtained is called velocity-time graph.

The instantaneous acceleration at any given instant can be obtained from the graph by finding the slop of the tangent at the point corresponding to the time

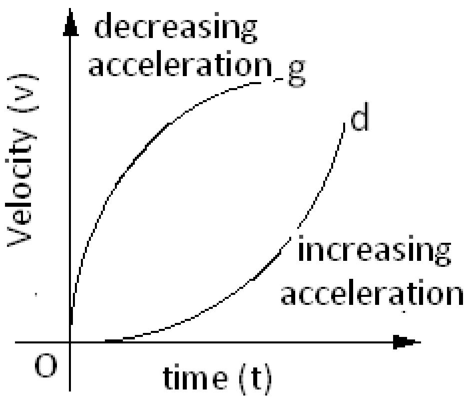


graph (C)

Graph AB is parallel straight indicate object is moving with constant velocity or acceleration is zero

Graph OA is oblique straight line slope is positive indicate object is uniformly accelerated

Graph BC is oblique straight line slope is negative indicated object is uniformly decelerated

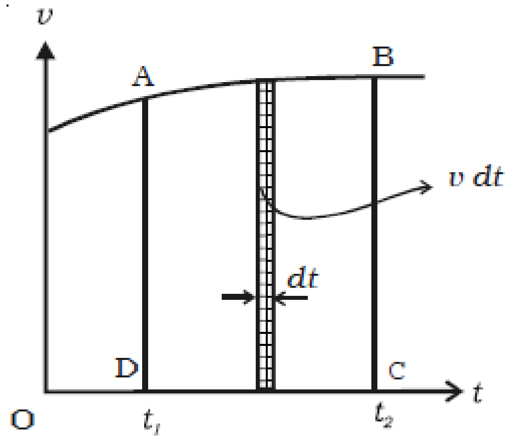


graph (D)

Graph Og represents decreasing acceleration as slop is decreasing with time

Graph Od represent increasing acceleration as slop is increasing with time

When the velocity of the particle is plotted as a function of time, it is velocity-time graph. Area under the curve gives displacement



We know that

$$v = \frac{dS}{dt}$$

$$dS = v \cdot dt$$

If displacements are S_1 and S_2 at time t_1 and t_2 then

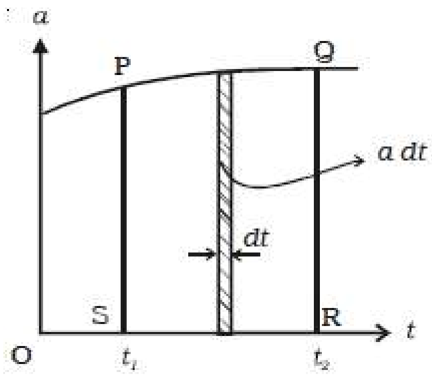
$$\int_{S_1}^{S_2} dS = \int_{t_1}^{t_2} v\, dt$$

$$S_2 - S_1 = \int_{t_1}^{t_2} v\, dt = Area\ ABCD$$

The area under the $v - t$ curve, between the given intervals of time, gives the change in displacement or the distance travelled by the particle during the same interval.

Acceleration – time graph

When the acceleration is plotted as a function of time, it is acceleration - time graph



$a = \frac{dv}{dt}$

dv = a dt

If v_1 and v_2 are the velocities at time t_1 and t_2 then

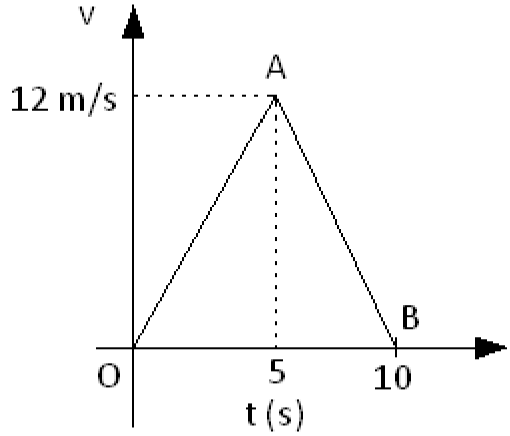
$$\int_{v_1}^{v_2} dy = \int_{t_1}^{t_2} adt$$

$$v_2 - v_1 = \int_{t_1}^{t_2} a\, dt = AreaPQRS$$

The area under the *a* – *t* curve, between the given intervals of time, gives the change in velocity of the particle during the same interval. If the graph is parallel to the time axis, the body moves with constant acceleration.

Solved numerical

Q) The v – t graph of a particle moving in straight line is shown in figure. Obtain the distance travelled by the particle from (a) t = 0 to t = 10s and from (b) t = 2s to 6s



Solution:

(a) Distance travelled in time period t = 0 to t = 10s is area of triangle OAB = (1/2)×10×12 = 60 m

(b)Distance in time period t = 2 to t = 6s

From graph slope of line OA is 2.4 m/s^2

Initial velocity at t = 2 sec u = 4.8 thus using formula

X = ut + (1/2)at^2 here time period is 3 sec

X_1 = (4.8)(3) + (1/2) (2.4) (3)^2 = 25.2

For segment A to B acceleration is 2.4 time period 1s u = 5

X_2 = (12) (1) – (1/2) (2.4) (1)^2 = 10.8

Thus distance = 25.2 +10.8 = 36 m

Vertical motion under gravity

When an object is thrown vertically upward or dropped from height, it moves in a vertical straight line. If the air resistance offered by air to the motion of the object is neglected, all objects moving freely under gravity will be acted upon by its weight only

This causes vertical acceleration g having value 9.8 m/s^2, so the equation for motion in a straight line with constant acceleration can be used.

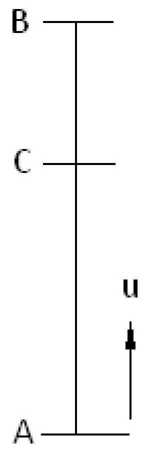
In some problems it is convenient to take the downward direction of acceleration as positive, in such case if the object is moving upward initial velocity should be taken as negative and displacement positive.

If object is moving downwards then, initial velocity should be taken as positive and displacement negative.

Projection of a body vertically upwards

Suppose an object is projected upwards from point A with velocity u

If we take down word direction of g as **Negative** then



(i) At a time t its velocity v = u –gt

(ii) At a time t, its displacement from A is gen by

$S = ut - (1/2)\,gt^2$

(iii) Its velocity when its displacement S is given by

$v^2 = u^2 - 2gS$

(iv) When it reaches the maximum height, its velocity v = 0.

This happens when t = u/g. The body is instantaneously rest

From formula

$V = u - gt$

$t = v/g$

(v) The maximum height reached. At maximum height final velocity v = 0 and S = H thus

From equation

$$v^2 = u^2 - 2gS$$
$$0 = u^2 - 2gH$$
$$H = \frac{u^2}{2g}$$

(vi) Total time to go up and return to the point of projection

Displacement S = 0 Thus from formula

$S = ut - (1/2)\,gt^2$

$0 = ut - (1/2)\,gt^2$

$T = 2u/g$

(vii) At any point C between A and B, where AC = s, the velocity v is given by

$$v = \pm\sqrt{u^2 - 2gS}$$

The velocity of body while crossing C upwards =

$$v = +\sqrt{u^2 - 2gS}$$

The velocity of body while crossing C downwards

$$v = -\sqrt{u^2 - 2gS}$$

Magnitudes of velocities are same

Solved numerical

Q) A body is projected upwards with a velocity 98 m/s.
Find (a) the maximum height reached
(b) the time taken to reach maximum height
(c) its velocity at height 196 m from the point of projection
(d) velocity with which it will cross down the point of projection and
(e) the time taken to reach back the point of projection

Solution:

(a) Maximum height

$$H = \frac{u^2}{2g} = \frac{(98)^2}{2 \times 9.8} = 490\ m$$

(b) Time taken to reach maximum height

T= u/g = 9.8/9.8 = 10s

(c) Velocity at a height of 196 m from the point of projection

$$v = \pm\sqrt{u^2 - 2gS}$$

$$v = \pm\sqrt{(98)^2 - 2(9.8)(196)} = \pm 75.91\ m/s$$

+75.91 m/s while crossing the height upward and -75.91 m/ while crossing it downwards

(d)Velocity with which it will cross down the point of projection

Magnitude is same but direction is opposite hence V = -u =-98 m/s

(e)The time taken to reach back the point of projection

T= 2u/g = (2×98)/9.8 = 20 s

----------------END--------------

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Position of points in two dimensions is represented in vector form

$$\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} \;\; and \;\; \vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath}$$

If particle moves from r_1 position to r_2 position then

Displacement is final position – initial position

$$\vec{s} = \vec{r}_2 - \vec{r}_1$$

$$\vec{s} = (x_2\hat{\imath} + y_2\hat{\jmath}) - (x_1\hat{\imath} + y_1\hat{\jmath} \;\;)$$

$$\vec{s} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$$

If $x_2 - x_1 = dx$ and $y_2 - y_1 = dy$

$$\vec{s} = dx\hat{\imath} + dy\hat{\jmath}$$

Now

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$

$dx/dt = v_x$ is velocity of object along x-axis and $dy/dt = v_y$ is velocity along y-axis

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

Thus motion in two dimension is resultant of motion in two independent component motions taking place simultaneously in mutually perpendicular directions.

Magnitude of velocity

$$v = \sqrt{v_x^2 + v_y^2}$$

Angle between velocity and x-axis is

$$tan\theta = \frac{v_y}{v_x}$$

$$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}$$

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$$

Magnitude of acceleration

$$a = \sqrt{a_x^2 + a_y^2}$$

Equation of motion in vector form

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{r} = \vec{r}_0 + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$average\ velocity = \frac{\vec{u} + \vec{v}}{2}$$

$$v^2 - u^2 = 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$$

Solved Numerical

Q1) A particle starts its motion at time t = 0 from the origin with velocity 10**j** m/s and moves in X-Y plane with constant acceleration 8**i**+2**j**

(a) At what time its x-coordinate becomes 16m? And at that time what will be its y co-ordinate?

(b) What will be the speed of this particle at this time?

Solution:

From equation of motion

$$\vec{r} = \vec{r}_0 + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$x\hat{\imath} + y\hat{\jmath} = (10\hat{\jmath})t + \frac{1}{2}(8\hat{\imath} + 2\hat{\jmath})t^2$$

$$x\hat{\imath} + y\hat{\jmath} = (4t^2)\hat{\imath} + (10t + t^2)\hat{\jmath}$$

Thus

$$x = 4t^2$$

When x = 16

16 = $4t^2$, t = 2 sec

Now y = $10t+t^2$ at t = 2 sec

Y = 20+4 = 240 m

b)

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{v} = 10\hat{j} + (8\hat{i} + 2\hat{j})t$$

t =2 sec

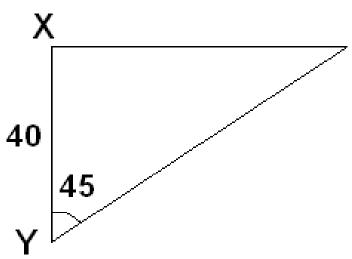
$$\vec{v} = 10\hat{j} + (8\hat{i} + 2\hat{j}) \times 2$$

$$\vec{v} = 16\hat{i} + 14\hat{j}$$

$$|\vec{v}| = \sqrt{16^2 + 14^2} = 21.26\ sec$$

Q2) A bomber plane moves due east at 100km/hr over a town X at a certain time, Six minutes later an interceptor plans sets off from station Y which is 40km due south of X and flies north east. If both maintain their courses, find the velocity with which interceptor must fly in order to take over the bomber

Solution:



Let velocity of interceptor speed be V . Now components of velocity along North direction is Vcos45 and along East direction is Vsin45

To intercept the bomber , interceptor has to travel a distance of 40 km in t seconds along North direction Thus 40 = Vcos45t

Vt= 40√2 ---- eq(1)

Now total distance travelled by bomber = 100t + distance travelled in 6 minutes before interceptor takeoff

Distance travelled by bomber along East direction = 100t +10 km

Thus distance travelled by interceptor

100t + 10 = Vsin45t

100t +10 = Vt / √2

From eq(1)

100t +10 =40

t = 3/10 Hr

Substituting above value of t in equation (1) we get

$$V = \frac{40\sqrt{2} \times 10}{3} = 188.56\ km/hr$$

Q3) A motor boat set out at 11a.m. from a position -6**i**-2**j** relative to a marker buoy and travels at steady speed of magnitude √53 on direct course to intercept a ship. This ship maintains a steady velocity vector 3**i**+4**j** and at 12 noon is at position 3**i**-4**j** from the buoy. Find (a) the velocity vector of the motorboat (b) the time of interception and (c) the position vector of point of interception from the buoy, if distances are measured in kilometers and speeds in km/hr

Solution

Let xi+vj be the velocity vector of the motor boat

$X^2 + y^2 = 53$ ------eq(1)

Position vector of motor boat after time t = -6**i**-2**j** + (x**i**+y**j**)t

Since position of ship is given 12 noon to find position of ship at time t, time for ship = (t-1)

Position vector of ship after time t = 3**i**-**j**+(3**i**+4**j**) (t-1)

The time of interception is given by

-6**i**-2**j**+(x**i**+y**j**)t = 3**i**-**j** +(3**i**+4**j**) (t-1)

∴-6+xt = 3t

∴xt = 6 +3t -------eq(2)

And -2yt = 4t-5

∴yt = 4t-3 ------eq(3)

From equation (1), (2) and (3)

$(6+3t)^2 + (4t-3)^2 = 53t^2$

$\therefore 36+9t^2+36t +16t^2-24t+9 = 53t^2$

∴28t-12t-45 = 0

∴(2t-3) (14t +15) = 0

t = (3/2) hr

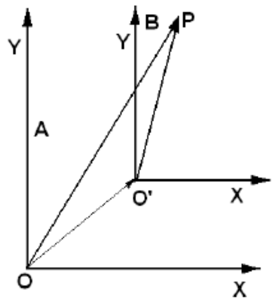
Time of interception = 12.30 pm

Solving equation(2) and (3) x = 7 and y = 2

Velocity vector of motor boat = 7**i**+2**j**

Point of interception = -6**i**-2**j** + (7**i** +2**j**) (3/2) = 9/2 **i** +**j**

Relative velocity



Two frame of reference A and B as shown in figure moving with uniform velocity with respect to each other. Such frame of references are called inertial frame of reference. Suppose two observers, one in from A and one from B study the motion of particle P.

Let the position vectors of particle P at some instant of time with respect to the origin O of frame A be

$$\vec{r}_{P,A} = \overrightarrow{OP}$$

And that with respect to the origin O′ of frame B be

$$\vec{r}_{P,B} = \overrightarrow{O'P}$$

The position vector of O′ w.r.t O is

$$\vec{r}_{B,A} = \overrightarrow{OO'}$$

From figure it is clear

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P} = \overrightarrow{O'P} + \overrightarrow{OO'}$$

$$\therefore \vec{r}_{P,A} = \vec{r}_{P,B} + \vec{r}_{B,A}$$

Differentiating above equation with respect to time we get

$$\frac{d}{dt}(\vec{r}_{P,A}) = \frac{d}{dt}(\vec{r}_{P,B}) + \frac{d}{dt}(\vec{r}_{B,A})$$

$$\therefore \vec{v}_{P,A} = \vec{v}_{P,B} + \vec{v}_{B,A}$$

Here $\mathbf{V}_{P,A}$ is the velocity of the particle w.r.t frame of reference A, $\mathbf{V}_{P,B}$ is the velocity of the particle w.r.t. reference frame B and $V_{B,A}$ is the velocity of frame of reference B with respect to frame A

Suppose velocities of two particles A and B are respectively $\mathbf{V}_A$ and $\mathbf{V}_B$ relative to frame of reference then velocity ($\mathbf{V}_{AB}$) of A with respect to B is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

And velocity **V**$_{BA}$ of B relative to A is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

Thus

$$\vec{v}_{AB} = -\vec{v}_{BA}$$

And

$$|\vec{v}_{AB}| = |\vec{v}_{BA}|$$

$$|\vec{v}_{AB}| = \sqrt{V_A^2 + V_B^2 - 2V_AV_B cos\theta}$$

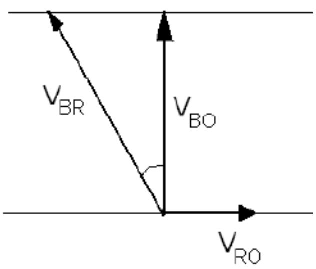
Also angle α made by the relative velocity with V_A is given by

$$tan\alpha = \frac{V_B sin\theta}{V_A - V_B cos\theta}$$

Solved Numerical

Q4) A boat can move in river water with speed of 8 km/h. This boat has to reach to a place from one bank of the river to a place which is in perpendicular direction on the other bank of the river. Then (i) in which direction should the boat has to be moved (ii) If the width of the river is 600m, then what will be the time taken by the boat to cross the river? The river flows with velocity 4 km/h

Solution



Suppose the river is flowing in positive X direction as shown in figure . To reach to a place in the perpendicular direction on the other bank, the boat has to move in the direction making angle θ with Y direction as shown in figure. . This angle should be such that the velocity of the boat relative to the opposite bank is in the direction perpendicular to the bank . Let V_{BR} = Velocity of boat with respect to river

V_{BO} = Velocity of Boat with respect to Observer on bank

V_{RO} = velocity of river with respect to observer on bank

$$\vec{V}_{BO} = \vec{V}_{BR} + \vec{V}_{RO}$$

In vector form

$$V_{BO}\hat{j} = -V_{BR}sin\theta\hat{\imath} + V_{BR}cos\theta\hat{j} + V_{RO}\hat{\imath}$$

$$V_{BO}\hat{j} = -8sin\theta\hat{\imath} + 8cos\theta\hat{j} + 4\hat{\imath}$$

Thus considering only x components

$$0 = -8sin\theta + 4$$

$$\Rightarrow \theta = 30^{O}$$

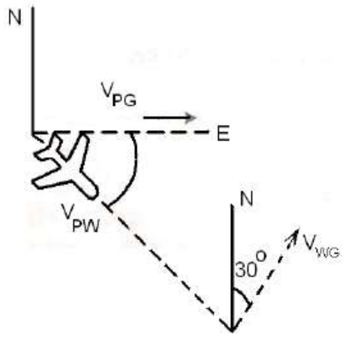
$$V_{BO} = 8cos\theta\ , \qquad V_{BO} = 8cos30$$

$$V_{BO} = 8\frac{\sqrt{3}}{2} = 4\sqrt{3} = 6.93\ km/h$$

Time taken to cross river

$$t = \frac{distance}{velocity} = \frac{0.6}{6.93} = 0.0865\ Hr = 5.2\ minutes$$

Q5) In figure a plane moves due east while pilot points the plane somewhat south of east, towards a steady wind that blows to the northeast. The plane has velocity V_{PW} relative to the wind, with an airspeed of 200km/hr directed at an angle θ south of east. The wind has velocity relative to the ground, with a speed 40√3 km/hr directed 30° east of north. What is the magnitude of the velocity of plane V_{PG} relative to the ground, and what is the value of θ



Solution

$$\vec{V}_{PG} = \vec{V}_{PW} + \vec{V}_{WG}$$

$$V_{PG}\hat{\imath} = V_{PW}cos\theta\hat{\imath} - V_{PW}sin\theta\vec{j} + V_{WG}sin30\hat{\imath} + V_{WG}cos30\hat{j}$$

$$V_{PG}\hat{\imath} = V_{PW}cos\theta\hat{\imath} + V_{WG}sin30\hat{\imath} + V_{WG}cos30\hat{j} - V_{PW}sin\theta\hat{j}$$

Comparing y co-ordinates

$$0 = V_{WG}cos30 - V_{PW}sin\theta$$

$$0 = 40\sqrt{3}\frac{\sqrt{3}}{2} - 200sin\theta$$

$$sin\theta = \frac{60}{200} = \frac{3}{10}$$

⇒ θ =17.45^{O}

Comparing x coordinates

$$V_{PG}\hat{\imath} = V_{PW}cos\theta\hat{\imath} + V_{WG}sin30\hat{\imath}$$

$$V_{PG} = 200cos17.45 + 40\sqrt{3}sin30$$

$$V_{PG} = 200cos17.45 + 40\sqrt{3}sin30 = 225\ km/hr$$

Projectile motion

A projectile is a particle, which is given an initial velocity, and then moves under the action of its weight alone.

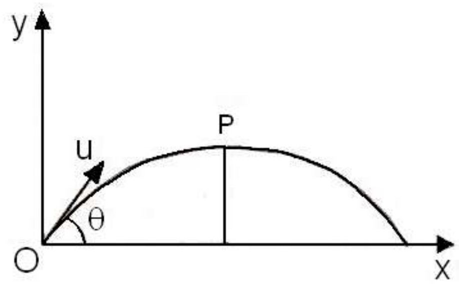
When object moves at constant horizontal velocity and constant vertical downward acceleration, such a two dimension motion is called projectile.

The projectile motion can be treated as the resultant motion of two independent component motion taking place simultaneously in mutually perpendicular directions. One component is along the horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to gravitational force.

Important terms used in projectile motion

When a particle is projected into air, the angle that the direction of projection makes with horizontal plane through the point of projection is called the angle of projection, the path, which the particle describes, is called the trajectory, the distance between the point of projection and the point where the path meets any plane draws through the point of projection is its range, the time that elapses in air is called as time of flight and the maximum distance above the plane during its motion is called as maximum height attained by the projectile

Analytical treatment of projectile motion



Consider a particle projected with a velocity u of an angle θ with the horizontal earth’s surface. If the earth did not attract a particle to itself, the particle would describe a straight line, on account of attraction of earth, however, the particle describes a curve path

Let us take origin at the point of projection and x-axis along the surface of earth and perpendicular to it respectively shown in figure

Here gravitational force is the force acting on the object downwards with constant acceleration of g downwards. There if no force along horizontal direction hence acceleration along horizontal direction is zero

Motion in x –direction

Motion in x-direction with uniform velocity

At t = 0, $X_0 = 0$ and $u_x = u\cos\theta$

Position after time t , $x = x_0 + u_x t$

$X = (u\cos\theta)\, t$ -----eq(1)

Velocity at t , $V_x = u_x$

$V_x = u\cos\theta$ ------eq(2)

Motion in y-direction:

Motion in y-direction is motion with uniform acceleration

When t = 0, $Y_0 = 0$, $u_y = u\sin\theta$ and $a_y = -g$

After time 't', $V_y = y_y + a_y t$

$V_y = u\sin\theta - gt$ -----eq(3)

$$Y = Y_0 + u_y t + \frac{1}{2} a_y t^2 \quad ---eq(4)$$

Also,

$$V_y^2 = u_y^2 + 2a_y t^2$$

$$V_y^2 = u^2 sin^2\theta - 2gy \quad ----eq(5)$$

Time of flight(T) :

Time of flight is the time duration which particle moves from O to O'.

i.e t = T, Y = 0

From equation (4)

$$0 = usin\theta T - \frac{1}{2}gT^2$$

$$T = \frac{2usin\theta}{g} \qquad ----eq(6)$$

Range of projectile (R):

Range of projectile distance travelled in time T,

i.e. R = V_x × T

R = ucosθT

$$R = ucos\theta \frac{2usin\theta}{g}$$

Since 2sinθcosθ = sin2θ

$$R = \frac{u^2 sin2\theta}{g} \qquad ----eq(7)$$

Maximum height reached (H):

At the time particle reaches its maximum height velocity of particle becomes parallel to horizontal direction i.e. V_y = 0, when Y = H

From equation (5)

$$0 = u^2 sin^2\theta - 2gH$$

$$H = \frac{u^2 sin^2\theta}{2g}$$

Equation of trajectory:

The path traced by a particle in motion is called trajectory and it can be known by knowing the relation between X and Y

From equation (1) and (4) eliminating time t we get

$$Y = Xtan\theta - \frac{1}{2}\frac{gX^2}{u^2}sec^2\theta$$

This is trajectory of path and is equation of parabola. So we can say the path of particle is parabolic

Velocity and direction of motion after a given time:

After time ‘t’

V_y = ucosθ and V_y = usinθ – gt

Hence resultant velocity

$$V = \sqrt{V_X^2 + V_Y^2}$$

$$V = \sqrt{u^2cos^2\theta + (usin\theta - gt)^2}$$

If direction of motion makes an angle α with horizontal

$$tan\alpha = \frac{V_y}{V_x} = \frac{usin\theta - gt}{ucos\theta}$$

$$\alpha = tan^{-1}\left(\frac{usin\theta - gt}{ucos\theta}\right)$$

Velocity and direction of motion at a given height

At height ‘h’ , V_x = ucosθ

And

$$V_y = \sqrt{u^2sin^2\theta - 2gh}$$

Resultant velocity

$$V = \sqrt{V_x^2 + V_y^2}$$

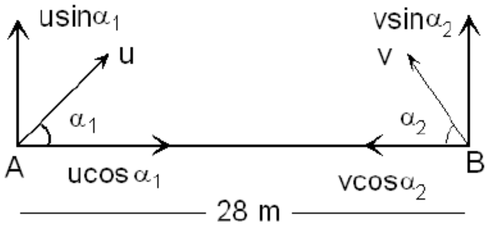
$$V = \sqrt{u^2 - 2gh}$$

Note that this is the velocity that a particle would have at height h if it is projected vertically from ground with u

Solved Numerical

Q6) Two particles are projected at the same instant from point A and B on the same horizontal level where AB = 28 m, the motion taking place in a vertical plane through AB. The particle from A has an initial velocity of39 m/s at an angle sin^{-1} (5/13) with AB and the particle from B has an initial velocity 25 m/s at an angle sin^{-1}(3/5) with BA. Show that the particle would collide in mid air and find when and where the impact occurs

Solution



If both the particles have same initial vertical velocity v then only particles will collide so that for collision both particles must be at same height

Vertical velocity of particle at A = $u\sin\alpha_1$ =39(5/13) =15 m/s

Vertical velocity of particle at B = $v\sin\alpha_2$ = 25(3/5) = 15 m/s

Since both particles have same vertical velocity they will collide

Since both the particles have travelled in all horizontal distance of 28 m

Therefore 28 = [$u\cos\alpha_1$ + $v\cos\alpha_2$]t

Now $\sin\alpha_1$ = 5/13 thus $\cos\alpha_1$ = (12/13)

$\sin\alpha_2$ = 3/5 thus $\cos\alpha_2$ = 4/5

Substituting values in equation (1) we get

28 = [39(12/13) + 25(4/5)] t

28 = (56) t

t = 0.5 sec. Particles will collide after 0.5 sec

Now Horizontal distance travelled by particle from A =[$u\cos\alpha_1$]t =[39 (12/13)]0.5= 18 m

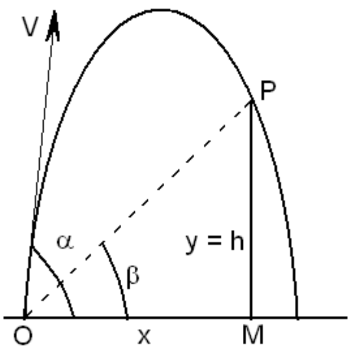
And vertical distance travelled =[$u\sin\alpha_1$]t – (1/2)gt^2

=[39(5/13)](0.5) – [0.5(9.8)$(0.5)^2$] = 6.275 m

Hence particle will collide at height of 6.275 m and from 18 m from point A

Q7) A particle is projected with velocity v from a point at ground level. Show that it cannot clear a wall of height h at a distance x from the point of projection if

$v^2 \geq g\left(h+\sqrt{x^2+h^2}\right)$



Solution:

From the equation for path of projectile

$$y = h = x\tan\alpha - \frac{gx^2}{2v^2}\frac{1}{\cos^2\alpha}$$

$$h = x\tan\alpha - \frac{gx^2}{2v^2}(1+\tan^2\alpha)$$

This is a quadratic equation in xtanα which is real. Therefore the discriminate cannot be negative.

$$\frac{gx^2}{2v^2}tan^2\alpha - xtan\alpha + \frac{gx^2}{2v^2} + h = 0$$

Here

$$a = \frac{g^2}{2v^2}, \qquad b = 1, \qquad c = \frac{gx^2}{2v^2} + h$$

Now Discriminate D = b^2-4ac ≥ 0

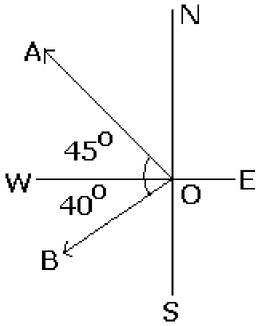
$$1 - 4\left(\frac{g}{2v^2}\right)\left(\frac{gx^2 + 2v^2h}{2v^2}\right) \geq 0$$

$$1 \geq 4\left(\frac{g}{2v^2}\right)\left(\frac{gx^2 + 2v^2h}{2v^2}\right)$$

$$v^4 \geq g^2x^2 + 2v^2hg$$

$$v^2 \geq g\left(h + \sqrt{x^2 + h^2}\right)$$

Q8) Two ships leave a port at the same time. The first steams north-west at 32 km/hr and the second 40^O south of west at 24 km/hr. After what time will they be 160 km apart?



Solution

As show in figure ship goes with 32 km/hr and ship goes with 24 km/hr

We will find relative velocity of Ship A with respect to B so we will get at what speed ships are going away say V_R

$$\vec{V}_R = \vec{V}_A - \vec{V}_B$$

$$V_R^2 = V_A^2 + V_B^2 - 2V_AV_BCos85$$

$$V_R = \sqrt{1024 + 576 - 2(32)(24)(0.0872)}$$

$$V_R = 38.3\frac{km}{hr}$$

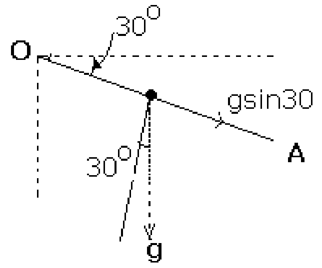
Let t be the time for ships to become 160km apart

38.3×t=160 ∴ t= 4.18 hr

Q9) A object slides off a roof inclined at an angle of 30^O to the horizontal for 2.45m. How far horizontally does it travel in reaching the ground 24.5 m below

Solution

As shown in figure component of gravitational acceleration along the inclined roof is, a=gsinθ = 9.8×(1/2) = 4.9 m/s^2



$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2\times4.9\times2.45 \Rightarrow v = 4.9m/s$

Now horizontal component of velocity which causes horizontal displacement.

$$V_h = Vcos30 = 4.9 \times \frac{\sqrt{3}}{2}$$

Vertical component $V_v = Vsin30 = 4.9 \times \frac{1}{2} = 2.45\ m/s$

Vertical displacement $h = ut + \frac{1}{2}at^2$

$$24.5 = 2.45 \times t + \frac{1}{2} \times 9.8 \times t^2$$

$$2t^2+t-10 = 0$$

Roots are t = 2.5s and -2s

Neglecting negative time

$$Horizontal\ displacement = 2.5 \times 4.9 \times \frac{\sqrt{3}}{2} = 10.6\ m$$

Q10) Two seconds after the projection, a projectile is moving in a direction at 30° to the horizontal, after one more second, it is moving horizontally. Determine the magnitude and direction of the initial velocity.

Solution

Given that in two second angle changed to zero from 30°.

Let y component of velocity at angle 30^O by v'_y and final velocity = zero.

$0 = V'_y - gt$

$V'_y = g$

Now if V' is the velocity at 30°, then $V'_y = V'\sin30$

∴ g= V' (1/2) or V' = 2g

And x component which remains unchanged = vcos30

$$V_x = 2g \times \frac{\sqrt{3}}{2} = g\sqrt{3}$$

Now in 3 seconds after projection velocity along y axis becomes zero

$0 = V_y - gt$ but x component remains unchanged

$V_y = 3g$

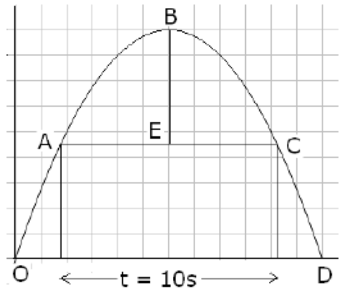
$$V = V_x^2 + V_y^2$$

$$V^2 = \left(g\sqrt{3}\right)^2 + (3g)^2 = 2g\sqrt{3}$$

Angle $tan\theta = \frac{V_y}{V_x} = \frac{3g\sqrt{3}}{g\sqrt{3}} = \sqrt{3}$ θ=60°

Q11) A projectile crosses half its maximum height at a certain instant of time and again 10s later. Calculate the maximum height. If the angle of projection was 30°, calculate the maximum range of the projectile as well as the horizontal distance it travelled in above 10 s.

Solution



Time taken by projectile to travel from point B to C is 5 sec.

And during this time it travels H/2 distance show by BE. Since point B is highest point no vertical component of velocity hence it is a free fall thus

$$\frac{H}{2} = \frac{1}{2} \times 9.8 \times t^2 \quad \therefore H = 245 \text{ m}$$

Angle of projection 30° given, from the formula for height

$$H = \frac{u^2 sin^2\theta}{2g}$$

$$245 = \frac{u^2 sin^2 30}{2g} = \frac{u^2}{8g}$$

u = 138.59

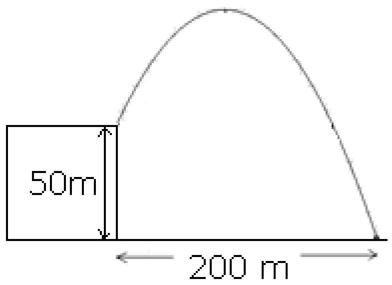
From the formula for range

$$R = \frac{u^2 sin2\theta}{2g} = 1697.4$$

In 10s horizontal distance covered by the particle = ucos30 ×10 ≈1200 m

Q12) A projectile is fired into the air from the edge of a 50-m high cliff at an angle of 30 deg above the horizontal. The projectile hits a target 200 m away from the base of the cliff. What is the initial speed of the projectile, *v*?

Solution



Let v be the velocity of projectile

Velocity along x-axis, V_x = vcosθ

Let t be the time of flight

Displacement along x-axis X, 200= (vcosθ) t

$$t = \frac{200}{vcos\theta} --- eq(1)$$

Velocity along Y-axis vsinθ

Displacement along Y-axis $50 = -(v sin\theta)t + \frac{1}{2}gt^2 - - - eq(2)$

Substituting value of t from eq(1) in eq(2) we get

$$50 = -(v sin\theta)\frac{200}{v cos\theta} + \frac{1}{2}g\left(\frac{200}{v cos\theta}\right)^2$$

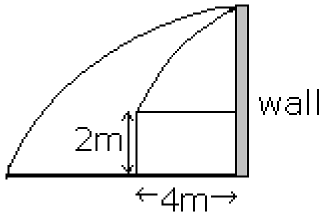
$$50 = -(tan\theta)200 + \frac{1}{2}9.8\left(\frac{200}{v cos\theta}\right)^2$$

$$50 + (tan30)200 = 4.9\left(\frac{200}{v cos30}\right)^2$$

$$v = \sqrt{\frac{4.9 \times 200^2}{cos^2 30 \times [50 + 200 tan30]}}$$

V= 39.74 m/s

Q13) A perfectly elastic ball is thrown on wall and bounces back over the head of the thrower, as shown in the figure. When it leaves the thrower's hand, the ball is 2m above the ground and 4m from the wall, and has velocity along x-axis and y-axis is same and is 10m/sec. How far behind the thrower does the ball hit the ground? [assume g = 10 m/s^2]



Solution

Displacement along x axis, 4 = 10(t), ∴ t= 0.4sec

Displacement along y-axis

$$y = 10(0.4) - \frac{1}{2}10(0.4)^2 = 3.2\ m$$

Thus ball struck the wall at height 2+3.2 = 5.2 m

Velocity along y-axis of ball when it struck the wall (use formula v=u+at) = 10-10(0.4) = 6m/s

When ball rebounds, direction of y component of velocity remains unchanged. But x component get's reversed.

Component of velocity after bounce are v_x = -10m/s and v_y = 6m/s

Angle with which it rebounds tanθ = 6/10 = 0.6, θ =35°,

With negative x-axis

Now H = 5.2, Time taken to travel H

5.2 = - 6t + ½ ×10×t^2

Root of equation t = 1.78 sec.

Distance travelled along x-axis = 1.78× 10 = 17.8 m

Ball falls behind person = 17.8 – 4 = 13.8 m

Q14) An aeroplane has to go to from a point O to another point A, 500 km away 30° east of north. A wind is blowing due north at a speed of 20 m/s. the air –speed of the plane is 150 m/s. [HC Verma, problem 49]

Solution

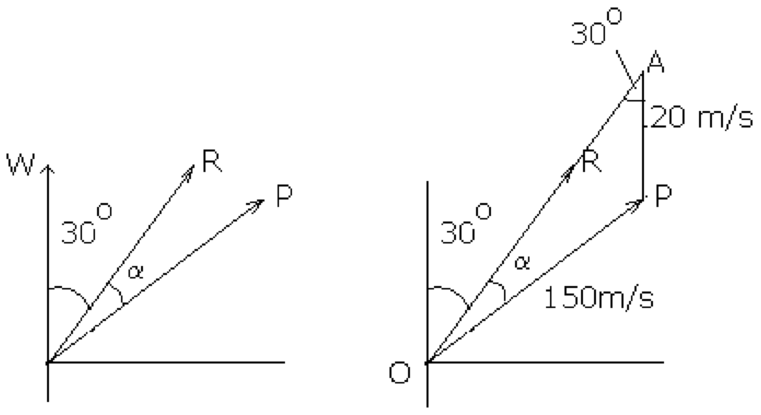


fig a fig b

In fig(a) Vector W represents direction of velocity of wind. Vector P represents direction of plane and vector R represents position where plane has to go.

In fig b. Wind velocity vector W is added to velocity vector of plane P. and R is resultant velocity. From the geometry of fig(b) ∠OAP = 30°.

By sine formula

$$\frac{150}{sin30} = \frac{20}{sin\alpha}$$

Sinα = 0.066 or α = 3°48′

Thus angle between Wind and Plane of velocity is 30+3° 48′ = 33°48′

Thus $R^2 = W^2 + P^2 + 2WP\cos 33°48'$

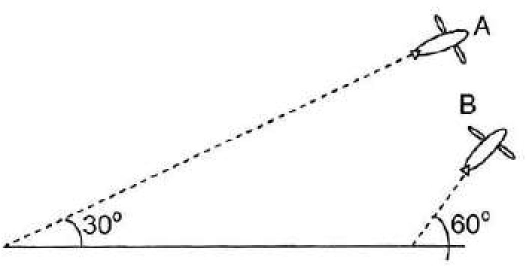
$R^2 = 20^2 + 150^2 + 2\times20\times150\ (0.8310) = 27886$

R = 167 m/s

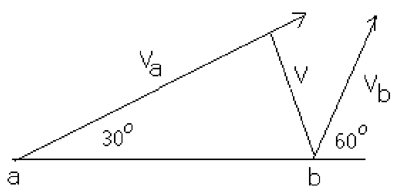
Distance is 500km time taken t

$$t = \frac{500{,}000}{167} = 2994\ s$$

Q15) Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in figure. The speed of A is $100\sqrt{3}\ ms^{-1}$. At time t = 0 s, an observer in A finds B at distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If a t = t_0, A just escapes being hit by B, t_0 in seconds is



Solution:



From the geometry of figure V = $V_a$tan30

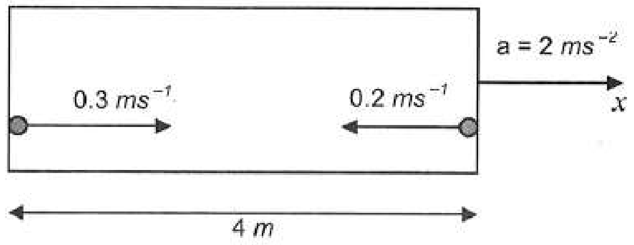
$$V = 100\sqrt{3} \times \frac{1}{\sqrt{3}} = 100m/s$$

Distance for V is 500 m

t_o = 500/100 = 5sec

Q16) A rocket is moving in a gravity free space with a constant acceleration of 2 ms^{-2} along +

x direction (see figure). The length of a chamber inside the rocket is 4 m. A ball is thrown from the left end of the chamber in + x direction with a speed of 0.3 ms^{-1} relative to the rocket. At the same time, another ball is thrown in −x direction with a speed of 0.2 ms^{-1} from its right end relative to the rocket. The time in seconds when the two balls hit each other is



Solution:

Let A have velocity 0.3m/s and B have velocity 0.2 m/s.

Ball A will face acceleration opposite to its motion as rocket is moving with positive acceleration Thus A ball will travel first in positive direction then negative direction. Ball B will travel up to the wall opposite for collision

For ball B direction of acceleration is in the direction of velocity i.e. 2m/s^2

$$4 = 0.2 \times t + \frac{1}{2} \times 2 \times t^2$$

Root is t = 1.99 ≈ 2s

Thus balls will collide after 2sec

Q17) A coin is drop in the elevator from height of 2m. Elevator accelerating down with certain acceleration. Coin takes 0.8 sec. find acceleration of elevator

Solution:

Since elevator is going down with positive acceleration.

Resultant acceleration of coin = g-a,

$$2 = 0 + \frac{1}{2}(g - a)t^2$$

$$2 = 0 + \frac{1}{2}(9.8 - a)(0.8)^2$$

a = 3.55 m/s^2

Q18) A particle starts its motion at time t = 0 from the origin with velocity 10**j** m/s and moves in X-Y plane with constant acceleration 8**i**+2**j**

(a) At what time its x-coordinate becomes 16m? And at that time what will be its y co-ordinate?

(b) What will be the speed of this particle at this time?

Solution:

From equation of motion

$$\vec{r} = \vec{r}_0 + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$x\hat{\imath} + y\hat{\jmath} = (10\hat{\jmath})t + \frac{1}{2}(8\hat{\imath} + 2\hat{\jmath})t^2$$

$$x\hat{\imath} + y\hat{\jmath} = (4t^2)\hat{\imath} + (10t + t^2)\hat{\jmath}$$

Thus

$$x = 4t^2$$

When x = 16

16 = 4t^2 , t = 2 sec

Now y = 10t+t^2 at t = 2 sec

Y = 20+4 = 240 m

b)

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{v} = 10\hat{\jmath} + (8\hat{\imath} + 2\hat{\jmath})t$$

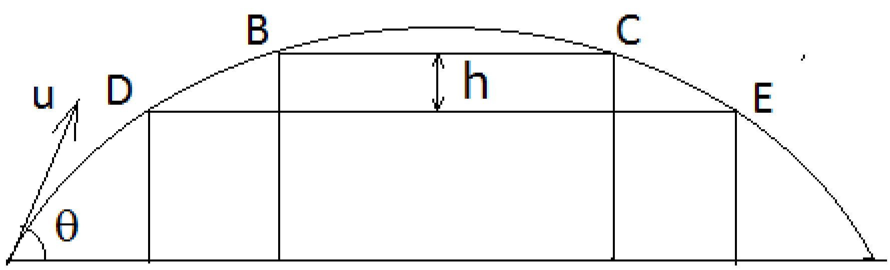
t =2 sec

$$\vec{v} = 10\hat{\jmath} + (8\hat{\imath} + 2\hat{\jmath}) \times 2$$

$$\vec{v} = 16\hat{\imath} + 14\hat{\jmath}$$

$$\overrightarrow{|v|} = \sqrt{16^2 + 14^2} = 21.26\ sec$$

Q19) A particle is projected with certain velocity as shown in figure.



If the time interval required to cross the point B and C is t_1

And the time interval required to cross D and E t_2

Find $t_2^2 - t_1^2$

Solution : We know that any projectile passes through same height for two time. Velocity of projectile along Y axis vertical is usinθ

Let point B and C be at height h_1. Thus time difference can be calculated as follows by forming equation of motion

$$h_1 = usin\theta t - \frac{1}{2}gt_1^2$$

$$gt_1^2 - 2usin\theta t + 2h_1 = 0$$

Above equation is quadratic equation having two real roots which gives time require to cross same height while going up which will have smaller value than time required to pass same height while going down

In this equation a = g , b = 2usinθ and c = 2h$_1$

Discriminate $D = \sqrt{b^2 - 4ac}$

On substituting values of a,b c in above equation

$$\sqrt{D} = \sqrt{4u^2 sin\theta^2 - 8gh_1}$$

If α and β are the two roots then difference in two roots will give us the time difference between time taken by projectile to cross the same height

$$\alpha - \beta = \frac{\sqrt{D}}{a} = \frac{\sqrt{4u^2 sin\theta^2 - 8gh_1}}{g} = \frac{2\sqrt{u^2 sin\theta^2 - 2gh_1}}{g}$$

Thus

$$t_1 = \frac{2\sqrt{u^2 sin\theta^2 - 2gh_1}}{g}$$

Similarly

$$t_2 = \frac{2\sqrt{u^2 sin\theta^2 - 2gh_2}}{g}$$

$$t_2^2 - t_1^2 = \frac{4(u^2 sin\theta^2 - 2gh_2)}{g^2} - \frac{4(u^2 sin\theta^2 - 2gh_1)}{g^2}$$

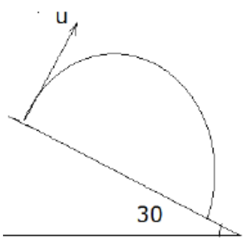
$$t_2^2 - t_1^2 = \frac{4}{g^2}(2gh_1 - 2gh_2)$$

$$t_2^2 - t_1^2 = \frac{8}{g}(h_1 - h_2)$$

Now h_1-h_2 = h as shown in figure

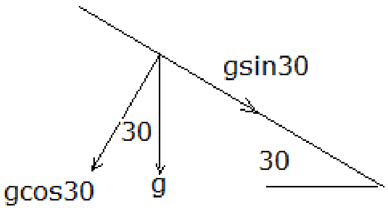
$$t_2^2 - t_1^2 = \frac{8h}{g}$$

Q20) A ball is projected from a point A with velocity 10m/s. Perpendicular to the inclined plane as shown in figure. Find the range of the ball on inclined plane



Solution:

We will turn the inclined plane and make it horizontal, and solve the problem in terms of regular projectile problem. Now gravitational acceleration will be, From figure component of gravitational acceleration perpendicular to x-axis is gcos30 and acceleration parallel to x-axis will be gsinθ



When we turn the plane then component of velocity along the plane will be zero

Now when projectile reaches the ground displacement along vertical is zero

$$0 = 10t - \frac{1}{2}gcos30t^2$$

$$t = \frac{40}{g\sqrt{3}} = \frac{40}{10\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Thus time of flight of projectile is 4/√3

Now gravitation have component along slope is gsin30 as show in figure

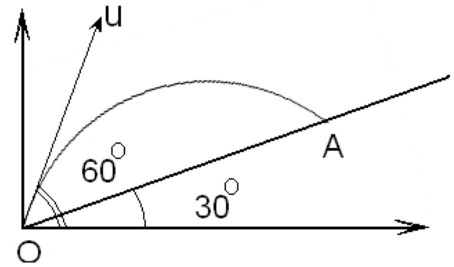
Now displacement along the slope = 0 +1/2gsinθt^2

$$R = ut + \frac{1}{2}gsin30t^2$$

$$R = 0 + \frac{1}{2}gsin30\frac{16}{3}$$

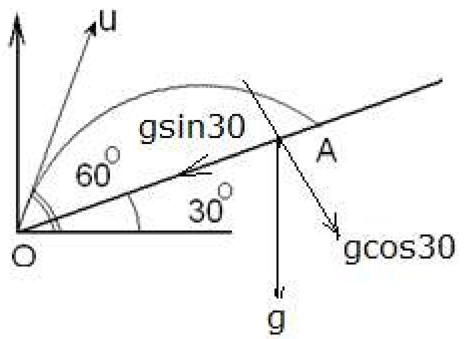
If g=10 then R= 40/3 m

Q21) A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is α =30° and the angle of barrel to the horizontal is β = 60°. The initial velocity of shell is 21 m/s. Find the distance from gun to the point at which the shell falls



Solution:

Asked to find OA



Note that projectile fired with angle of 30° with slope. Now if we rotate the plane then

Velocity along x-axis is ucos30 and velocity along y axis is usin30

Component of gravitational acceleration perpendicular to x axis is gcos30 and parallel will be gsin30

This time of flight. Displacement along y-axis is zero

$$0 = usin30t - \frac{1}{2}gcos30t^2$$

$$t = \frac{2u}{g\sqrt{3}}$$

Displacement along x axis Range

$$R = ucos30t - \frac{1}{2}gsin30t^2$$

$$R = u\frac{\sqrt{3}}{2}\frac{2u}{g\sqrt{3}} - \frac{1}{2}g\frac{1}{2}\frac{4u^2}{3g^2}$$

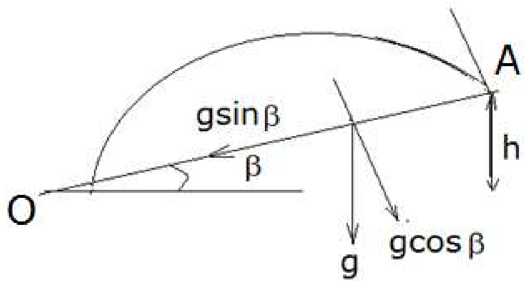
$$R = \frac{u^2}{g} - \frac{u^2}{3g}$$

$$R = \frac{u^2}{g}\left(1 - \frac{1}{3}\right) = \frac{2u^2}{3g}$$

$$R = \frac{2(21^2)}{3 \times 9.8} = 30\, m$$

Q22) A particle is projected with velocity u strikes at right angle a plane inclined at angle β with horizontal . find the height of the strike point height

Solution



As show in figure let object strike at point height h, initial velocity be u and angle of projection be θ with horizontal

Now gravitational acceleration perpendicular to inclined to plane.

If we rotate the inclined plane make it horizontal then angle of projection will be (θ – β)

Component of velocity along x aixs will be ucos(θ – β) and along y axis is usin (θ – β)

Gravitational acceleration perpendicular to plane is gcosβ and parallel to plane is gsinβ

Particle strike at 90^O with horizontal . i.e velocity along x axis is zero, and displacement along vertical is zero. Thus we will form two equations

At the end of time of flight t velocity along x-axis is zero form equation of motion

V = u – at

$$0 = ucos(\theta - \beta) - gsin\beta t$$

$$t = \frac{ucos(\theta - \beta)}{gsin\beta} - - - eq(1)$$

Displacement along x-axis That is OA or range

$$R = ucos(\theta - \beta)t - \frac{1}{2}gsin\beta t^2$$

Substituting value of t from equation 1 we get

$$R = ucos(\theta - \beta)\frac{ucos(\theta - \beta)}{gsin\beta} - \frac{1}{2}gsin\beta\ \frac{u^2cos^2(\theta - \beta)}{g^2sin^2\beta}$$

$$R = \frac{u^2cos^2(\theta - \beta)}{gsin\beta} - \frac{u^2cos^2(\theta - \beta)}{2gsin\beta}$$

$$R = \frac{1}{2}\frac{u^2cos^2(\theta - \beta)}{gsin\beta} - - - eq(2)$$

Since θ is assumed must be eliminated by following equation

Displacement along Y axis zero

$$0 = usin(\theta - \beta)t - \frac{1}{2}gcos\beta t^2$$

$$t = \frac{2usin(\theta - \beta)}{gsin\beta} - - - eq(3)$$

From equation 1 and 3 we get

$$\frac{2usin(\theta - \beta)}{gsin\beta} = \frac{ucos(\theta - \beta)}{gsin\beta}$$

$$tan(\theta - \beta) =$$

$$0 = ucos(\theta - \beta) - 2usin(\theta - \beta)$$

$$tan(\theta - \beta) = \frac{1}{2}cot\beta - - - eq(4)$$

From trigonometric identity

$$sec^2(\theta - \beta) = 1 + tan^2(\theta - \beta)$$

From equation 4

$$sec^2(\theta - \beta) = 1 + \frac{1}{4}cot^2\beta$$

As $\sec\theta = 1/\cos\theta$ thus

$$cos^2(\theta - \beta) = \frac{4}{4 + cot^2\beta} - - - eq(5)$$

Substituting equation 5 in equation 3 for range we get

$$R = \frac{1}{2}\frac{u^2 4}{gsin\beta(4 + cot^2\beta)}$$

$$R = \frac{2u^2}{gsin\beta(4 + cot^2\beta)}$$

Now from figure h = OAsinβ

$$h = \frac{2u^2}{gsin\beta(4 + cot^2\beta)}sin\beta = \frac{2u^2}{g(4 + cot^2\beta)}$$

Cot may be converted in terms of sine and cos using identity

Q23) To a man walking on straight path at speed of 3km/hr rain appears to fall vertically now man increased its speed to 6km/hr, it appears that rain is falling at 45^o with vertical. Find the speed of rain Ans 3√2

Solution

It is not given the angle of rain let it be θ with vertical.

V_{rm} = Velocity of rain with respect to man

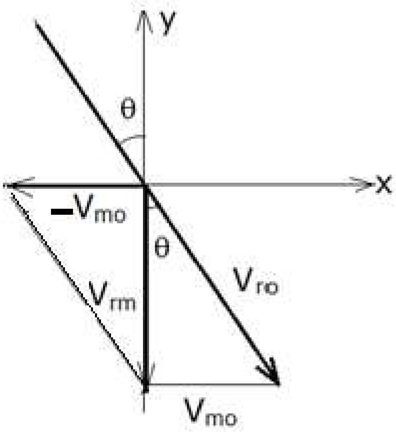
V_{ro} = Velocity of man with respect to observer

V_{mo}= Velocity of man with respect to observer

Now

$V_{rm} = V_{ro} + V_{om}$ or $V_{rm} = V_{ro} - V_{mo}$. Now refer to vector diagram

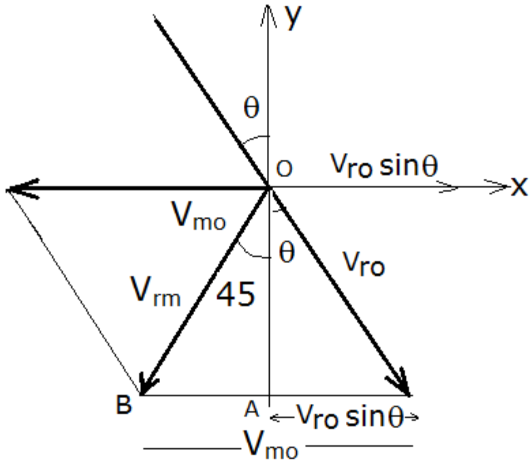
Case I When speed of man V_{mo} =3 km/hr



From figure $\sin\theta = V_{mo}/V_{rm} = 3/V_{rm}$

Or $V_{rm}\sin\theta = 3$ eq(1)

CaseII When speed of man V_{mo} = 6km/hr



From adjacent figure Resolution of V_{ro} along X-axis and Y axis are $V_{ro}\sin\theta$ and $V_{ro}\cos\theta$

Also note that BA = V_{mo} - $V_{ro}\sin\theta$

BA = 6 - $V_{ro}\sin\theta$ And OA = $V_{ro}\cos\theta$

$$tan45 = \frac{BA}{OA} = \frac{6 - V_{ro}sin\theta}{V_{ro}cos\theta}$$

As tan45 = 1 ∴ $V_{ro}\cos\theta = 6 - V_{ro}\sin\theta$

from caseI equation we know that $V_{ro}\sin\theta = 3$

$V_{ro}\cos\theta = 6\text{-}3 = 3$ eq(2). By squaring and adding equation 1 and 2 we get $V_{ro} = 3\sqrt{2}$

Q24) Two cars C and D moving in the same direction on a straight road with velocity 12m/s and 10m/s respectively. When the separation in then was 200m. D starts accelerating so as to avoid accident.

Solution

Relative velocity of C with respect to D = $V_c - V_D$ = 12-10 = 2m/s or u =2m/s

Initial separation or distance is 200m

To avoid accident velocity of D should increased to 12m/s, so that relative velocity of C with respect to D become zero or v = 0

Relative acceleration of C with respect to D a_r = 0 – a = -a

Using formula

$v^2 = u^2 + 2a_r s$ we get

0 = 4 - 2a(200)

a_r = 0.1 m/s

Q25) A ball rolls off the top of a staircase with a horizontal velocity u m/s. if the steps are h m high and w m wide the ball will hit the edge of which step.

Solution:

By forming two individual equation in one dimension and by connecting them through time we will get necessary equation for n

Let ball hit the n^{th} step, there no initial component of velocity along vertical direction, as ball rolls in horizontal direction

Horizontal displacement =nw eq(1)

Horizontal displacement=ut eq(2)

from eq(1) and eq(2) we get ut=nw

t=nw/u eq(3)

Vertical displacement=nh eq(4)

vertical displacement=½ (g) t^2

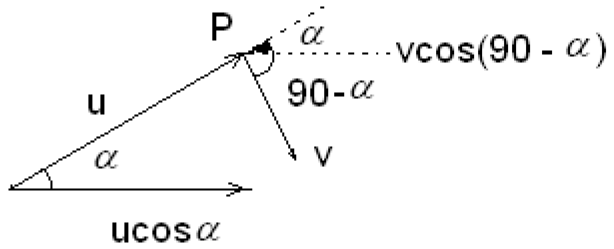
nh=½ (g) t^2 eq(5)

substituting value of time t from eq(3) in eq(5) we get

nh=½ (g) $(nw/u)^2$

$$n = \frac{2hu^2}{gw^2}$$

Q26) A particle is projected from a point O with velocity u in a direction making an angle α upward with the horizontal. At P, it is moving at right angle to its initial direction of projection, What will be velocity at P.
Solution



As shown in figure component of u along x-axis is ucosα

At point P it changes its direction by 90° let velocity be v

Then component of v along x-axis is vcos(90-α) = vsinα

Thus ucosα = vsinα

v = ucotα

LAWS OF MOTION

Frame of reference

A “frame of reference” is just a set of coordinates: something you use to measure the things that matter in Newtonian problems, that is to say, positions and velocities, so we also need a clock.
Or A place and situation from where an observer takes his observation is called frame of reference.
A point in space is specified by its three coordinates (x, y, z) and an “event” like, say, a little explosion, by a place and time: (x, y, z, t).

An inertial frame is defined as one in which Newton’s law of inertia holds—that is, anybody which isn’t being acted on by an outside force stays at rest if it is initially at rest, or continues to move at a constant velocity if that’s what it was doing to begin with.
Example of inertial frame of reference is observer on Earth for all motion on surface of earth. Car moving with constant velocity
An example of a non-inertial frame is a rotating frame, such as an accelerating car,

Accelerated frame of reference is defined as one in which Newton’s law of inertia does not hold good. Example When bus starts suddenly from rest we experience backward jerks, although no force is acted on us.
Any frame of reference which is moving with acceleration are called accelerated frame of reference.

Newton’s first law of motion

If a body is observed from an inertial frame which is at rest or moving with uniform velocity then it will remain at rest or continue to move with uniform velocity until an external force is applied on it
The property due to which a body remains or continues its motion with uniform velocity is called as inertia
Force is a push or pull that disturbs or tends to disturb inertia of rest or inertia of uniform motion with uniform velocity of a body.
Hence first law of motion defines inertia, force and inertial frame of reference.
The inertia is of three types
(i) Inertia of rest
It is the inability of the body to change its state of rest by itself.
Examples
(a) A person standing in a bus falls backward when the bus suddenly starts moving. This is because, the person who is initially at rest continues to be at rest even after the bus has started moving.

(b) A book lying on the table will remain at rest, until it is moved by some external agencies.
(c) When a carpet is beaten by a stick, the dust particles fall off vertically downwards once they are released and do not move along the carpet and fall off.

(ii) Inertia of motion
Inertia of motion is the inability of the body to change its state of motion by itself.
Examples
(a) When a passenger gets down from a moving bus, he falls down in the direction of the motion of the bus.
(b) A passenger sitting in a moving car falls forward, when the car stops suddenly.
(c) An athlete running in a race will continue to run even after reaching the finishing point

(iii) Inertia of direction.
It is the inability of the body to change its direction of motion by itself.
Examples
When a bus moving along a straight line takes a turn to the right, the passengers are thrown towards left. This is due to inertia which makes the passengers travel along the same straight line, even though the bus has turned towards the right.

Force

From the first law, we infer that to change the state of rest or uniform motion, an external agency called, the force is required.
Force is defined as that which when acting on a body changes or tends to change the state of rest or of uniform motion of the body along a straight line.

A force is a push or pull upon an object, resulting in the change of state of a body. Whenever there is an interaction between two objects, there is force acting on each other. When the interaction ceases, the two objects no longer experience a force. Forces exist only as a result of an interaction.

There are two broad categories of forces between the objects, contact forces and non–contact forces resulting from action at a distance.
Contact forces are forces in which the two interacting objects are physically in contact with each other.
Tensional force, normal force, force due to air resistance, applied forces and frictional forces are examples of contact forces.
Action-at-a-distance forces (non- contact forces) are forces in which the two interacting objects are not in physical contact which each other, but are able to exert a push or pull despite the physical separation.
Gravitational force, electrical force and magnetic force are examples of non- contact forces.

Solved Numerical

Q) Two forces **F_1** and **F_2** , mutually perpendicular, acts on a 5.0 kg mass. If F_1 = 20 N and F_2 = 15N, find the acceleration and direction of resultant force

Solution

Force is vector thus

F = **F_1** + **F_2** angle between force F_1 and F_2 is 90 given

Magnitude of F

$$\left|\vec{F}\right| = \sqrt{F_1^2 + F_2^2 + F_1 F_2 cos90}$$

$$F = \sqrt{(20)^2 + (15)^2} = 25N$$

Direction of force

$$tan\alpha = \frac{F_2 sin\theta}{F_1 + F_1 cos\theta}$$

$$tan\alpha = \frac{15}{20} \Rightarrow \alpha = 37^o$$

Direction of resultant force makes an angle of 37^O with direction of **F_1**

Acceleration = F/m = 25/5 = 5 m/s^2

Momentum of a body

It is observed experimentally that the force required to stop a moving object depends on two factors: (i) mass of the body and (ii) its velocity

A body in motion has momentum. *The momentum of a body is defined as the product of its mass and velocity.* If *m* is the mass of the body and *v* its velocity, the linear momentum of the body is given by

$$\vec{p} = m\vec{v}$$

Momentum has both magnitude and direction and it is, therefore, a vector quantity. The momentum is measured in terms of *kg m s*$^{-1}$ and its dimensional formula is MLT^{-1}.

When a force acts on a body, its velocity changes, consequently, its momentum also changes. The slowly moving bodies have smaller momentum than fast moving bodies of same mass. If two bodies of unequal masses and velocities have same momentum, then,

$$\vec{p}_1 = \vec{p}_2$$

$$m_1\vec{v}_1 = m_2\vec{v}_2$$

$$\frac{m_1}{m_2} = \frac{\vec{v}_2}{\vec{v}_1}$$

Newton's second law of motion

Newton's second law of motion deals with the behaviour of objects on which all existing forces are not balanced.

According to this law, *the rate of change of momentum of a body is directly proportional to the external force applied on it and the change in momentum takes place in the direction of the force.*

If **p** is the momentum of a body and ***F*** the external force acting on it, then according to Newton's second law of motion

$$\vec{F} \propto \frac{d\vec{p}}{dt}$$

Or

$$\vec{F} = k\frac{d\vec{p}}{dt}$$

Unit of force is chosen in such a manner that the constant *k* is equal to unity. (i.e) *k* =1.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

If a body of mass *m* is moving with a velocity ***v*** then, its momentum is given by

p = m**v**

$$\therefore \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Here **a** is the acceleration produced in the body given by a=dv/dt

The force acting on a body is measured by the product of mass of the body and acceleration produced by the force acting on the body.

The second law of motion gives us a measure of the force. The acceleration produced in the body depends upon the inertia of the body (i.e) greater the inertia, lesser the acceleration.

One Newton is defined as that force which, when acting on unit mass produces unit acceleration. Force is a vector quantity. The unit of force is *kg m* s^{-2} or *Newton*. Its dimensional formula is MLT^{-2}.

Impulsive force and Impulse of a force

(i) Impulsive Force

An impulsive force is very great force acting for a very short time on a body, so that the change in the position of the body during the time the force acts on it may be neglected.

(e.g.) The blow of a hammer, the collision of two billiard balls etc.

(ii) Impulse of a force

The impulse J of a constant force F acting for a time t is defined as the product of the force and time.

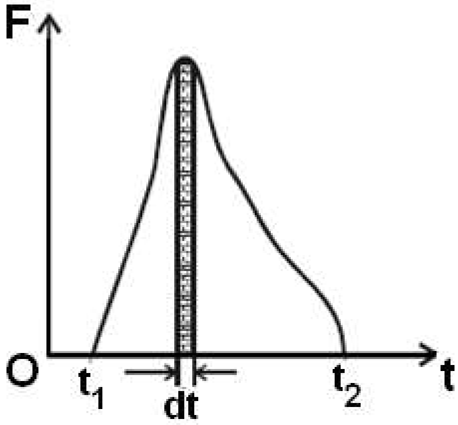
(i.e) Impulse = Force × time

J = F × t

The impulse of force *F* acting over a time interval *t* is defined by the integral,

$$J = \int_0^t F dt \qquad -----(1)$$

The impulse of a force, therefore can be visualized as the area under the force versus time graph as shown in Fig.



When a variable force acting for a short interval of time, then the impulse can be measured as,

$J = F_{average} \times dt$

Impulse of a force is a vector quantity and its unit is N s.

Principle of impulse and momentum

By Newton's second law of motion, the force acting on a body = *m a* where *m* = mass of the body and *a* = acceleration produced

The impulse of the force = $F \times t = (m\,a)\,t$

If *u* and *v* be the initial and final velocities of the body then, a = (v-u)/t

Therefore, impulse of the force =

$$J = m \times \frac{(v-u)}{t} t = m(v-u) = mv - mu$$

Impulse = final momentum of the body – initial momentum of the body.

(*i.e*) Impulse of the force = Change in momentum

The above equation shows that *the total change in the momentum of a body during a time interval is equal to the impulse of the force acting during the same interval of time. This is called principle of impulse and momentum*

Examples

(i) A cricket player while catching a ball lowers his hands in the direction of the ball.

If the total change in momentum is brought about in a very short interval of time, the average force is very large according to the equation,

$$F = \frac{mv - mu}{t}$$

By increasing the time interval, the average force is decreased. It is for this reason that a cricket player while catching a ball, to increase the time of contact, the player should lower his hand in the direction of the ball , so that he is not hurt.

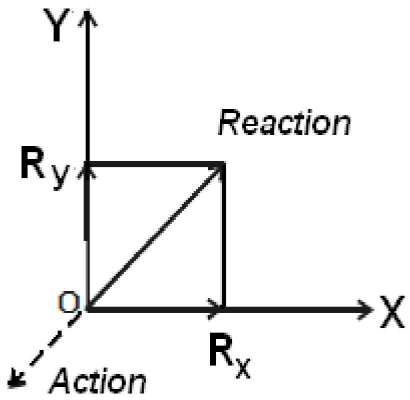
(ii) A person falling on a cemented floor gets injured more where as a person falling on a sand floor does not get hurt. For the same reason, in wrestling, high jump etc., soft ground is provided.
(iii) The vehicles are fitted with springs and shock absorbers to reduce jerks while moving on uneven or wavy roads.

Newton's third Law of motion

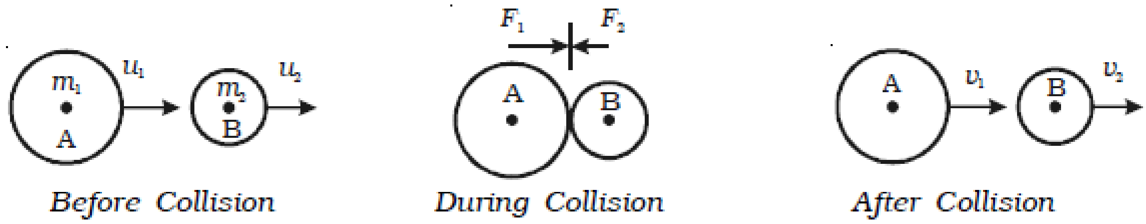
It is a common observation that when we sit on a chair, our body exerts a downward force on the chair and the chair exerts an upward force on our body.
There are two forces resulting from this interaction:
A force on the chair and a force on our body. These two forces are called action and reaction forces. Newton's third law explains the relation between these action forces.
It states that *for every action, there is an equal and opposite reaction.*
(i.e.) whenever one body exerts a certain force on a second body, the second body exerts an equal and opposite force on the first.
Newton's third law is sometimes called as the law of action and reaction. Let there be two bodies 1 and 2 exerting forces on each other.
Let the force exerted on the body 1 by the body 2 be $\mathbf{F}_{12}$ and the force exerted on the body 2 by the body 1 be $\mathbf{F}_{21}$
Then according to third law of motion $\mathbf{F}_{12} = -\mathbf{F}_{21}$
One of these forces, say $\mathbf{F}_{12}$ may be called as the action whereas the other force $\mathbf{F}_{21}$ may be called as the reaction or vice versa.
This implies that we cannot say what the cause (action) is or which the effect (reaction) is.
It is to be noted that always the action and reaction do not act on the same body; they always act on different bodies.
The action and reaction never cancel each other and the forces always exist in pair.
The effect of third law of motion can be observed in many activities in our everyday life.

The examples are
(i) When a bullet is fired from a gun with a certain force (action), there is an equal and opposite force exerted on the gun in the backward direction (reaction).
(ii) When a man jumps from a boat to the shore, the boat moves away from him. The force he exerts on the boat (action) is responsible for its motion and his motion to the shore is due to the force of reaction exerted by the boat on him.
(iii) The swimmer pushes the water in the backward direction with a certain force (action) and the water pushes the swimmer in the forward direction with an equal and opposite force (reaction).
(iv) We will not be able to walk if there were no reaction force. In order to walk, we push our foot against the ground. The Earth in turn exerts an equal and opposite force.
This force is inclined to the surface of the Earth. The vertical component of this force balances our weight and the horizontal component enables us to walk forward.
(v) A bird flies by with the help of its wings. The wings of a bird push air downwards (action). In turn, the air reacts by pushing the bird upwards (reaction).

(vi) When a force exerted directly on the wall by pushing the palm of our hand against it (action), the palm is distorted a little because, the wall exerts an equal force on the hand (reaction)



Law of conservation of momentum
From the principle of impulse and momentum, impulse of a force,
$J = mv - mu$
If $J = 0$ then $mv - mu = 0$ (or) $mv = mu$
(i.e) final momentum = initial momentum
In general, *the total momentum of the system is always a constant* (*i.e*) *when the impulse due to external forces is zero, the momentum of the system remains constant. This is known as law of conservation of momentum.*
We can prove this law, in the case of a head on collision between two bodies.
Proof
Consider a body A of mass m_1 moving with a velocity u_1 collides head on with another body B of mass m_2 moving in the same direction as A with velocity u_2 as shown in Fig



After collision, let the velocities of the bodies be changed to v_1 and v_2 respectively, and both moves in the same direction. During collision, each body experiences a force.
The force acting on one body is equal in magnitude and opposite in direction to the force acting on the other body. Both forces act for the same interval of time.
Let F_1 be force exerted by A on B (action), F_2 be force exerted by B on A (reaction) and t be the time of contact of the two bodies during collision.
Now, F_1 acting on the body B for a time t, changes its velocity from u_2 to v_2.
$\boldsymbol{F_1}$ = mass of the body B × acceleration of the body B

$$F_1 = m_2\left(\frac{v_2 - u_2}{t}\right) \quad ----eq(1)$$

Similarly, F_2 acting on the body A for the same time t changes its velocity from u_1 to v_1
$\boldsymbol{F_2}$ = mass of the body A × acceleration of the body A

$$F_2 = m_1\left(\frac{v_1 - u_1}{t}\right) \quad ----eq(2)$$

Then by Newton's third law of motion $F_1 = -F_2$

$$m_2\left(\frac{v_2 - u_2}{t}\right) = -m_1\left(\frac{v_1 - u_1}{t}\right)$$
$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

(i.e) total momentum before impact = total momentum after impact.
(i.e) total momentum of the system is a constant.
This proves the law of conservation of linear momentum

Applications of law of conservation of momentum
The following examples illustrate the law of conservation of momentum.
(i) Recoil of a gun
(ii) Explosion of a bomb
Suppose a bomb is at rest before it explodes. Its momentum is zero. When it explodes, it breaks up into many parts, each part having a particular momentum. A part flying in one direction with a certain momentum, there is another part moving in the opposite direction with the same momentum. If the bomb explodes into two equal parts, they will fly off in exactly opposite directions with the same speed, since each part has the same mass.

Solved Numerical

Q) A person of mass 60kg is standing on a raft of mass 40 kg in a lake. The distance of the person from the bank is 30m. If the person starts running towards the bank with velocity 10m/s, then what will his distance be from the bank after one second
Solution
Initial momentum of raft + man = 0
Let M = mass of man , m = mass of raft, u = velocity of man wrt raft
Let v be the velocity of raft after man runs on raft which will be negative as raft will move backward
Velocity of man wrt observer on ground = u-v
Mv + m(v-u) = 0
40v + 60(v-10) = 0
V = 6 m/s is the velocity with which raft is moving back
Thus velocity of man wrt bank = 10-6=4 m/s
Person travelled distance in 1 sec = 4 m
So person is at 30-4 = 26 m from bank

Q) Two balls, each of mass 80 g, moving towards each other with velocity $5 ms^{-1}$, collide and rebound with the same speed. What will be the impulse of force on each ball due to the other? What is the value of change in momentum of each ball?

Solution

Initial velocity of ball =- 5 ms^{-1} final velocity = $5ms^{-1}$ (as ball rebounds)

Change in velocity = final – initial velocity = 5 –(-5) = 10 m/s

Change in momentum = m(change in velocity) = 0.08×10 =0.8 kg ms^{-1}

Impulse = change in momentum = 0.8 Ns

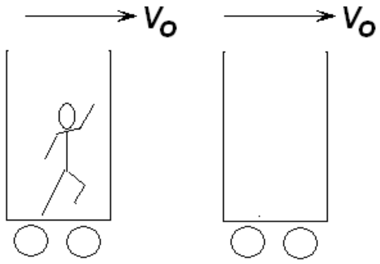
Q) Two identical boggies move one after the other due to inertia (without friction) with the same velocity v_0. A man of mass m rides the rear boggy. At a certain moment the man jumps into the front boggy with a velocity u relative to this boggy. The mass of each boggy is M. Find the velocity with which the buggies will move afterwards.

Solution

We will use law of conservation of momentum

all velocities must be with reference to stationary observer.

Let m be the mass of person . Let V_1 be the velocity of boggy as man jumps then velocity of man = (V_1 +u) w.r.t. observer then momentum of man = m(V_1+u)



Before Jumping

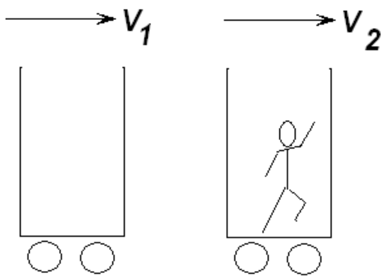
From conservation of momentum

Now momentum before jumping = momentum after jumping from boggy 1 rear boggy.

$$(M+m)V_0 = MV_1 + m(V_1 + u)$$

Note here V_0 +u is the velocity of man with respect to observer

$$V_1 = V_0 - \frac{m}{M+m}u$$



After Jumping

Initial momentum of boggy 2 front boggy = MV_0

From conservation of momentum

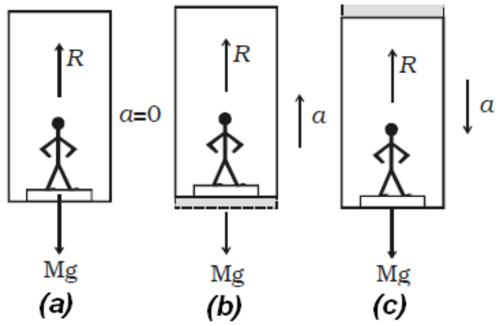
$MV_0 + m(V_1+u) = (M+m)V_2$

$$V_2 = V_0 + \frac{mM}{(M+m)^2}u$$

Applications of Newton's third law of motion

(i) Apparent loss of weight in a lift

Let us consider a man of mass *M* standing on a weighing machine placed inside a lift. The actual weight of the man = *Mg.* This weight (action) is measured by the weighing machine and in turn, the machine offers a reaction R. This reaction offered by the surface of contact on the man is the apparent weight of the man.



Case (i)

When the lift is at rest:

The acceleration of the man = 0

Therefore, net force acting on the man = 0

From figure(a), $R - Mg = 0$ (or) $R = Mg$

That is, the apparent weight of the man is equal to the actual weight.

Case (ii)

When the lift is moving uniformly in the upward or downward direction:

For uniform motion, the acceleration of the man is zero. Hence, in this case also the apparent weight of the man is equal to the actual weight.

Case (iii)

When the lift is accelerating upwards:

If a be the upward acceleration of the man in the lift, then the net upward force on the man is $F = Ma$

From Fig the net force

$F = R - Mg = Ma$ (or) $R = M(g + a)$

Therefore, apparent weight of the man is greater than actual weight.

Case (iv)

When the lift is accelerating downwards:

Let a be the downward acceleration of the man in the lift, then the net downward force on the man is $F = Ma$

From Fig. c, the net force

$F = Mg - R = Ma$ (or) $R = M(g - a)$

Therefore, apparent weight of the man is less than the actual weight.

When the downward acceleration of the man is equal to the acceleration due to the gravity of earth, (i.e) $a = g$

$\therefore R = M(g - g) = 0$

Hence, the apparent weight of the man becomes zero. This is known as the weightlessness of the body.

(ii) Working of a rocket and jet plane

The propulsion of a rocket is one of the most interesting examples of Newton's third law of motion and the law of conservation of momentum.

The rocket is a system whose mass varies with time. In a rocket, the gases at high temperature and pressure, produced by the combustion of the fuel, are ejected from a nozzle. The reaction of the escaping gases provides the necessary thrust for the launching and flight of the rocket.

From the law of conservation of linear momentum, the momentum of the escaping gases must be equal to the momentum gained by the rocket. Consequently, the rocket is propelled in the forward direction opposite to the direction of the jet of escaping gases. Due to the thrust imparted to the rocket, its velocity and acceleration will keep on increasing. The mass of the rocket and the fuel system keeps on decreasing due to the escaping mass of gases.

Let m be the mass of rocket at time t

V_{RO} =be the velocity of rocket with respect to observer

dm =be the mass of fule burnt in time dt .

V_{GR} = Ejected gas has velocity u relative to rocket

V_{GO} = denote the velocity of gases with respect to observer.

Now according to law of conservation of momentum

$mV_{RO} = (m - dm)(V_{RO} + dv) - dmV_{GO}$

$dmV_{RO} = mdV_{RO} - dmV_{GO}$

$mdV_{RO} = dm(V_{RO} - V_{GO})$

$mdV_{RO} = dm(V_{RO} + V_{OG})$

$mdV_{RO} = dm\, V_{RG}$

$mdV_{RO} = dm\,(-V_{GR})$

$$-V_{GR}\frac{dm}{m} = dV_{RO}$$

Integrating on both sides

$$-V_{GR}\int_{m_0}^{m}\frac{dm}{m} = \int_{v_0}^{v} dV_{RO}$$

$$-V_{GR}[lnm]_{m_0}^{m} = [V]_{v_0}^{v}$$

$$-V_{GR}ln\left(\frac{m}{m_0}\right) = v - v_0$$

$$V_{GR}ln\left(\frac{m_0}{m}\right) = v - v_0$$

If initial velocity $v_o = 0$ the

$$v = V_{GR}ln\left(\frac{m_0}{m}\right)$$

$$v = 2.303V_{GR}log\left(\frac{m_0}{m}\right)$$

Solved Numerical

Q) The mass of a rocket is 2.8×10^6 kg, at launch time of this 2×10^6 kg is fuel. The exhaust speed is 2500 m/s and the fuel is ejected at the rate of 1.4×10^4 kg/s

(a) Find thrust on the rocket

(b) What is initial acceleration at launch time? Ignore air resistance

Solution:

(a) The magnitude of thrust is given by

(b) $F_{thrust} = \frac{dM}{dt}V$

The direction of thrust will be opposite to the direction of relative velocity of gas as mass is decreasing i.e. upward

= $(1.4\times10^4$ kg/s) (2500 m/s) = 3.5×10^7 N

(c) To find the acceleration, we can use

$F_{External} + F_{trust} = Ma$

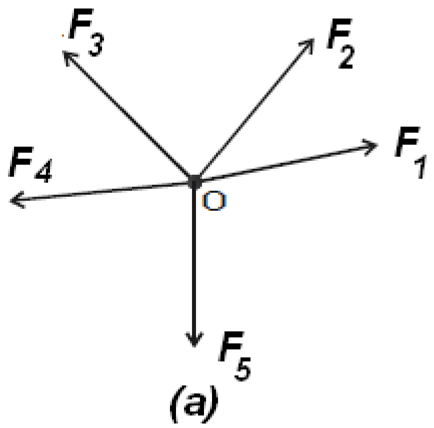
Here external force = weight acting downward and thrust is upward

$-Mg + T_{thrust} = Ma$

$a = - g + F_{thrust}/ M$

$$a = \left(-9.8 + \frac{3.5\times10^7}{2.8\times10^6}\right) = 2.7\ ms^{-2}$$

Concurrent forces and Coplanar forces



Force is a vector quantity and can be combined according to the rules of vector algebra. A force can be graphically represented by a straight line with an arrow, in which the length of the line is proportional to the magnitude of the force and the arrowhead indicates its direction.

A force system is said to be concurrent, if the lines of all forces intersect at a common point (Figure a).

A force system is said to be coplanar, if the lines of the action of all forces lie in one plane.

Equilibrium of concurrent forces

The condition, in which the resultant (net) force of all the external forces acting on a particle is zero, is called equilibrium.

Thus the steady state and the state of motion with uniform velocity of the body are both equilibrium states

Thus for equilibrium state

$$\sum \vec{F} = 0$$

If more than one force is acting on the object then for equilibrium , their vector addition must be zero

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \cdots = 0$$

Since force is a vector quantity the sum of the corresponding components of the force should also become zero

$$\sum F_x = 0\,, \quad \sum F_y = 0 \quad \sum F_z = 0$$

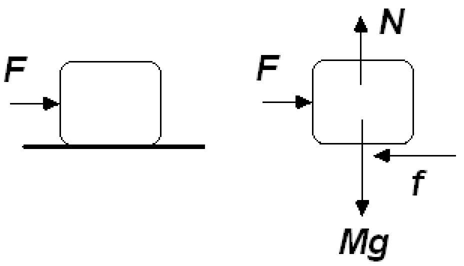
For particle remaining in equilibrium, under the effect of three forces **F_1, F_2, F_3,** the vector sum of the two forces (**$F_1 + F_2$**) has the magnitude equal to that of F_3 but direction opposite to it so that

$F_1 + F_2 + F_3 = 0$

$F_1 + F_2 = - F_3$

Friction

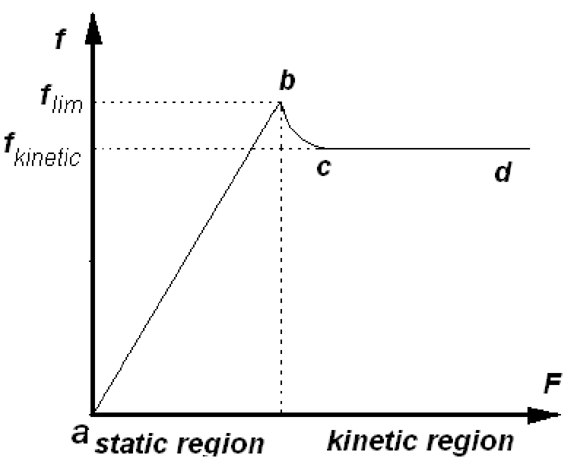
Friction plays a dual role in our life. It impedes the motion of an object, causes abrasion and wear and converts other form of energy in to heat. On the other hand, without it we could not walk, drive cars, climb rope or use nails. Friction is contact force that opposes the relative motion or tendency of relative motion of two bodies.



Consider a block on a horizontal table as shown in figure. If we apply a force, acting to the right, the block remains stationary if F is not too large. The force that counteracts F and keep the block from moving is called frictional force.

If we keep on increasing the force, the block will remain at rest and for a particular value of the applied force, the body comes to a state of about to move. Now if we slightly increase the force from this value, the block starts its motion with a jerk and we observe that to keep the block moving we need less effort than to start its motion.

So from this observation, we see that we have three states of block, first block does not move, second block is about to move and third block starts moving. The friction force acting in three states are called static frictional force, limiting frictional force and kinetic frictional force respectively. If we draw the graph between applied force and frictional force for the observation its nature is shown in figure.



Static frictional force

When there is no relative motion between the contact surfaces, frictional force is called static frictional force. It is a self-adjusting force, it adjusts its value according to requirement (of no relative motion)

Limiting frictional force

This frictional force acts when body is about to move. This is the maximum frictional force that can exist at the contact surface. We calculate its value using laws of friction

Laws of friction

(i) The magnitude of limiting frictional force is proportional to the normal force at the contact surface

$f_{lim} \propto N \Longrightarrow f_{lim} = \mu_S N$

Here μ_S is constant, the value of which depends on the nature of surface in contact and is called as ' coefficient of static friction'. Typical values of μ range from 0.05 to 1.5

(ii) The magnitude of limiting frictional force is independent of area of contact between the surfaces

Kinetic frictional force

One relative motion starts between the surface in contact, the frictional force is called as kinetic frictional force. The magnitude of kinetic force is also proportional to normal force.

$f_K = \mu_K N$

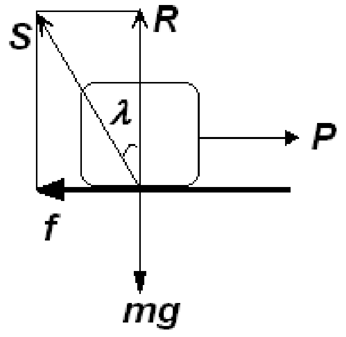
From the previous observation we can say that $\mu_K < \mu_S$

Although the coefficient of kinetic friction varies with speed, we shall neglect any variation. i.e. once relative motion stats, a constant frictional force starts opposing its motion

Angle of friction

The resultant of normal reaction R and the frictional force f is S which makes an angle λ with R. Now tan λ = f/R = μR/R = μ

The angle λ is called the angle of friction



Solved Numerical

If coefficient of friction is 0.6. Calculate the angle of friction.

Solution:

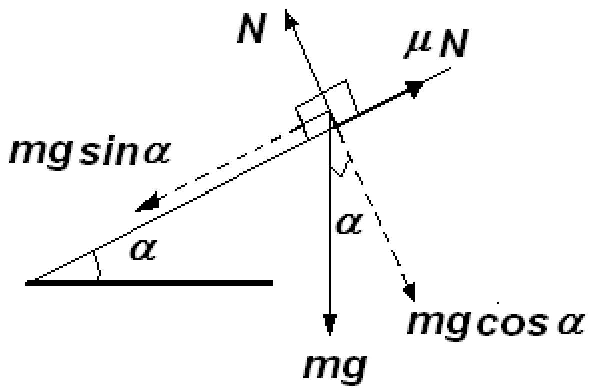
$\tan\lambda = \mu$

$\tan\lambda = 0.6$

$\lambda = 31^o$

Angle of repose

This is concerned with an inclined plane on which a body rests exerting its weight on the plane. The angle of repose of an inclined plane with respect to a body in contact with it is the angle of inclination of the plane with horizontal when the block just starts sliding down the plane under its own weight.



The limiting equilibrium of a body resting on the inclined plane is shown in figure
The forces acting are (its weight mg downward (ii) Normal reaction (iii) the force of limiting friction. Taking α as the angle of repose and resolving the forces along the plane and perpendicular to the plane, we get for equilibrium

$$Mg\cos\alpha = N \quad \text{----eq(1)}$$

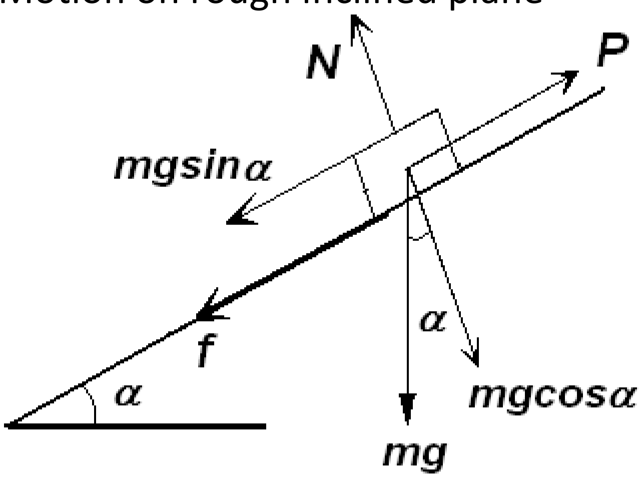
$$Mg\sin\alpha = f = \mu N \quad \text{-----eq(2)}$$

Dividing equation (1) by (2)

$$\mu = \tan\alpha$$

$$\therefore \text{ angle of repose} = \alpha = \tan^{-1}\mu$$

Motion on rough inclined plane



Suppose a motion up the plane takes place under the action of pull P acting parallel to the plane

$N = mg\cos\alpha$

Frictional force acting down the plane

$F = \mu\, mg\cos\alpha$

Applying Newton's second law for motion up the plane

$P - (mg\sin\alpha + F) = ma$

$P - mg\sin\alpha - \mu\, mg\cos\alpha = ma$

If P = 0, the block may slide downwards with an acceleration a. The frictional force would then act up the plane

$$mg\sin\alpha - F = ma$$

$$mg\sin\alpha - \mu\, mg\cos\alpha = ma$$

Motion of connected bodies

Commonly used forces

(i) Weight of a body : It is the force with which Earth attracts a body towards its centre. If M is the mass of the body and g is acceleration due to gravity, weight of the body is Mg. We take its direction vertically downwards

(ii) Normal force: Let us consider a block resting on the table. It is acted upon by its weight is vertically downward direction and is at rest. It means there is another force acting on the block in opposite direction, which balances its weight. This force is provided by the table and we call it as normal force. Hence, if two bodies are in contact a contact force arises. If the surface is smooth the direction of force is normal to the plane of contact. We call this force as Normal force. We take its direction towards the body under consideration

(iii) Tension in string : Let a block is hanging from a string weight of the block is acting in vertically downward but it is not moving hence its weight is balanced by a force due to string. This force is called 'tension in string'. Tension is a force in stretched string. Its direction is taken along the string and away from the body under consideration

Using free body diagram (FBD), we can find acceleration of connected bodies, unknown forces on the bodies

Following steps are needed while solving such questions

Step1: identify the unknown accelerations and unknown forces involved in the question

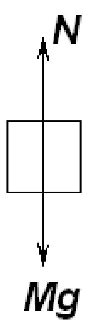
Step2: Draw free body diagram of different bodies in the given system

FBD : It is a diagram that shows forces acting on the body making it free from other bodies applying forces on the body under consideration. Hence free body diagram will include the forces like weight of the body normal force tension in the string and applied force.

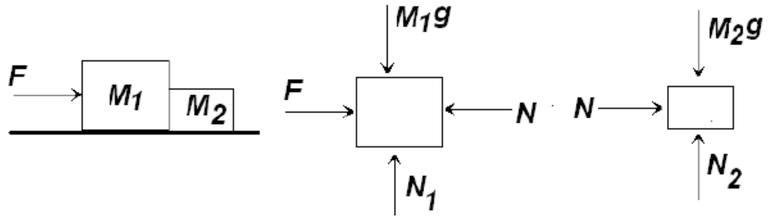
The important thing while drawing FBD is the shape of the body should be taken under consideration and force should be shown in a particular way. For example weight should be applied from centre of gravity of body, normal force should be applied on the respective surfaces, tension should be applied on the sides of string

Example

(i) Free body diagram of a book resting on table

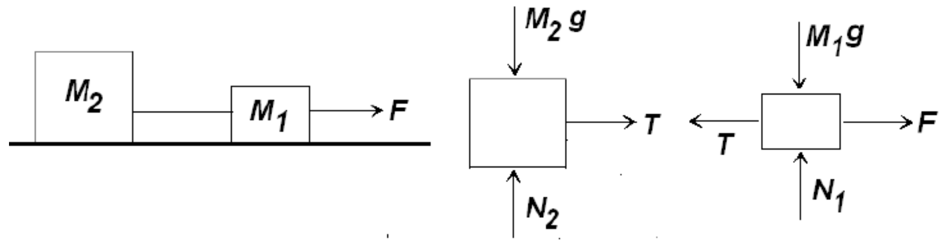


(ii) Free body diagram of bodies in contact and moving together on smooth surface



Note that, normal force is taken normal to the surface of contact and towards the body under consideration

(iii) Free body diagram of bodies connected with string and moving under the action of external force, on smooth surface



Note that, tension is acting along the string and away from the body under consideration

Step 3: Identify the direction of acceleration and resolve the forces along this direction and perpendicular to it

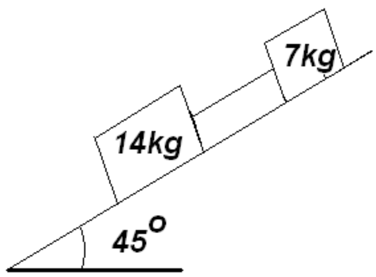
Step4: Find net force in the direction of acceleration and apply F = Ma to write equation of motion in that direction. In the direction of equilibrium take zero net force

Step5: If needed write relation between accelerations of bodies given in the situation

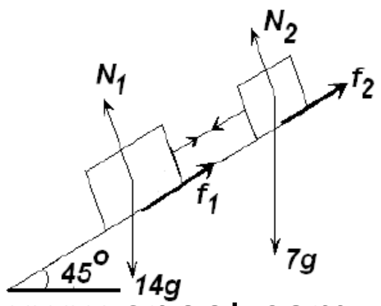
Step6: Solve the Written equations in step 4 and 5 to find unknown acceleration and force

Solved Numerical

Q) Two masses 14kg and 7kg connected by a flexible inextensible string rest on an inclined plane , inclined at 45^O with the horizontal as shown in figure. The coefficient of friction between the plane and the 14kg mass is ¼ and that between the plane and the 7kg mass is 3/8. Find the tension in the connecting string



Solution:



The force diagram of the masses placed on the inclined plane is shown in figure. Considering the motion of 14kg mass the equation of motion can be written as

$14g\sin45 - f_1 - T = 14a$ ---eq(1)

Where a is the acceleration down the plane

N_1 = 14gcos45 -----eq(2)
$f_1 = \mu N_1$ = (1/4)×14gcos45 ----eq(3)
∴ 14gsin45 – (1/4))×14gcos45 – T = 14a

$$\frac{14\times 9.8}{\sqrt{2}} - \frac{1}{4}\times\frac{14\times 9.8}{\sqrt{2}} - T = 14a \quad ----eq(4)$$

The equation of motion for 7kg mass can be written similarly considering the motion of 7kg mass separately
T+7gsin45 – f_2 = 7a ------eq(5)
N_2 = 7gcos45 -------eq(6)

$$f_2 = \mu N_2 = \frac{3}{8}\times 7g\cos 45^o \quad -----eq(7)$$

$$\therefore T + 7g\sin 45 - \frac{3}{8}\times 7g\cos 45 = 7a \quad -----eq(8)$$

$$T + \frac{7\times 9.8}{\sqrt{2}} - \frac{3}{8}\times\frac{7\times 9.8}{\sqrt{2}} = 7a$$

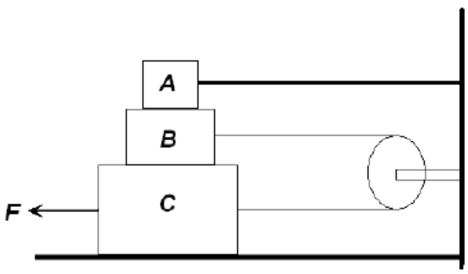
$$T + \frac{7\times 9.8}{\sqrt{2}}\times\frac{5}{8} = 7a$$

From equation(4),

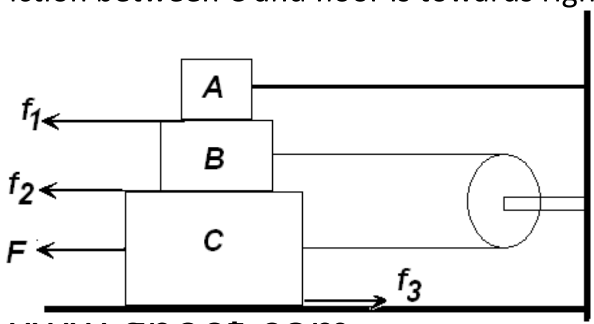
$$\frac{14\times 9.8}{\sqrt{2}}\times\frac{3}{4} - T = 14a$$

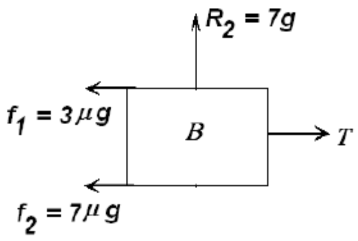
Solving above equation for T we get
T = 4.03 N

Q) In figure shown, block AB and C weigh 3kg, 4kg and 8kg respectively. The coefficient of sliding friction between any two surfaces is 0.25. A held at rest by a massless rigid rod mixed on the wall while B and C are connected by a string passing round a frictionless pulley. Find the force needed to drag C along the horizontal surface to left at constant speed Assume the arrangement shown in figure is maintained all through



Solution:
Note friction is opposite to applied force Thus for surface between B and C friction will be towards left. Block A is fixed thus direction of friction between A and B is towards left . Fiction between C and floor is towards right.





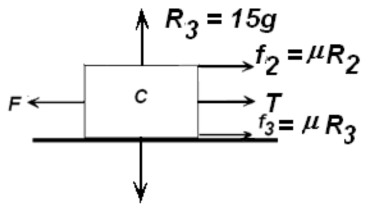
Block B and block A total mass = 2+4 = 7 exerts a force of 7g on block C thus normal force N= 7g upwards

Frictional force is $\mu N = 7\mu g$ between surface of C and B.

Frictional force between block A and B is 3 μg

Thus $T = f_1+f_2 = 3\mu g+7\mu g =10\mu g$

$T = 10\times0.25\times9.8 = 24.5N$



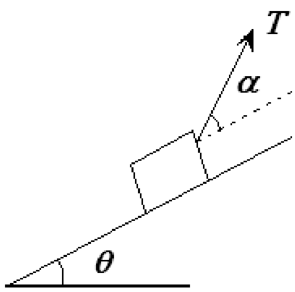
With reference to block C, f_2 will be towards right as it opposes the motion of block c

Now $F = f_2 + f_3 +T$

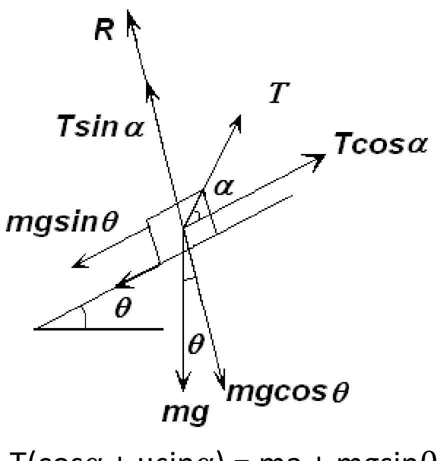
$F = 7\mu g +15\mu g+10\mu g=32\mu g$

$F= 32\times0.25\times9.8 = 78.4N$

Q) A block of mass m is pulled upward by means of a thread up an inclined plane forming an angle θ with horizontal as shown in figure. The coefficient of friction is μ. Find the inclination of the thread with horizontal so that the tension in the thread is minimum. What is the value of the minimum tension.



Solution:



The different forces acting on the mass are shown in figure. Let the mass move up the plane with an acceleration ‘a’. Writing the equation of motion.

$R + T\sin\alpha = mg\cos\theta$

$R = mg\cos\theta - T\sin\alpha$ -----eq(1)

$T\cos\alpha - mg\sin\theta - f = ma$ ------eq(2)

Where f is the force of friction

$f = \mu (mg\cos\theta - T\sin\alpha)$ ------eq(3)

Sunstituting the value of f from equation (3) in equation (2)

$T\cos\alpha - mg\sin\theta - \mu mg\cos\theta + \mu T\sin\alpha = ma$

$T(\cos\alpha + \mu\sin\alpha) = ma + mg\sin\theta + \mu mg\cos\theta$

$$T = \frac{ma + mgsin\theta + \mu mgcos\theta}{cos\alpha + \mu sin\alpha} \quad -----eq(4)$$

For T minimum

$$\frac{dT}{d\alpha} = 0$$

For T to be minimum $(\cos\alpha + \mu\sin\alpha)$ should be maximum

$$\frac{d}{d\alpha}(cos\alpha + \mu sin\alpha) = 0$$

$$-sin\alpha + \mu cos\alpha = 0$$

$$sin\alpha = \mu cos\alpha \quad \text{------eq}(5)$$

Since

$$\frac{d^2}{d\alpha^2}(cos\alpha + \mu sin\alpha) = -ve$$

Thus T will be maximum when $-sin\alpha + \mu cos\alpha = 0$

$$tan\alpha = \mu \quad \text{-------eq}(6)$$

$$\alpha = tan^{-1}(\mu)$$

or

T will have minimum value when a = 0 and $\alpha = tan^{-1}(\mu)$

From equation (4)

$$T_{min} = \frac{mgsin\theta + \mu mgcos\theta}{cos\alpha + \mu sin\alpha}$$

Now from equation (5)

$$cos\alpha + \mu sin\alpha = cos\alpha + \mu(\mu cos\alpha)$$

$$cos\alpha + \mu sin\alpha = cos\alpha\ (1 + \mu^2)$$

$$cos\alpha + \mu sin\alpha = \frac{1 + \mu^2}{sec\alpha}$$

$$cos\alpha + \mu sin\alpha = \frac{1 + \mu^2}{\sqrt{1 + tan^2\alpha}}$$

From equation(6)

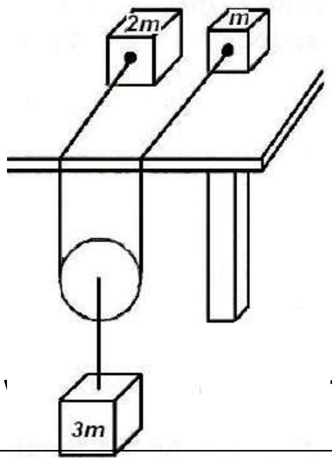
$$cos\alpha + \mu sin\alpha = \frac{1 + \mu^2}{\sqrt{1 + \mu^2}} = \sqrt{1 + \mu^2}$$

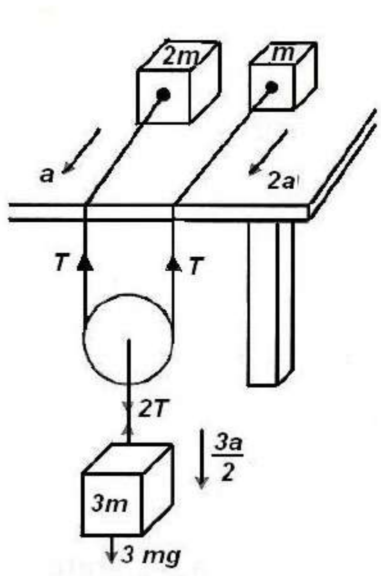
Thus

$$T_{min} = \frac{mgsin\theta + \mu mgcos\theta}{\sqrt{1 + \mu^2}}$$

Q) Two particles of masses m and 2m are placed on a smooth horizontal table. A string, which joins them, hang over the edge supporting a light pulley, which carries a mass 3m

Two parts of the string on the table are parallel and perpendicular to the edge of the table. The parts of the string outside the table are vertical. Show that the acceleration of the particle of mass 3m is 9g/17





Let T be the tension in the string, a be the acceleration of mass 2m, 2a be the acceleration of mass m

T = (m) (2a) ---eq(1)

The mass 3m will come down with acceleration

a' = (a+2a)/2 = 3a/2

$$\therefore 3mg - 2T = 3m \cdot \frac{3a}{2}$$

From equation (1)

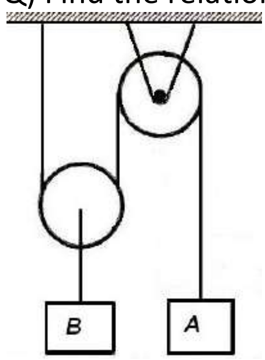
$$3mg - 2(2ma) = 3m \cdot \frac{3a}{2}$$

a = (6/17)g

∴ the acceleration of 3m mass = (3/2)a

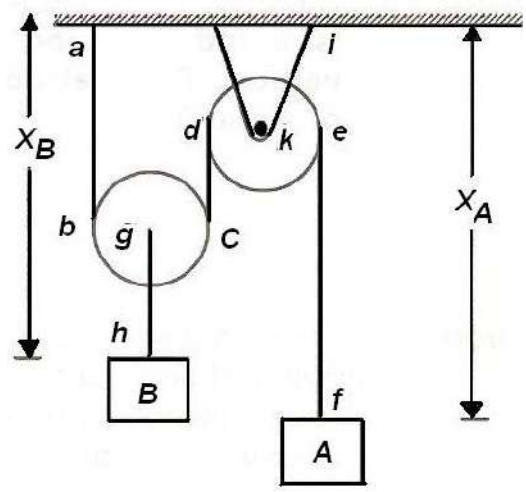
A' = (3/2)(6/17)g = (9/17)g

Q) Find the relation between acceleration of blocks A and B



Solution

Length of the string in given problem remains constant.



Thus we will expresses all the distances in terms of X_A and X_B

Now length gh = l_1 and length ie = l_2 are constant

arc bc and arc de are constant

Length of string =

ab+arc bc +cd+arc de + ef = constant ---- eq(1)

We will express all variable length in terms of X_A and X_B, l_1 and l_2

ab = X_B – gh = X_B – l_1

cd = X_B - gh - ik = X_B – gh – l_2

ef = X_A – ie = X_A – l_2

substituting above values in equation (1) we get

$(X_B - l_1)$ + arc bc + $(X_B - gh - l_2)$ + arc de + $X_A - l_2$ = constant

$\therefore 2X_B + X_A$ = constant --- eq(2)

taking derivative w.r.t time

$$2\frac{dX_B}{dt} + \frac{dX_A}{dt} = 0$$

If B is assumed to be moving down is positive then V_A will be negative

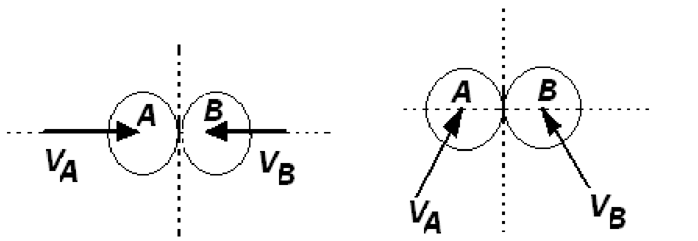
$2V_B = V_A$

Taking derivative with time

$2a_B = a_A$

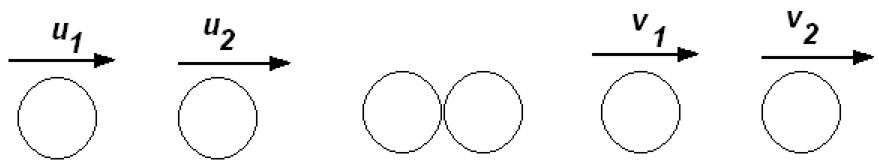
Impact

A collision between two bodies which occurs in a very short interval of time and during which the two bodies exert relatively large force on each other is called impact. The common normal to the surfaces in contact during the impact is called the line of impact. If centres of mass of the two bodies are located on this line, the impact is a central impact. Otherwise, the impact is said to be eccentric.



Direct central impact *Oblique central impact*

Direct central impact or head on impact



Let m_1 and m_2 masses having velocity u_1 and u_2 undergoes head on impact. Velocity after collision be v_1 and v_2

According to law of conservation of energy

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ -------eq(1)

To obtain values of v_1 and v_2 it is necessary to establish a second relation between v_1 and v_2. For this purpose, we use Newton's law of restitution according to which velocity of separation after impact is proportional to the velocity of approach before collision. In the present situation

$(v_2 - v_1) \propto (u_1 - u_2)$

Or $(v_2 - v_1) = e(u_1 - u_2)$ ------eq(2)
Here 'e' is a constant called as coefficient of restitution. Its value depends on the type of collision. The value of the coefficient 'e' is always between 0 and 1. It depends to a large extent on the two materials involved, but it also varies considerably with the impact velocity, the shape and size of the two colliding bodies
Multiply equation (2) by m_1

$m_1v_2 - m_1v_1 = e\,m_1 u_1 - e\,m_1u_2$
$e\,m_1 u_1 - e\,m_1u_2 = -m_1v_1 + m_1v_2$ -----eq(3)

Add eq(1) and eq(3)
$m_1u_1 + m_2u_2 + e\,m_1 u_1 - e\,m_1u_2 = m_2v_2 + m_1v_2$
$m_1u_1(1+e) + u_2(m_2 - em_1) = v_2(m_2 + m_1)$

$$v_2 = \frac{m_1(1+e)}{m_2+m_1}u_1 + \frac{m_2 - em_1}{m_2+m_1}u_2 \quad ----eq(4)$$

Multiply eq(2) by m_2

$m_2v_2 - m_2v_1 = e\,m_2 u_1 - e\,m_2u_2$

$e\,m_2 u_1 - e\,m_2u_2 = m_2v_2 - m_2v_1$ --------eq(5)

Subtract eq(5) from eq(1)

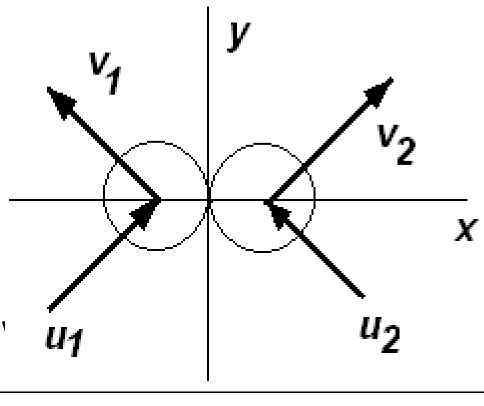
$m_1u_1 + m_2u_2 - e\,m_2 u_1 + e\,m_2u_2 = m_1v_1 + m_2v_1$

$u_1(m_1 - e\,m_2) + m_2u_2 = (m_1 + m_2)v_1$

$$v_1 = \frac{m_1 - em_2}{m_2+m_1}u_1 + \frac{m_2}{m_2+m_1}u_2 \quad ----eq(6)$$

(i) **Elastic collision**, collision in which kinetic energy is conserved , e = 1
(ii)**Perfectly inelastic** collision e = 0

Oblique central impact or indirect impact



Here X axis is line of impact and y axis is tangent to the spherical surfaces. Colliding bodies are directed along the line of impact

Velocities of v_1 and v_2 of the particles are unknown in magnitude and direction, their determination will require the use of four independent equations

(i)The component along y axis of the momentum of each particle, considered separately is conserved, thence the component of the velocity of each particle remains unchanged, we can write

$(u_1)_y = (v_1)_y$ and $(u_2)_y = (v_2)_y$

(ii)The component along the x-axis of the total momentum of the two particles is conserved we write

$m_1(u_1)_x + m_2(u_2)_x = m_1(v_1)_x + m_2(v_2)_x$

(iii)The component along the x-axis of the relative velocity of the two particles after impact is obtained by multiplying the x component of their relative velocity before impact by the coefficient of restitution

$(v_1)_x + (v_2)_x = e[(u_1)_x - (u_2)_x]$

We have thus obtained four independent equations, which can be solved for the components of the velocities of A and B after impact.

Solved numerical

Q) A block of mass 1.2kg moving at a speed of 20cm/s collides head on with a similar block kept as rest. The coefficient of restitution is 0.6. Find the loss of kinetic energy during collision

Solution: Let u_1 be the velocity of first block = 20cm/c, $u_2 = 0$, $m_1 = m_2 = 1.2$

$$v_1 = \frac{m_1 - em_2}{m_2 + m_1}u_1 + \frac{m_2}{m_2 + m_1}u_2$$

$$v_1 = \frac{1.2 - 0.6 \times 1.2}{1.2 + 1.2} \times 20 + \frac{1.2}{1.2 + 1.2} \times 0$$

V_1 = 4cm/s = 0.04 m/s

$$v_2 = \frac{m_1(1 + e)}{m_2 + m_1}u_1 + \frac{m_2 - em_1}{m_2 + m_1}u_2$$

$$v_2 = \frac{1.2(1 + 0.6)}{1.2 + 1.2}20 + \frac{1.2 - (0.6 \times 1.2)}{1.2 + 1.2}0$$

V_2 = 16cm/s = 0.16 m/s

Kinetic energy before collision =

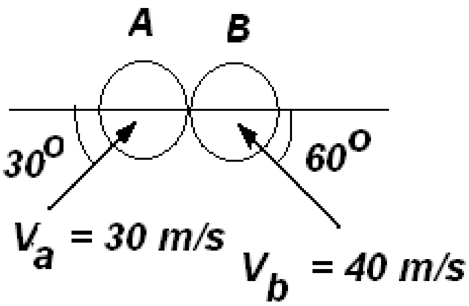
$\frac{1}{2}m_1u_1^2 = \frac{1}{2} \times 1.2 \times (0.2)^2 = 0.024\ J$

Kinetic energy after collision

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}\times 1.2\times(0.04)^2 + \frac{1}{2}\times 1.2\times(0.16)^2 = 0.01629\,J$$

Loss of kinetic energy = 0.024 – 0.1629 = 7.7×10^{-3} J

Q) The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are shown . Assume e = 0.90, determine the magnitude and direction of the velocity of each ball after impact



Solution

x component of velocity $(V_A)_x$ = 30cos30 = +26.0m/s

y component of velocity $(V_A)_y$ = 30sin30 = +15.0 m/s

x component of velocity $(V_B)_y$ = -40cos60 = -20.0 m/s

y component of velocity $(V_B)_y$ = 40sin60 = +34.6 m/s

Since the impulsive force are directed along the line of impact along x axis

The y component of the momentum remains unchanged. We have

$(V'_A)_y$ = 15 m/s and $(V'_B)_y$ = +34.6 m/s

Now according to law of conservation of momentum

$m(V_A)_x + m(V_B)_x = m(V'_A)_x + m(V'_B)_x$

$m(26.0) + m(-20) = m(V'_A)_x + m(V'_B)_x$

$m(V'_A)_x + m(V'_B)_x = 6.0$ ------eq(1)

Using law of restitution

$(V'_A)_x + (V'_B)_x = e[(V_A)_x + (V_B)_x]$

$(V'_A)_x + (V'_B)_x = 0.9[26-(-20)]$

$(V'_A)_x + (V'_B)_x = 41.4$ -----eq(2)

Solving equation (1) nad (2)we get

$(V'_A)_x$ = -17.7 m/s and $(V'_B)_x$ = +23.7 m/s

Resultant Motion : Adding vectorially the velocity components of each ball,

We obtain

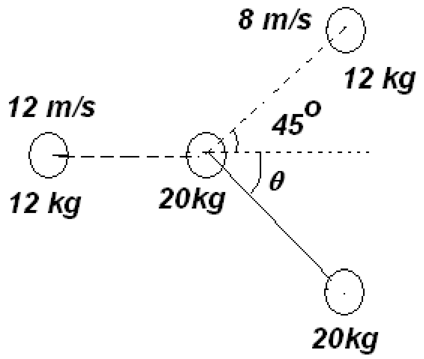
$\tan A = (V'_A)_x / (V'_A)_y$ = 15/(-17.7) = -0.8474 = 40.3^O negative sign indicates angle is with negative x axis in second quarter

magnitude V'_A = 23.2

Similarly resultant magnitude V'_B = 41.9.2 and angle is 55.6^O

Q) A sphere of steel of mass 15kg moving with a velocity of 12m/s, along x-axis collides with a stationary sphere of mass 20kg, If velocity of the first sphere after the collision is 8m/s and is moving at an angle of 45^O with x-axis, find the magnitude and direction of the second sphere after the collision.

Solution:



m_1 = 12 kg, u_1 = 12 m/s; m_2 = 20kg , u_2 = 0

v_1 = 8m/s

From law of conservation of momentum

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

12×12i = 12(8cos45$\hat{\imath}$)+12(8sin45$\hat{\jmath}$) + 20($v_2$cosθ $\hat{\imath}$- $v_2$sinθ$\hat{\jmath}$)

Comparing x coordinates

144 = 48√2+20$v_2$cosθ

20$v_2$cosθ = 76.12 -------eq(1)

Comparing y coordinates

0 = 48√2 -20 $v_2$sinθ

20 $v_2$sinθ = 67.88 --------eq(2)

From equation (1) and (2) we get

tanθ = 67.88/76.12 = 0.89

θ = 41^O 44'

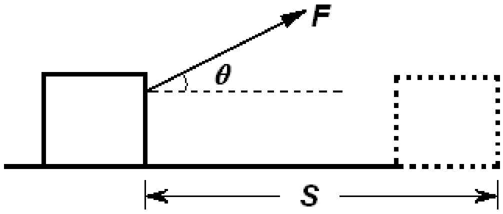
From equation (2)

20v_2 sin(41^O44') = 67.88

V_2 = 6.37 m/s

Work done by a constant force:

Consider an object undergoes a displacement S along a straight line while acted on a force F that makes an angle θ with S as shown



The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement

$W = FS\cos\theta$

Work done is a scalar quantity and S.I. unit is N-m or Joule (J). Its dimensional formula is $M^1L^2T^{-2}$

We can also write; work done as a scalar product of force and displacement

$$W = \mathbf{F} \cdot \mathbf{S}$$

From this definition, we conclude the following points:

(i) Work done by a force is zero, if point of application of force does not move (S=0)

(ii) Work done by a force is zero if displacement is perpendicular to the force ($\theta=90^O$)

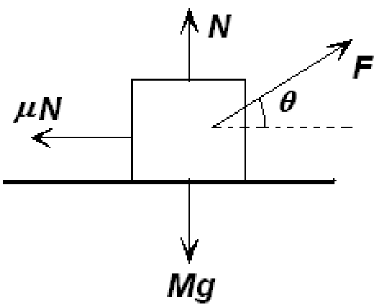
(iii) If angle between force and displacement is acute ($\theta < 90^O$), we say that work done by the force is positive or work is done on the object

(iv) If angle between force and displacement is obtuse ($\theta > 90^O$), we say that work done by the force is negative or work is done by the object

Solved Numerical

Q) A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block

Solution



The forces acting on the block as shown in figure Force F will resolve as Fsinθ along normal while Fcosθ will be opposite to friction. Thus we get

$F\cos\theta = \mu N$ ----- (1)

And

$N + F\sin\theta = Mg$ -----eq(2)

Eliminating N from equation (1) and (2)

$F\cos\theta = \mu (Mg - F\sin\theta)$

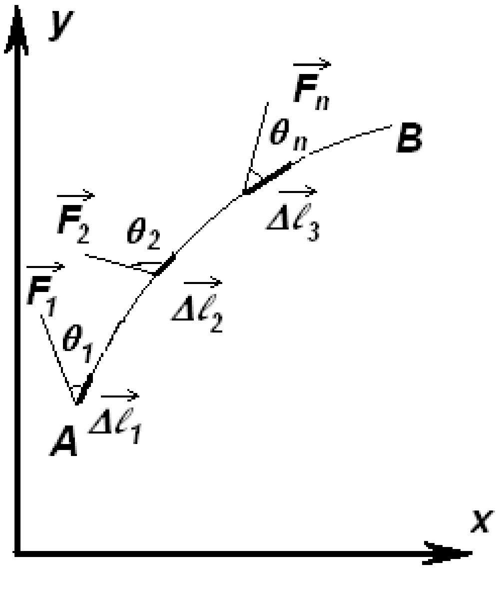
$$F\cos\theta + F\sin\theta = \mu mg$$

$$F = \frac{\mu Mg}{\cos\theta + \sin\theta}$$

Work done by this force during displacement d

$$W = Fd = \frac{\mu Mgd}{\cos\theta + \sin\theta}$$

Work done by Variable force



(a)

Consider a particle being displaced along the curved path under the action of a varying force, as shown in figure. In such situation, we cannot use W= (Fcosθ)S to calculate the work done by the force because this relationship applies when F is constant in magnitude and direction

However if we imagine that the particle undergoes a very small displacement Δ**l₁** , shown in figure(a), then F is approximately constant over this interval and we can express the work done by the force for this small displacement as $W_1 = F_x\Delta$ **l_1**

In order to calculate work done, the whole curved path is assumed to be divided in small segments Δ **l_1, Δ l_2, Δ l_3, ---------Δ l_n**

Let **F_1**, **F_2**, **F_3**, --------- **F_n** be the force at respective segments. The force over each such segment can be considered as constant because the segments are very small.

Total work done

W = **F_1**·Δ **l_1** + **F_2**·Δ **l_2** + **F_3**·Δ **l_3** + ………..+ **F_n**·Δ **l_n**

$$W = \sum_{A}^{B} \vec{F}_i \overrightarrow{\Delta l}_i$$

If we take |Δ**l**|→ 0 , the above summation gets converted into an integral

$$W = \int_{A}^{B} \vec{F} \cdot \overrightarrow{dl} = \int_{A}^{B} F cos\theta dl$$

Solved Numerical

Q) A particle moves from x = 0 to x = 10m on X-axis under the effect of force $F(x) = (3x^2 – 2x +7)$**i** N.

Calculate the work done

Solution: since direction of force and displacement is same θ = 0

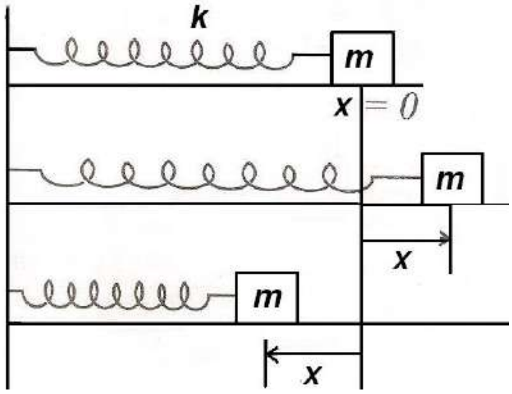
$$W = \int_{0}^{10} F\ dx$$

$$W = \int_{0}^{10} (3x^2 - 2x + 7)\ dx$$

$$W = \left[\frac{3x^3}{3}\right]_0^{10} - \left[\frac{2x^2}{2}\right]_0^{10} + [7x]_0^{10}$$

W = 1000 - 100 + 70 = 970 J

Work done by a spring



A common physical system for which the force varies with position is a spring-block as shown in figure. If the spring is stretched or compressed by a small distance from its unstitched or compressed by a small distance from its unscratched configuration, the spring will exert a force on the block given by F = -kx, where x is compression or elongation in spring, k is a constant called spring constant whose value depends inversely on un-stretched length and the nature of material of spring.

Negative sign in above equation indicates that the direction of the spring force is opposite to x , the displacement of the free end.

Consider a spring block system as shown in figure and let us calculate work done by the spring when block is displaced by x_0

At any moment if elongation is x, then force on block by spring is kx towards left.

Therefore, work done by the spring when block further displaced by dx

dw = - kxdx (Negative sign indicates displacement is opposite to spring force)

Total work done by the spring

$$W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2$$

Similarly, work done by the spring when it is given a compression x_0 is

$$-\frac{1}{2}kx_0^2$$

We can also say that work done by external agent

$$\frac{1}{2}kx_0^2$$

Power

If external force is applied to an point like object and if the work done by this force is ΔW in the time interval Δt, then the average power during this interval is defined as

$$P = \frac{\Delta W}{\Delta t}$$

The work done on the object contributes to increasing energy of the object. A more general definition of power is the time rate of energy transfer. This instantaneous power is the limiting value of the average power as Δt approaches zero

$$P = \frac{dW}{dt}$$

Where we have represented the infinitesimal value of the work done by dW (even though it is not a change and therefore not a differential). We know that

$dW = \vec{F} \cdot \vec{S}$

Therefore the instantaneous power can be written as

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{S}}{dt} = \vec{F} \cdot \vec{v}$$

The SI unit of power is Joule per second (J/s), also called watt (W)

1W = 1 J/s = 1kgm^2s^{-3}

Energy

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up.

Conversely if some work is done upon an object, the object will be given some energy. Energy and work are mutually convertible.

Kinetic energy

Kinetic energy (K.E.) is the capacity of a body to do work by virtue of its motion

If a body of mass m has velocity v its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest to its velocity v.

The numerical value of the kinetic energy can be calculated from the formula

$$K.E. = \frac{1}{2}mv^2$$

This formula can be derived as follows:

Consider a **constant force F** which acting on a mass m initially at rest, particle accelerate with constant velocity and attend velocity v after displacement of S .

For the formula

$v^2 - u^2 = 2as$

Initial velocity is zero

$v^2 = 2as$

Multiply both the sides by m

$mv^2 = 2mas$

$mv^2 = 2W$ [As work = FS = mas]

$W = (1/2)mv^2$

But Kinetic energy of body is equivalent to the work done in giving the velocity to the body

Hence K.E = $(1/2)mv^2$

Since both m and v^2 are always positive K.E is always positive and does not depend up on the direction of motion of body. Another equation for kinetic energy

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{1}{2}\frac{p^2}{m}$$

Potential energy

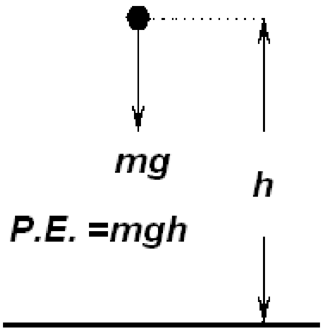
Potential energy is the energy due to position. If a body is in a position such that if it were released it would begin to move, it has potential energy

There are two common forms of potential energy, gravitational and elastic

Gravitational potential energy

It is possessed by virtue of height

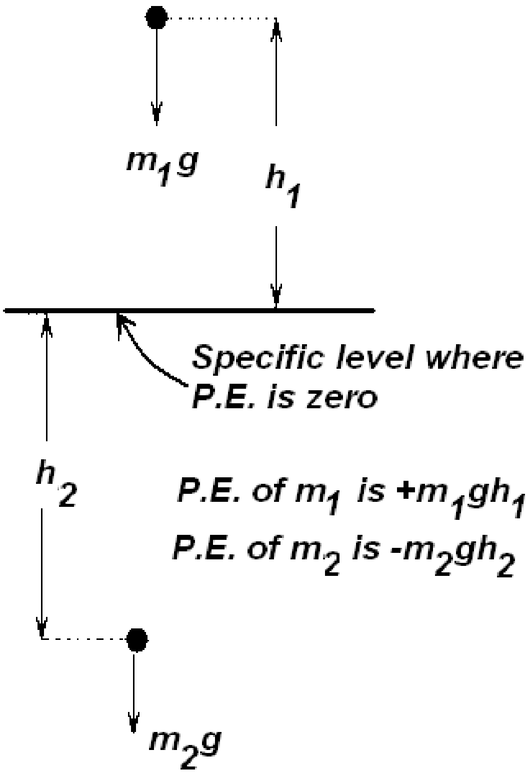
When an object is allowed to fall from one level to a lower level it gains speed due to gravitational pull, i.e. it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its gravitational potential energy into kinetic energy.



The gravitational potential energy is equivalent to the negative of the amount of work done by the weight of the body in causing the descent.

If a mass m is at a height h above a lower level, the P.E. possessed by the mass is (mg) (h)

Since h is the height of an object above a specific level, an object below the specified level has negative potential energy



Therefore Gravitational Potential Energy = $\pm$ mgh

- The chosen level from which height is measured has no absolute position. It is important therefore to indicate clearly the zero P.E. level in any problem in which P.E. is to be calculated.
- Gravitational Potential Energy = $\pm$ mgh is applicable only when h is very small in comparison to the radius of earth.

Elastic potential Energy

It is a property of stretched or compressed springs.
The end of a stretched elastic spring will begin to move if it is released. The spring therefore possesses potential energy due to its elasticity (i.e. due to change in its configuration)
The amount of elastic potential energy stored in a spring of natural length a and spring constant k when it is extended by a length x is equal to the amount of work necessary to produce the extension
Work done = (1/2)kx^2 so
Elastic Potential energy = (1/2) kx^2
Elastic potential energy is never negative whether the spring is extended or compressed

Work energy theorem

When a body is acted upon by force acceleration is produced in it. Thus velocity of the body changes and hence the kinetic energy of the body also changes. Also force acting on a body displaces the body and so work is said to be done on the body by force. These facts indicate that there should be some relation between the work done on body and change in its kinetic energy.
The work done by the force F
W= F S
W = ma
W = mas
Also $v^2 - u^2 = 2as$
Multiplying both sides by m
$m(v^2 - u^2) = 2ams$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = mas$$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = W$$

Here u and v are the speeds before and after application of force.
The left hand side of above equation gives change in kinetic energy while right hand gives the work done
Thus ΔK = W
The work done by the resultant force on a body is equal to change in kinetic energy of the body. This statement is known as work energy theorem.

Work energy theorem for variable force
Work-energy theorem is valid from variable force
Suppose position dependent force F(x) acts on a body of mass m
Work done under the influence of force

$$W = \int_i^f F(x)dx$$

$$W = \int_i^f m\frac{dv}{dt}dx$$

$$W = \int_i^f m\frac{dx}{dt}dv$$

$$W = \int_i^f mv\,dv \quad [\because \frac{dx}{dt} = v]$$

$$W = m\int_i^f v\,dv$$

If initial velocity of the body and final velocity of the body are v_i and v_f

$$W = m\int_{v_i}^{v_f} v\,dv = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$$

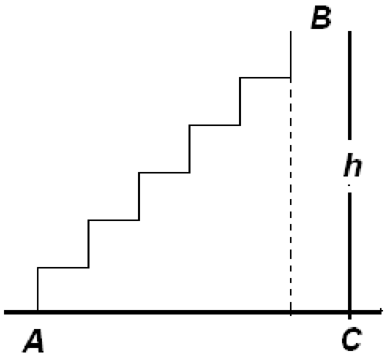
$$W = \frac{m}{2}\left[v_f^2 - v_i^2\right]$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Conservative and non-conservative force

Conservative force

A conservative force may be defined as on for which work done in moving between two points A and B is independent of the path taken between two points. Work done to move particles through stairs is equal to moving particle vertically .The implication of "conservative" in this context is that you could move it from A and B by one path and returns to A by another path with no net loss of energy – any close return path A takes net work zero. Or mechanical energy is conserved



A further implication is that the energy of an object which is subject only to that conservative force is dependent upon its position and not upon the path by which it reached that position. This makes it possible to define a potential energy function which depends upon position only

If a force acting on an object is a function of position only, it is said to be a conservative force and it can be represented by potential energy function which for a one-dimensional case satisfies the derivative condition

$$F(x) = -\frac{dU}{dx}$$

Example for verification

(a) Gravitational potential energy = -mgh

Thus

$$F(h) = -\frac{d(-mgh)}{dh}$$

F(h) = mg

(b) Spring potential energy = (1/2)kx^2

$$F(x) = -\frac{1}{2}k\frac{d(x^2)}{dx}$$

$$F(x) = -kx$$

Non-conservative force

Consider a body moving on a rough surface from A to B and then back from B to A. Work done against frictional forces only add up because in both the displacement work is done against frictional force only. Hence frictional force cannot be considered as a conservative force. It is non-conservative force

Conservation of mechanical energy

Kinetic and potential energy both are forms of mechanical energy. The total mechanical energy of a body or system of bodies will be changed in values if

(a) An external force other than weight causes work to be done(work done by weight is potential energy and is therefore already included in the total mechanical energy)

(b) Some mechanical energy is converted into another form of energy (e.g. sound, heat , light) such a conversion of energy usually takes place when a sudden change in the motion of the system occurs. For instance, when two moving objects collide some mechanical energy is converted into sound energy, which is heard as a bang at the impact.

If neither (a) nor (b) occurs, then the total mechanical energy of a system remains constant.

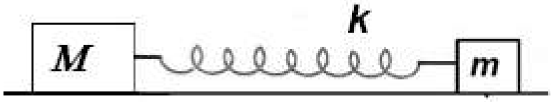
This is the principle of Conservation of Mechanical Energy and can be expressed as

The total mechanical energy of a system remains constant provided that no external work is done and no mechanical energy is converted into another form of energy

When this principle is used in solving problems, a careful appraisal must be made of any external forces, which are acting. Some external forces do work and hence cause a change in the total energy of the system.

Solved Numerical

Q) A spring of force constant k is kept in compressed condition between two blocks of masses m and M on the smooth surface of table as shown in figure. When the spring is released both the blocks move in opposite directions. When the spring attains its original (normal) position, both the blocks lose the contacts with spring. If x is the initial compression of the spring find the speed of block while getting detached from the spring.



Solution

According to law of conservation of energy

Spring Potential energy = Sum of kinetic energy off block

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$$

From law of conservation of momentum

$mv_1 = Mv_2$

$$v_2 = \frac{m}{M}v_1$$

$$kx^2 = mv_1^2 + M\left(\frac{m}{M}v_1\right)^2$$

$$kx^2 = v_1^2\left(m + \frac{m^2}{M}\right)$$

$$kx^2 = v_1^2\left(\frac{mM + m^2}{M}\right)$$

$$v_1^2 = \frac{kx^2M}{m(M+m)}$$

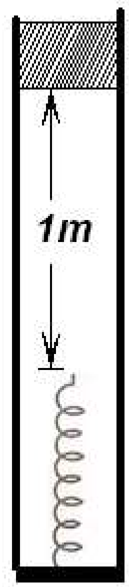
$$v_1 = \sqrt{\frac{kM}{m(M+m)}} \cdot x$$

Similarly

$$v_2 = \sqrt{\frac{km}{M(M+m)}} \cdot x$$

Q) A 20kg body is released from rest, so as to slide in between vertical rails and compresses a spring having a force constant k = 1920 N/m. the spring is 1m below the starting position of the body. The rail offers a resistance of 36N to the motion of the body. Find (i) the velocity of the body just before touching the spring (ii) the distance ,ℓ through which the spring is compressed (iii) the distance 'h' through which the body rebounds up

Solution



(i) Let velocity of the body just before touching the spring be v

Change in k.E = work done

$$\frac{1}{2}mv^2 - 0 = mg \times 1 - 36 \times 1$$

$$\frac{1}{2} \times 20 \times v^2 = 20 \times 9.8 \times 1 - 36 \times 1$$

v = 4 m/s

(ii) Let x be maximum compression of the spring. Then effective height for calculation of potential energy = 1+x

From conservation of energy

Spring potential energy = Change in P.E - Work done against friction

$$\frac{1}{2}kx^2 = mg(1+x) - 36(1+x)$$

$$\frac{1}{2} \times 1920 \times x^2 = 20 \times 9.8 \times (1+x) - 36(1+x)$$

X = 0.5 m

(iii) Let object bounce up to height h

Potential energy of object = spring potential energy – work against friction

$$mgh = \frac{1}{2}kx^2 - 36h$$

$$20 \times 9.8 \times h = \frac{1}{2} \times 1920 \times (0.5)^2 - 36h$$

$$h = \frac{240}{232} = \frac{30}{29} = 1.03m$$

Q) if work is done on a particle at constant rate, prove that the velocity acquired in describing a distance from rest varies as $x^{1/3}$

Solution

Power is constant, P = F .V = constant (say k)

Now mav = K

$$av = \frac{k}{m}$$

$$v\frac{dv}{dt} = \frac{k}{m}$$

$$v\frac{dv}{dt}\frac{dx}{dx} = \frac{k}{m}$$

$$v^2\frac{dv}{dx} = \frac{k}{m}$$

$$v^2 dv = \frac{k}{m}dx$$

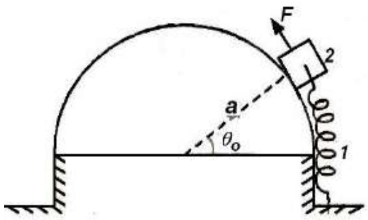
$$\int_0^v v^2 dv = \frac{k}{m}\int_0^x dx$$

$$\frac{v^3}{3} = \frac{k}{m}x$$

$$v^3 \propto x$$

$$v \propto x^{1/3}$$

Q) In the position shown in figure, the spring constant k is undeformed. Find the work done by the variable force F, which is always directed along the tangent to the smooth hemispherical surface on the small block of mass m to shift it from the position 1 to position 2 slowly.



Solution:

From the condition of the equilibrium of the block at any arbitrary angular position $\theta < \theta_o$

F = mgcosθ + kx

Work done in displacing the block through a distance dx = dW

dW = Fdx = (mgcosθ + kx)dx

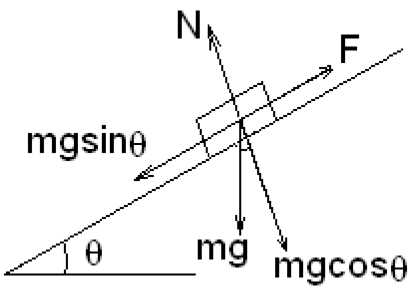
where x = aθ, dx = adθ

Total work done by the force F on the small block of mass m to shift from the position 1 to position 2 is

$$W = \int F dx = \int_{\theta}^{\theta_0} (\mathrm{mgcos\theta} + \mathrm{ka\theta})\mathrm{ad\theta}$$

$$W = mgasin\theta_0 + \frac{ka^2}{2}\theta_0^2$$

Q) A block of mass 2 kg is pulled up on a smooth incline of angle 30^{O} with horizontal. If the block moves with an acceleration of 1 m/s^2, find the power delivered by the pulling force at a time 4 seconds after motion starts. What is the average power delivered during these four seconds after the motion starts?



Solution:

To find power delivered by force at t=4 we have to calculate velocity at t=4 and use formula P = force × velocity

i) Calculation of velocity at t = 4 s

$$V = u + at$$

$$a=1 m/s^2 \; t = 4 \text{ sec given}$$

$$V = 4 \text{ m/s}$$

ii) Calculation of resultant force

Given resultant acceleration $a = 1 \text{ m/s}^2, \theta = 30^O, g = 9.8 \text{ m/s}^2$

Thus from the diagram and resolving forces we get equation

$F – mg\sin\theta = ma$

$F = mg\sin\theta + ma$

$F = 2\times 9.8\times \sin 30 + 2 \times 1 = 11.8$ N

By substituting values of F and v in equation of power

P = Fv

P = 11.8 × 4 = 47.2 W

To find average power

We have to find total work done by using formula W = FS and then use formula P = W/t

But S is not given , can be calculated using $v^2 = u^2 + 2as$

We have already calculated v = 4m/s, u = 0, a = $1 m/s^2$

Thus

$$16 = 0 + 2\times 1\times S$$

$$S = 8 \text{ m}$$

Now work done in 4 seconds = Force × displacement

Work done in 4 seconds = 11.8 ×8 = 94.4 J

Average power = Work / time

Average power = 94.4 / 4 = 23.6 W

Q) A block of mass m released from rest onto an ideal non-deformed spring of spring constant k from a negligible height. Neglect the air resistance, find the compression d of the spring.

Solution

(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

Block is just kept on spring not allow to fall on spring. Thus Weight of block will press the spring and restoring force will oppose the compression. And equilibrium will be establish.

Let compression be 'd' thus potential energy lost by the block = mgd

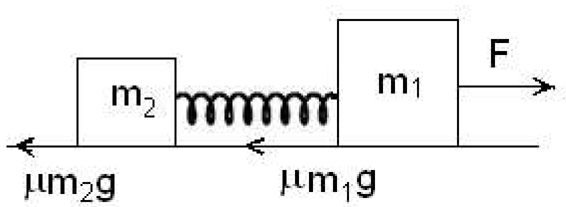
Potential energy gain by spring = $(/1/2)kd^2$

Thus potential energy lost by the block = potential energy gain by spring

$$mgd = (1/2)kd^2$$

$$d = 2mg/k$$

Q) Two masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between bars and surface is μ. What minimum constant force has to be applied in the horizontal direction to the mass m_1 in order to shift the other mass m_2



Solution:
Note that acceleration of both the masses will be different. Because acceleration of mass m_2 is due to restoring force of spring.
Problem can be solved using law of conservation of energy.
Fist consider mass m_2 do not move and is stationary.
Let x be the displacement of mass m_1 then Work done by force = Fx
This work done is used to overcome friction of mass m_1 and remaining stored as potential energy of spring
Now work done to overcome friction = Frictional force × displacement
$= \mu\, m_1 g\, x$
Energy stored in spring since mass m_1 have moved by distance 'x', stretching is spring is 'x' as we have already stated motion of block m_2 is due to restoring force of spring
Thus PE. Of Spring = $(1/2)kx^2$

$$Fx = \mu\, m_1 g\, x + (1/2)kx^2 \quad \text{-- --- eq(1)}$$

Since block m_2 moves due to restoring force of spring thus restoring force = frictional force

$$Kx = \mu\, m_2 g.$$

Substituting value of Kx in equation (1) we get

$$Fx = \mu\, m_1 g\, x + (1/2)\, \mu\, m_2 g x$$

$$F = \mu\, g\, (\, m_1 + m_2 /2\,)$$

Q) A block of mass M is attached with a vertical relaxed spring of spring constant k. if the block is released, find maximum elongation in spring.
Solution: Let x be the elongation.
Thus potential energy lost by mass = mgx
Energy gain by spring = $(1/2)\, kx^2$
From law of conservation of energy
Potential energy loss by mass M = energy gain by spring
$Mgx = (1/2)kx^2$
$X = 2Mg/k$
(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

Q) A horse pulls a wagon of 5000 kg from rest against a constant resistance of 90N. the pull exerted initially is 600N and it decreases uniformly with the distance covered to 400N at a distance of 15m from start. Find the velocity of wagon at this point.

Solution:

Force is varying Initially 600 N and goes down to 400 N. Thus average force applied for pull = (600 +400) /2 = 500 N

Resistive force is constant 90N

Effective force = Average force – resistive force

Effective force = 500 – 90 = 410 N

Displacement = 15 m

Thus work done by the force = 410 × 15 = 6150 J

This work by force produces kinetic energy

∴ Kinetic energy of object = work done by force

$$\therefore \frac{1}{2}mv^2 = 6150$$

V= 1.57 m/s

Q) A block of mass 5.0 kg is suspended from the end of a vertical spring, which is stretched by 10cm under the lead of the block. The block is given is given a sharp impulse from below so that it acquires an upward speed of 2.0 m/s. How high will it rise? Take g = 10 m/s^2

Solution:

For equation mg = kx

5 ×10= k ×0.1

K = 500 N

Spring is already elongated by 0.1m thus it already have some potential energy, block attached to spring also have potential energy.

when sharp impulse is given to block gain more potential energy and spring gain potential energy

Thus After sharp impulse given Total energy of system

= previously store potential energy of spring due to 0.1m elongation + given kinetic energy+ potential energy of mass

[Here potential energy of mass is taken positive as below equilibrium point of spring before attaching mass]

When spring gets compressed say by x

= Potential energy of spring + potential energy of block

= $(1/2)kx^2$ - mg(x)

[Here potential energy taken negative as object moved above the equilibrium position of spring before attaching mass]

Thus from law of conservation of energy

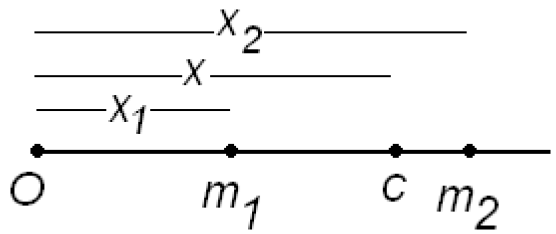
$(1/2)\ k(0.1)^2 + (1/2)\ mv^2 + mg(0.1) = (1/2)kx^2 - mg(x)$

$(1/2)500(0.01) + (1/2)5(2)^2 + 5(10)(0.1) = (1/2)(500)x^2 - 5(10)x$

X = 0.1 m and height to which block raise = 0.1 +0.1= 0.2m from equilibrium point before attaching the mass.

CENTRE OF MASS

As shown in figure consider two particles having mass m_1 and m_2 lying on X-axis at distance of x_1 and x_2 respectively from the origin (O). The centre of mass of this system is that point whose distance from origin O is given by



$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Here, x is the mass-weight average position of x_1 and x_2.

The centre of mass of the two particles of equal mass lies at the centre (on the line joining the two particles between the two particles)

Consider a set of n particles whose masses are m_1 m_2, m_3, … m_n and whose vector relative to an origin O are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, …….. $\mathbf{r}_n$ respectively
The centre of mass of this set of particles is defined as the point with position vector $\mathbf{r}_{CM}$

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$$

Here M is the total mass of the body.

Solved numerical

Q) Three particles of mass 2kg, 5kg, and 3kg are situated at points with position vectors (i + 4j – 7k)m, (3i – 2j +k)m and (I – 6j +13k) m respectively. Find the position vector of centre of mass
Solution:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

$$\vec{r}_{CM} = \frac{2\left(\hat{\imath} + 4\hat{\jmath} - 7\hat{k}\right) + 5\left(3\hat{\imath} - 2\hat{\jmath} + \hat{k}\right) + 3\left(\hat{\imath} - 6\hat{\jmath} + 13\hat{k}\right)}{2 + 5 + 3}$$

$$\vec{r}_{CM} = \left(2\hat{\imath} - 2\hat{\jmath} + 3\hat{k}\right) m$$

Centre of mass of continuous bodies

For calculating centre of mass of continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice is usually determined by the symmetry of the body.

Consider an element dm of the body having position vector r, the quantity m_ir_i can be replaced by dm$\mathbf{r}_i$, direct sum over particles becomes integral over the body

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}dm$$

In component form, this equation can be written as

$$x_{cm} = \frac{1}{M}\int x\,dm$$

$$y_{cm} = \frac{1}{M}\int y\,dm$$

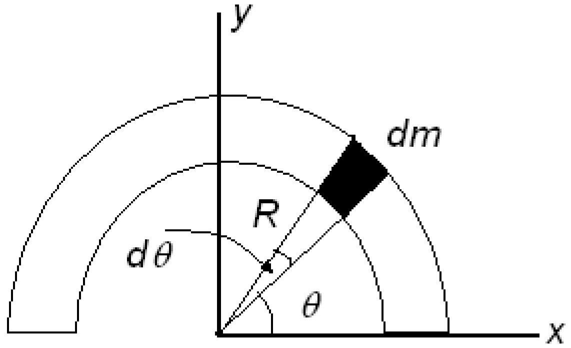
$$z_{cm} = \frac{1}{M}\int z\,dm$$

To evaluate the integral we must express the variable m in terms of spatial coordinates x, y, z or **r**

Solved Numerical

Q) Locate the centre of mass of a uniform semicircular rod of radius R and linear density σ kg/m

Solution



From the symmetry of the body we see at once that the centre of mass of the body must lie along y-axis. So X_{CM} = 0.

In this case it is convenient to express the mass element in terms of the angle θ, measured in radian.

The element, which subtends an angle dθ at the origin, has a length Rdθ and a mass dm = σRdθ. Its y coordinate is y = Rsinθ

Therefore

$$y_{CM} = \int_0^{\pi} \frac{y\,dm}{M}$$

$$y_{CM} = \int_0^\pi \frac{\sigma R^2 sin\theta d\theta}{M}$$

$$y_{CM} = \frac{\sigma R^2}{M}[-cos\theta]_0^\pi$$

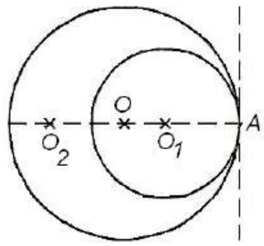
$$y_{CM} = \frac{2\sigma R^2}{M}$$

Total mass of ring M = πRσ

$$\therefore y_{CM} = \frac{2R}{\pi}$$

Q) A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42cm is removed from one edge of the plate as shown in figure. Find the centre of mass of the remaining portion

Solution



Let O be the centre of circular plate and O_1, the centre of circular portion removed from the plate. Let O_2 be the centre of mass of the remaining part.

Area of original plate = $\pi R^2 = (28)^2\ \pi\ cm^2$

Area removed from circular plate = $\pi r^2 = (21)^2\ \pi\ cm^2$

Let σ be the mass per cm^2. Then

Mass of the original plate m = $(28)^2\pi\sigma$

Mass of the removed part $m_1 = (21)^2\ \pi\sigma$

Mass of the remaining part $m_2 = (28)^2\pi\sigma - (21)^2\ \pi\sigma = 343\pi\sigma$

Now the masses m_1 and m_2 may be supposed to be concentrated at O_1 and O_2 respectively. Their combined centre of mass is at O. Taking O as origin we have form definition of centre of centre of mass.

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$x_1 = OO_1 = OA - O_1A = 28 - 21 = 7cm$

$x_2 = OO_2 = ?$, $x_{cm} = 0$

$$0 = \frac{(21)^2\pi\sigma \times 7 + 343\pi\sigma \times x_2}{m_1 + m_2}$$

$$(21)^2\pi\sigma \times 7 + 343\pi\sigma \times x_2 = 0$$

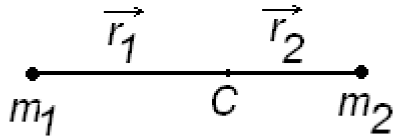
$x_2 = -9$ cm

Q) The distance between two particles of mass m_1 and m_2 is r. If the distances of these particles from the centre of mass of the system are r_1 and r_2 respectively, show that

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right] \quad and \;\; r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Solution

Centre of mass will be in between the line joining the two masses as shown in figure



let coordinates of centre of mass C be (0,0) thus vector r_1 will be negative and vector r_2 is positive.

Thus $m_1 r_1 = m_2 r_2$

$$r_1 = \frac{m_2}{m_1} r_2$$

also $r = r_1 + r_2$

$$\therefore r = \frac{m_2}{m_1} r_2 + r_2$$

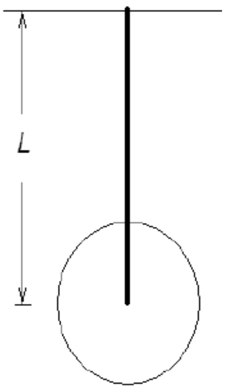
$$\therefore r = \frac{m_2 + m_1}{m_1} r_2$$

$$r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Similarly it can be obtained

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right]$$

Q) A thin rod of length L and uniform cross-section is suspended vertically as shown in figure. A circular disc is attached at the lower end of the road such that the lower end of the rod is at the centre of the disc. Find the position of C.M. of the system with respect to the point of suspension. Let M_1 and M_2 be the masses of the rod and the disc respectively



Solution :

As shown in figure centre of rod must be at the distance L/2 from the point of suspension. And centre of mass of disc is at distance L from the point of suspension

Suppose centre of mass is at distance r_{cm} from the point of suspension then

$$\vec{r}_{CM} = \frac{M_1 \frac{L}{2} + M_2 L}{M_1 + M_2}$$

Difference between Centre of mass (CM) and centre of gravity (CG)

The center of gravity is based on weight, whereas the center of mass is based on mass. So, when the gravitational field across an object is uniform, the two are identical. However, when the object enters a spatially-varying gravitational field, the CG will move closer to regions of the object in a stronger field, whereas the CM is unmoved.

More practically, the CG is the point over which the object can be perfectly balanced; the net torque due to gravity about that point is zero. In contrast, the CM is the average location of the mass distribution. If the object were given some angular momentum, it would spin about the CM.

Clearly if gravitational acceleration is uniform $r_{Cm} = r_{CG}$

If gravitational field is not uniform $r_{Cm} \neq r_{CG}$

Velocity of centre of mass

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Momentum of centre of mass

$$\vec{P} = M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

Acceleration of centre of mass

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Force on centre of mass

$$\vec{F} = M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n$$

Equation shows that the system moves under the influence of the resultant external force F as if the whole mass of the system is concentrated at its centre of mass

Law of conservation of momentum

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

Above equation shows that "If resultant external force acting on a system of particle is zero, then the total linear momentum of the system remain constant" this statement is known as the law of conservation of linear momentum.

Solved numerical

Q) A man weighing 70kg is standing at the centre of a flat boat of mass 350 kg. The man who is at a distance of 10m from the shore walks 2m towards it and stops. How far will he be from the bank? Assume the boat to be of uniform thickness and neglect friction between boat and water.

Solution

Man and boat form a system. This system is not acted by any external force

Thus according to law of conservation of momentum

Centre of mass of system will remain unchanged with reference to observer on the bank

Now man is standing at the centre of boat thus CM is at 10 m from bank

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

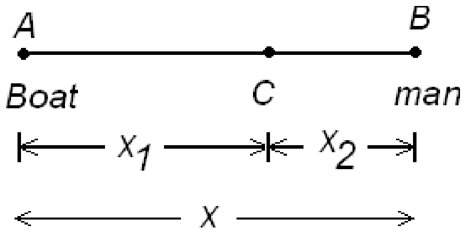
$$10 = \frac{70x_1 + 350x_2}{70 + 350}$$

$420 = 7x_1 + 35x_2$
$60 = x_1 + 5x_2$ ---- eq(1)
Since man has walked 2 m distance between the CM of Boat and Man is 2m
Also $x_1 - x_2 = 2$ -----eq(2)
From equation (1) and (2) we get
$X_1 = 25/3 = 8.33$ m

Q) A person is standing on a stationary raft in a lake. The distance of the person from bank of the lake is 30m. The masses of the person and the raft are 60kg and 40kg respectively. Now, the person starts running on the raft towards the bank at the speed of 10m/s with respect to the raft. How far from the bank, would be the person be after one second
Solution
Since no external force acts on the system, the position of centre of mass of the system should remain unchanged. If we take coordinates as (0,0) then

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Here x_{Cm} = (0,0)



Let m_1 = mass of boat ,
m_2 = mass of man

$$0 = \frac{40(-x_1) + 60x_2}{m_1 + m_2}$$

$$40x_1 = 60x_2$$

Let $x_1 + x_2 = x$
$x_1 = x - x_2$

$$40(x - x_2) = 60x_2$$

$$40x = 100x_2$$

$$x_2 = \frac{2}{5}x$$

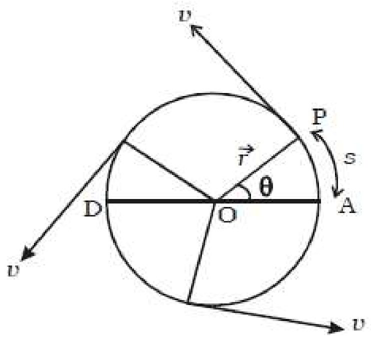
So centre of mass C is at a distance of (2/5) from the centre of boat
In one second, person will run through a distance of 10m towards the bank.
So raft must move (2/5)×10 away from the bank. So that centre of mass of system remains unchanged.
So, person distance from bank = 30 - (2/5)×10 = 26 m

Q) Two skaters A and B of mass M and 1.5M are standing together on a frictionless ice surface. They push each other apart. The skater B moves away from A with a speed of 2m/s relative to ice. What will be the separation between two skaters after 8 seconds?
Solution.
Since no external force acts on the system, Centre of mass with respect to observer on ice remains unchanged.

Thus $Mv = (1.5M)\times 2$
Thus velocity of A with respect to ice = 3 m/s
Now relative velocity = 3 +2 m/s
$\therefore$ Relative separation = 5×8 = 40 m

7.1 Circular Motion

When a particle moves on a circular path with a constant speed, then its motion is known as uniform circular motion in a plane. The magnitude of velocity in circular motion remains constant but the direction changes continuously.

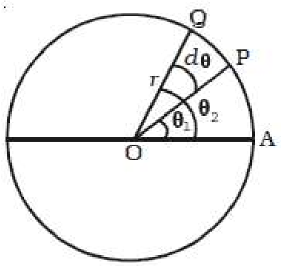


Let us consider a particle of mass *m* moving with a velocity *v* along the circle of radius *r* with centre O as shown in Fig

P is the position of the particle at a given instant of time such that the radial line OP makes an angle θ with the reference line DA. The magnitude of the velocity remains constant, but its direction changes continuously. The linear velocity always acts tangentially to the position of the particle (i.e) in each position, the linear velocity **v** is perpendicular to the radius vector **r**.

Angular displacement

Let us consider a particle of mass *m* moving along the circular path of radius *r* as shown in Fig.



Let the initial position of the particle be A. P and Q are the positions of the particle at any instants of time *t* and *t* + *dt* respectively. Suppose the particle traverses a distance *ds* along the circular path in time interval *dt.* During this interval, it moves through an angle $d\theta = \theta_2 - \theta_1$. *The angle swept by the radius vector at a given time is called the angular displacement of the particle.*

If *r* be the radius of the circle, then the angular displacement is given by $d\theta = ds / r$

The angular displacement is measured in terms of radian.

Angular velocity

The rate of change of angular displacement is called the angular velocity of the particle

Let $d\theta$ be the angular displacement made by the particle in time *dt* , then the angular velocity of the particle is

$$\omega = \frac{d\theta}{dt}$$

Its unit is rad s^{-1} and dimensional formula is T^{-1}.

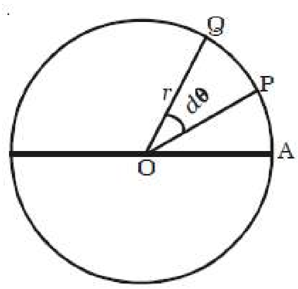
For one complete revolution, the angle swept by the radius vector is 360° or 2π radians. If *T* is the time taken for one complete revolution, known as period, then the angular velocity of the particle is

$$\omega = \frac{2\pi}{T}$$

If particle makes n revolution per second then frequency is n

7.2 Relation between linear velocity and angular velocity

Let us consider a body P moving along the circumference of a circle of radius *r* with linear velocity *v* and angular velocity ω as shown in Fig.. Let it move from P to Q in time *dt* and *d*θ be the angle swept by the radius vector.



Let PQ = *ds,* be the arc length covered by the particle moving along the circle, then the angular displacement *d* θ is expressed as

$$d\theta = \frac{ds}{r}$$

But ds = v dt

$$\therefore d\theta = \frac{vdt}{r}$$

$$\therefore \frac{d\theta}{dt} = \frac{v}{r}$$

$$\therefore \omega = \frac{v}{r} \; or \; v = \omega r$$

In vector notion

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Thus, for a given angular velocity ω, the linear velocity *v* of the particle is directly proportional to the distance of the particle from the centre of the circular path (i.e) for a body in a uniform circular motion, the angular velocity is the same for all points in the body but linear velocity is different for different points of the body.

7.2 Angular acceleration

If the angular velocity of the body performing rotatory motion is non-uniform, then the body is said to possess angular acceleration.

The rate of change of angular velocity is called angular acceleration.

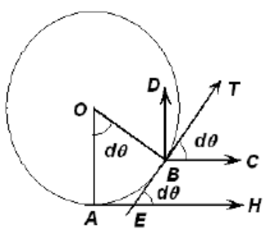
If the angular velocity of a body moving in a circular path changes from ω_1 to ω_2 in time *t* then its angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

The angular acceleration is measured in terms of rad s^{-2} and
its dimensional formula is T^{-2}.

7.3 Centripetal acceleration



The speed of a particle performing uniform circular motion remains constant throughout the motion but its velocity changes continuously due to the change in direction (i.e) the particle executing uniform circular motion is said to possess an acceleration.

Consider a particle executing circular motion of radius *r* with linear velocity *v* and angular velocity ω. The linear velocity of the particle acts along the tangential line. Let *d*θ be the angle described by the particle at the centre when it moves from A to B in time *dt*.

At A and B, linear velocity *v* acts along

AH and BT respectively. In Fig. ∠*AOB* = *d*θ = ∠*HET* (∵ angle subtended by the two radii of a circle = angle subtended by the two tangents).

The velocity *v* at B of the particle makes an angle dθ with the line BC and hence it is resolved horizontally as *v cos d*θ along BC and vertically as *v* sin *d* θ along BD.

∴ The change in velocity along the horizontal direction = *v cos d*θ −*v*

If *d*θ is very small, *cos d*θ = 1

∴Change in velocity along the horizontal direction = *v* − *v* = 0

(i.e) there is no change in velocity in the horizontal direction.

The change in velocity in the vertical direction (i.e along AO) is

dv = *v sin d*θ − *0* = *v sin d*θ

If *d*θ is very small, *sin d*θ = *d*θ

∴ The change in velocity in the vertical direction (i.e) along radius of the circle *dv* = *v.d*θ

But linear acceleration

$$a = \frac{dv}{dt} = v\frac{d\theta}{dt} = v\omega$$

We know that *v* = *r* ω

$$a = \frac{v^2}{r}$$

Hence, the acceleration of the particle producing uniform circular motion is along AO (i.e) directed towards the centre of the circle. This acceleration is directed towards the centre of the circle along the radius and perpendicular to the velocity of the particle. This acceleration is known as centripetal or radial or normal acceleration.

Solved Numerical

Q1) A particle moves in a circle of radius 20cm. Its linear speed at any time is given by v = 2t where v is in m/s and t is in seconds. Find the radial and tangential acceleration at t = 3 sec and hence calculate the total acceleration at this time

Solution

The linear speed at 3 sec is v = 2×3 = 6 m/s

The radial acceleration at 3 sec

$$a_r = \frac{v^2}{r} = \frac{6\times 6}{0.2} = 180\ ms^{-2}$$

The tangential acceleration is given by dv/dt

Thus tangential acceleration $a_t = 2$ as v = 2t

Total acceleration

$$\sqrt{a_r^2 + a_t^2} = \sqrt{180^2 + 2^2} = \sqrt{32404}\ ms^{-2}$$

7.4 Centripetal force

According to Newton's first law of motion, a body possesses the property called directional inertia (i.e.) the inability of the body to change its direction. This means that without the application of an external force, the direction of motion cannot be changed. Thus when a body is moving along a circular path, some force must be acting upon it, which continuously changes the body from its straight-line path (Fig 2.40). It makes clear
that the applied force should have no component in the direction of the motion of
the body or the force must act at every point perpendicular to the direction of motion of the body.
This force, therefore, must act along the radius and should be directed towards the
centre. Hence for circular motion, a constant force should act on the body, along the radius towards the centre and perpendicular to the velocity of the body. This force is known as centripetal force.
If m is the mass of the body, then the magnitude of the centripetal force is given by
F = mass × centripetal acceleration

$$F = \frac{mv^2}{r} = mr\omega^2$$

Examples

Any force like gravitational force, frictional force, electric force, magnetic force etc. may act as a centripetal force. Some of the examples of centripetal force are :
(i) In the case of a stone tied to the end of a string whirled in a circular path, the centripetal force is provided by the tension in the string.
(ii) When a car takes a turn on the road, the frictional force between the tyres and the road provides the centripetal force.
(iii) In the case of planets revolving round the Sun or the moon revolving round the earth, the centripetal force is provided by the gravitational force of attraction between them
(iv) For an electron revolving round the nucleus in a circular path, the electrostatic force of attraction between the electron and the nucleus provides the necessary centripetal force.

7.5 Non uniform circular motion

If the speed of the particle moving in a circle is not constant the acceleration has both radial and tangential components. The radial and tangential accelerations are

$$a_r = \frac{v^2}{r} \quad and\ a_t = \frac{dv}{dt}$$

The magnitude of the resultant acceleration will be

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Solved numerical

Q2) A certain string which is 1m long will break if the load on it is more than 0.5kg. A mass of 0.05kg is attaché to one end of it and the particle is whirled round in a horizontal circle by holding the free end of the string by one hand. Find the greatest number of revolutions per minute possible without breaking the string

Solution

Centrifugal force = breaking force

$Mr\omega^2$ = breaking force (mg)

$0.05 \times 1 \times \omega^2 = 0.5 \times 9.8$

$\omega = \sqrt{98}$

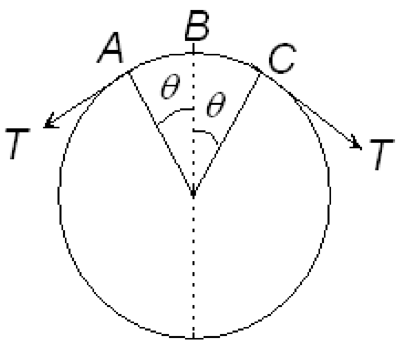
Now $\omega = 2\pi f$ here f = number of revolution per second

$2\pi f = 9.899$

f = 1.567 rev/sec

f = 1.567×60 = 94.02 rev/min

Q3) A metal ring of mass m and radius R is placed on a smooth horizontal table and is set to rotate about its own axis in such a way that each part of ring moves with velocity v. Find tension in the ring



Solution

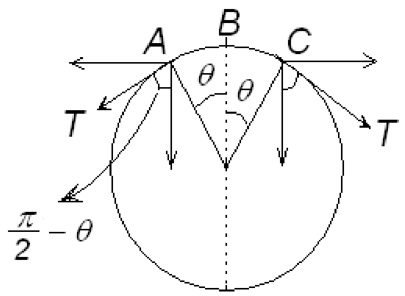
If we cut the ring along the imaginary diameter then left half will go left and right half will go on right side . Thus tension on left and right are as shown in figure

Now let the mass of small segment ABC be Δm

$$\Delta m = \frac{m}{2\pi R} R(2\theta) \quad --- \text{eq}(1)$$

As the elementary portion ACB moves in a circle of radius R at speed v its acceleration towards centre is

$$\frac{(\Delta m)v^2}{R}$$



Resolving the tension along x and y axis. We have considered BAC as very small segment then vertically down ward components will be along the radius. Producing tension on the ring

Thus

$$T\cos\left(\frac{\pi}{2} - \theta\right) + T\cos\left(\frac{\pi}{2} - \theta\right) = \frac{(\Delta m)v^2}{R}$$

$$2Tsin\theta = \frac{(\Delta m)v^2}{R} \quad ----eq(2)$$

Putting value of Δm from equation (1) in equation(2)

$$2Tsin\theta = \frac{\mathrm{m}}{2\pi\mathrm{R}}\mathrm{R}(2\theta)\frac{v^2}{R}$$

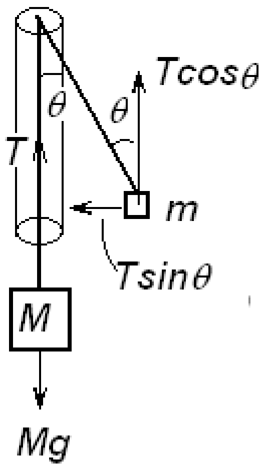
$$T = \frac{\mathrm{m}}{2\pi}\left(\frac{2\theta}{2\sin\theta}\right)\frac{v^2}{R}$$

Since θ is very small

$$\frac{2\theta}{2\sin\theta} = 1$$

$$T = \frac{\mathrm{m}v^2}{2\pi R}$$

Q4) A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in figure. The mass m moves around in a circular path, which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is l and θ is the angle this length makes with vertical. What should be the frequency of rotation of mass m so that M remains stationary



Solution

Let m follows a circular path of radius r and angular frequency ω

The force acting on M is Mg downwrds

Thus tension in string T = Mg -----(1)

Force acting on m for circular motion is $mr\omega^2$ from diagram

Component of tension in string towards centre is Tsinθ

Thus $T\sin\theta = mr\omega^2$ -------(2)

From eq(1) and eq(2)

$$Mg\sin\theta = mr\omega^2$$

$$\omega = \sqrt{\frac{Mgsin\theta}{mr}} \quad ----eq(3)$$

From the geometry of figure r = lsinθ thus

$$\omega = \sqrt{\frac{Mgsin\theta}{mlsin\theta}} = \sqrt{\frac{Mg}{ml}}$$

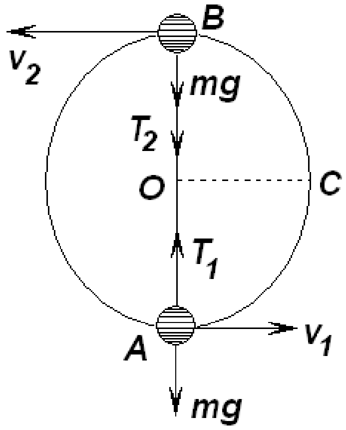
Now ω=2πf

$$2\pi f = \sqrt{\frac{Mg}{ml}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$$

7.6 Motion in vertical circle

Body is suspended with the help of a string.



Imagine an arrangement like a simple pendulum where a mass m is tied to a string of length r, the other end of the string being attached to a fixed point O.

Let the body initially lie at the equilibrium position A. Let it be given a velocity v_1 horizontally. If the velocity is small then the body would make oscillations in the vertical plane like a simple pendulum

Let us now find the velocity of projection v_1 at the lowest point A required to make the body move in a vertical circle.

The following points can be noted while considering the motion

(i)The motion along the vertical circle is non-uniform since velocity of body changes along the curve. Hence the centripetal acceleration must also change.

(ii)The resultant force acting on the body (in the radial direction) provides the necessary centripetal force.

(iii) The velocity decreases as the body up from A to B, the topmost point

When the body is at A, the resultant force acting on it is T_1 – mg, where T, is the tension in the string The centripetal force

$$T_1 - mg = \frac{mv_1^2}{r}$$

$$T_1 = mg + \frac{mv_1^2}{r} \quad ---eq(1)$$

The tension is always positive (even if v_1 is zero). Therefore, the string will be taut when is at A. The condition for the body to complete the vertical circle is that the string should be taut all time. i.e. the tension is greater than zero

The region where string is most likely become slack is about the horizontal radius OC. Also the tension would become the least when it is the topmost point B. Let it be T_2 at B and the velocity of the body be v_2. The resultant force on the body at B is

$$T_2 + mg = \frac{mv_2^2}{r}$$

$$T_2 = \frac{mv_2^2}{r} - mg \quad ----eq(2)$$

If the string is to taut at B, $T_2 \geq 0$

$$\frac{mv_2^2}{r} - mg \geq 0$$

$$v_2^2 \geq rg$$

$$v_2 \geq \sqrt{rg} \quad ---eq(3)$$

At A, the Kinetic Energy of the body

$$\frac{1}{2}mv_1^2$$

Potential energy of the body = 0

Total energy at point A

$$\frac{1}{2}mv_1^2$$

At B, the kinetic energy of the body =

$$\frac{1}{2}mv_2^2$$

Potential energy at B = mg(2r)

Total energy at B

$$2mgr + \frac{1}{2}mv_2^2$$

Using the principle of conservation of energy, we have

$$\frac{1}{2}mv_1^2 = 2mgr + \frac{1}{2}mv_2^2$$

$$v_1^2 = v_2^2 + 4gr \quad ----eq(4)$$

Combining equations (3) and (4)

$$v_1^2 \geq rg + 4gr$$

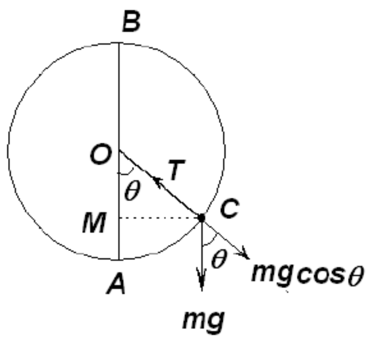
$$v_1 \geq \sqrt{5gr}$$

Hence, if the body has minimum velocity of √(5gr) at the lowest point of vertical circle, it will complete the circle.

The particle will describe complete circle if both v_2 and T_2 do not vanish till the particle reaches the highest point.

Important points

(i)If the velocity of projection at the lowest point A is less than √(2gr), the particle will come to instantaneously rest at a point on the circle which lies lower than the horizontal diameter. It will then move down to reach A and move on to an equal height on the horizontal diameter. It will then move down to reach A and move on to an equal height on the other side of A. Thus the particles execute oscillations. In case v vanishes before T does We may find an expression for tension in the string when it makes an angle θ with the vertical. At C, the weight of the body acts vertically downwards, and the tension in the string is towards the centre O.



The weight mg is resolved radially and tangentially.

The radial component in mgcosθ and the tangential component is mgsinθ

The centripetal force is T - mgcosθ

$$\mathrm{T} - \mathrm{mgcos\theta} = \frac{\mathrm{mv}^2}{\mathrm{r}}$$

Where v is the velocity at C

$$\therefore T = m\left(\frac{v^2}{r} + gcos\theta\right) \quad ----eq(1)$$

The velocity can be expressed in terms of v_1 at A

The total energy at A

$$\frac{1}{2}mv_1^2$$

The kinetic energy at C

$$\frac{1}{2}mv^2$$

Potential energy at C = mg(AM)

= mg (AO-MO)

=mg(r – r cosθ)

=mgr(1-cosθ)

Total energy at C =

$$\frac{1}{2}mv^2 + mgr(1 - cos\theta)$$

From conservation of energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1 - cos\theta)$$

$$v_1^2 = v^2 + 2gr(1 - cos\theta)$$

$$v^2 = v_1^2 - 2gr(1 - cos\theta)$$

Substituting value of v in equation (1)

$$T = m\left(\frac{v_1^2 - 2gr(1 - cos\theta)}{r} + gcos\theta\right)$$

$$T = \frac{mv_1^2}{r} + 3mg\left(cos\theta - \frac{2}{5}\right) \quad -----eq(2)$$

This expression gives the value of the tension in the string in terms of the velocity at the lowest point and the angle θ

Equation (1) shows that tension in the string decreases as θ increases, since the term gcosθ decreases as θ increases

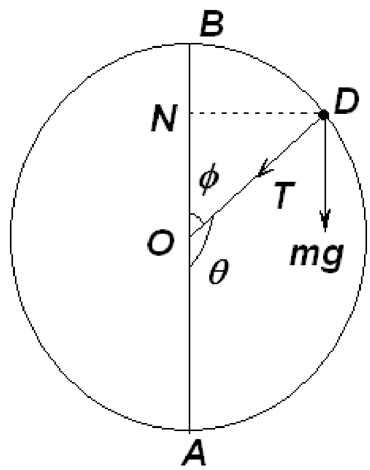
When θ is 90^{O}, cosθ = 0 and

$$T_H = \frac{mv^2}{r}$$

This is obvious because, the weight is vertically downwards where as the tension is horizontal. Hence the tension alone is the centripetal force

(ii) If the velocity of projection is greater than √(2gr) but less than √(5gr), the particle rises above the horizontal diameter and the tension vanishes before reaching the highest point.

(iii) if we make θ an obtuse angle.



At point D, the string OD makes an angle φ with vertical. The radial component of the weight is mgcosφ towards the centre O

$$T + mgcos\varphi = \frac{mv^2}{r}$$

$$T = m\left(\frac{v^2}{r} - gcos\varphi\right) \quad - - - - - eq(1)$$

Kinetic energy at D

$$\frac{1}{2}mv^2$$

Potential energy at D = mg(AN)

=mg(AO+ON)

= mg(r+rcosφ)

= mgr(1+cosφ)

From conservation of energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1 + cos\varphi)$$

$$v^2 = v_1^2 - 2mgr(1 + cos\varphi)$$

Substituting value of v² in eq(1) we get

$$T = m\left(\frac{v_1^2 - 2mgr(1 + cos\varphi)}{r} - gcos\varphi\right)$$

$$T = m\left[\frac{v_1^2}{r} - 2g(1 + cos\varphi) - gcos\theta\right]$$

$$T = m\left[\frac{v_1^2}{r} - 3g\left(cos\varphi + \frac{2}{3}\right)\right]$$

This equation shows that the tension becomes zero if

$$\frac{v_1^2}{r} = 3g\left(cos\varphi + \frac{2}{3}\right) \quad - - - - eq(2)$$

If the tension is not to become zero

$$> 3g\left(cos\varphi + \frac{2}{3}\right)$$

Equation (2) gives the value of φ at which the string becomes slack

$$cos\varphi + \frac{2}{3} = \frac{v_1^2}{3rg}$$

$$cos\varphi = \frac{v_1^2}{3rg} - \frac{2}{3}$$

7.7 A body moving inside a hollow tube or sphere

The same discussion holds good for this case, but instead of tension in the sting we have the normal reaction of the surface. If N is the normal reaction at the lowest point, then

$$N - mg = \frac{mv_1^2}{r}$$

$$N = m\left(\frac{v_1^2}{r} + g\right)$$

At highest point of the circle

$$N + mg = \frac{mv_1^2}{r}$$

$$N = m\left(\frac{v_1^2}{r} - g\right)$$

The condition $v_1 \geq \sqrt{(5rg)}$ for the body to complete the circle hold for this also

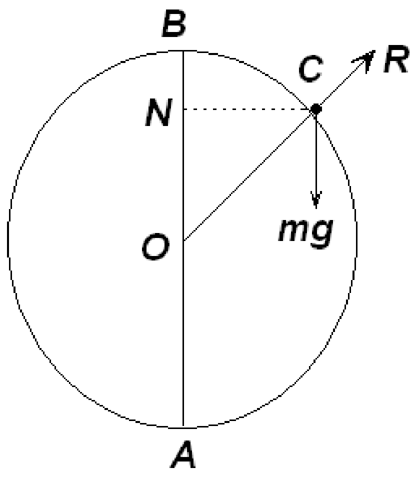
All other equations (can be) similarly obtained by replacing T by reaction R

7.8 Body moving on a spherical surface

The small body of mass m is placed on the top of a smooth sphere of radius r.

If the body slides down the surface, at what point does it fly off the surface?

Consider the point C where the mass is, at a certain instant. The forces are the normal reaction R and the weight mg.



The radial component of the weight is mgcosφ acting towards the centre. The centripetal force is

$$mgcos\varphi - R = \frac{mv^2}{r}$$

Where v is the velocity of the body at O

$$R = m\left(gcos\varphi - \frac{v^2}{r}\right)$$

The body flies off the surface at the point where R becomes zero

$$gcos\varphi = \frac{v^2}{r}$$

$$cos\varphi = \frac{v^2}{gr} \qquad -----eq(1)$$

To find v, we can use conservation of energy

$$\frac{1}{2}mv^2 = mg(BN)$$

$$= mg(OB - ON)$$

$$= mg(r - r\cos\varphi)$$

$$v^2 = 2rg(1-\cos\varphi)$$

$$\frac{v^2}{gr} = 2(1 - \cos\varphi) \quad ----eq(2)$$

From equation(1) and (2)

$$\cos\varphi = 2-2\cos\varphi$$

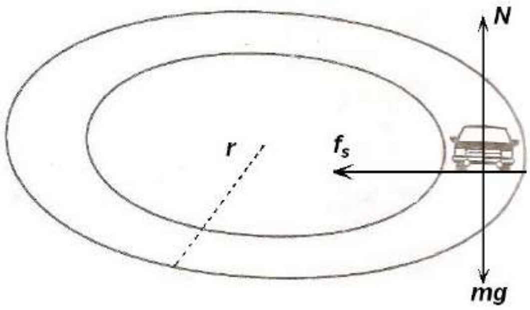
$$\cos\varphi = 2/3$$

This gives the angle at which the body goes of the surface. The height from the ground to that point = AN = r(1+cosφ)

Height from ground = r(1+2/3) = (5/3) r

7.9 Motion of a vehicle on a level circular path

Vehicle can move on circular path safely only if sufficient centripetal force is acting on the vehicle. Necessary centripetal force is provided by friction between tyres of the vehicle



Let v be the velocity of vehicle.

Forces on vehicle are

(1)Weight of vehicle (mg in downward direction)

(2) The normal reaction N by road in upward

(3) The frictional force f_S by road – parallel to the surface of the road

Since the vehicle has no acceleration in vertical direction

N-mg = 0

N = mg

The required centripetal force for the circular motion of the vehicle on this road must be provided by the friction f_S

$$f_s = \frac{mv^2}{r}$$

But $f_s = \mu N = \mu mg$

From this we can say that for safe speed on the road

Centripetal force ≤ maximum friction

$$\frac{mv^2}{r} \le \mu mg$$
$$v^2 \le \mu rg$$

And maximum safe speed

$$v_{max} = \sqrt{\mu rg}$$

If the speed of the vehicle is more than this v_{max} it will be thrown away from the road. Note that safe speed is independent of mass of vehicle

Solved Numerical

Q5) A car goes on a horizontal circular road of radius R, the speed increases at a rate dv/dt. The coefficient of friction between road and tyre is μ. Find the speed at which the car will skid

Solution

Here at any time t , the speed of car becomes v, the net acceleration in the plane of road is

$$\sqrt{\left(\frac{v^2}{r}\right)^2 + a^2}$$

This acceleration provided by frictional force

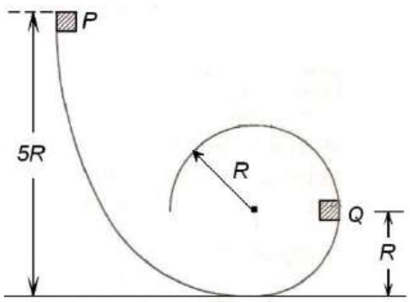
At the moment car will side if

$$M\sqrt{\left(\frac{v^2}{r}\right)^2 + a^2} = \mu Mg$$

$$v = [R^2(\mu^2 g^2 - a^2)]^{1/4}$$

Q6) A small block of mass m slides along the frictionless loop-to-loop track shown in figure. a)It starts from rest at P, what is the resultant force acting on it at Q (b) At what height above the block be released so that the force it exerts against the track at the top of the loop equals its weight?

Solution



Difference between Point P and Q is 4R

Thus according to law of conservation of energy

K.E. at Q = Loss of P.E

$$\frac{1}{2}mv^2 = mg\,(4R)$$

$$V^2 = 8gR$$

At Q, the only force acting on the block is mg downward and Normal of the track which is in radial direction provides centripetal force for circular motion.

$$N = \frac{mv^2}{R} = \frac{m \times 8gR}{R} = 8mg$$

The loop must exert a force on the block equal to eight time the block's weight

(b) For the block to exert a force equal to its weight against the track at the top of the loop

$$\frac{mv'^2}{R} = 2mg$$

$$v'^2 = 2gR$$

(b) block exerts the force equal to mass of the block on at the highest point of circular loop

At highest point centrifugal force acts outward and gravitational force acts down wards. Thus to get force exerted by the block on loop

F = centrifugal – gravitational force

$$mg = \frac{mv'^2}{R} - mg$$

$$2mg = \frac{mv'^2}{R}$$

$$v'^2 = 2gR$$

Thus Kinetic energy

$$\frac{1}{2}mv'^2 = mgR$$

Let the block be released from height H then loss of potential energy = mg(H-R)

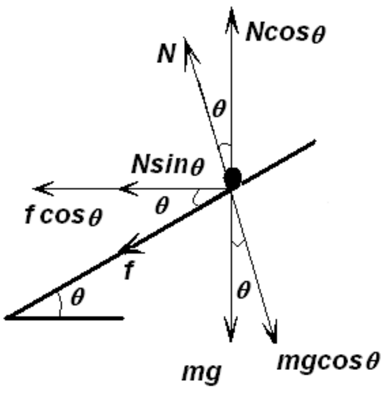
$$\therefore mg(H-R) = mgR$$

$$H = 2R$$

Thus block must be released from the height of 2R

7.10 Motion of vehicle on Banked road

To take sharp turns friction of the road may not be sufficient. If at the curvature of road if the road is banked then in required centripetal force for circular motion, a certain contribution can be obtained from the normal force (N) by the road and the contribution of friction can be decreased to certain extent.



In figure the section of road with the plane of paper is shown. This road is inclined with the horizontal at an angle θ. The forces acting on the vehicle are also shown in figure.

Forces on vehicle

(1) Weight (mg) downward direction

(2) Normal force(N) perpendicular to road

(3) frictional force parallel to road

As the acceleration of vehicle in the vertical direction is zero

$N\cos\theta = mg + f\sin\theta$

$\therefore mg = N\cos\theta - f\sin\theta$ --------eq(1)

In the horizontal direction, the vehicle performs a circular motion. Hence it requires centripetal force., which is provided by the horizontal components of f and N

$$\frac{mv^2}{r} = Nsin\theta + fcos\theta \qquad -----eq(2)$$

Dividing equation (2) by (1) we get

$$\frac{v^2}{rg} = \frac{Nsin\theta + fcos\theta}{Ncos\theta - fsin\theta}$$

Now f = μN thus

$$\frac{v^2}{rg} = \frac{Nsin\theta + \mu Ncos\theta}{Ncos\theta - \mu Nsin\theta}$$

$$\frac{v^2}{rg} = \frac{sin\theta + \mu cos\theta}{cos\theta - \mu sin\theta}$$

Dividing numerator and denominator by cosθ

$$\frac{v^2}{rg} = \frac{tan\theta + \mu}{1 - \mu tan\theta}$$

$$v = \sqrt{rg\left[\frac{tan\theta + \mu}{1 - \mu tan\theta}\right]}$$

Cases: (i) If μ = 0 then

$$v_0 = \sqrt{rgtan\theta}$$

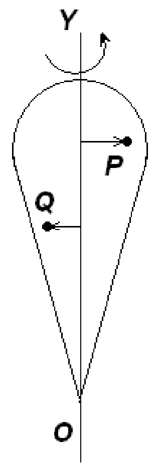
If we drive the vehicle at this speed on the banked curved road the contribution of friction becomes minimum in the enquired centripetal force, and hence wear and tear of the tyres can be minimized. This speed v_o is called the optimum speed

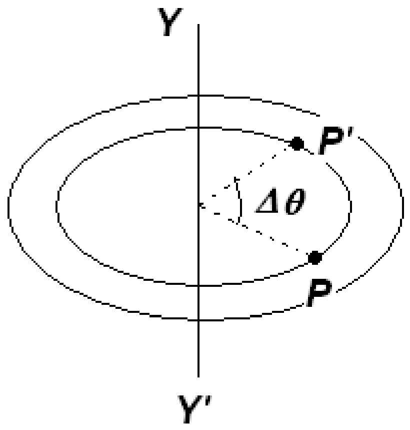
(ii) If $v < v_O$ then the frictional force will act towards the higher (upper edge of the banked road. The vehicle can be kept stationary. It can be parked on banked road, only if tanθ ≤ μ

7.11 ROTATIONAL MOTION OF RIGID BODIES

A rigid body is a body whose deformation is negligible when suspended to external force , in a rigid body the distance between any two points remains constant. A rigid body can undergo various types of motion. It may translate, rotate or any translate and rotate at the same time When a rigid body translates each particle of rigid body undergoes same displacement, has same velocity and same acceleration. To apply equation of translation, all its mass can be considered at centre of mass and we can use $\mathbf{F}_{ext} = M\mathbf{a}_{CM}$

Rotation





If all the particles of a rigid body perform circular motion and the centres of these circles are steady on a definite straight line called axis of rotation it is a geometrical line and the motion of the rigid body is called the rotational motion

In figure two particles P and Q of rigid body are shown. The rigid body rotates about axis OY. The circular paths of particles P and Q are in the plane perpendicular to axis of motion OY

Consider a rigid body undergoes rotation about axis YY'. A point P of the rigid body is undergoing circular motion of radius OP = r. If during a time interval Δt , the body rotates through an angle $\Delta\theta$, the arc PP' will subtend an angle $\Delta\theta$ at the centre of motion of its circular path. $\Delta\theta$ is angular displacement of rigid body as well as of point P.

Average angular speed during a time interval Δt is defined as

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular speed is defined as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

If the body rotates through equal angles in equal intervals of time, it is said as rotating uniformly. But if it's angular speed changes with time it is said to be accelerated and its angular acceleration is defined as

Average angular acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

In case of uniform rotation, angular displacement $\Delta\theta = \omega t$

In case of uniformly accelerated rotation following kinematic relations we use

$$\omega = \omega_0 + \alpha t$$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Linear velocity and linear acceleration of a particle in rigid body

Linear velocity and angular velocity is related by following relation

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Direction of ω is along the axis of rotation

On taking derivative of above equation we get

$$\vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$$

Thus linear acceleration have two components we find the direction using right hand screw rule

Radial component $\vec{\omega} \times \vec{v}$

Tangential component $\vec{\alpha} \times \vec{r}$

Magnitude Since angle between ω and v, α and r is π/2 and v = ωr

$$a = \sqrt{(\omega^2 r)^2 + \alpha^2 r^2}$$

If resultant acceleration makes an angle β with Op, the

$$tan\beta = \frac{\alpha r}{\omega^2 r}$$

If the rigid body is rotating with constant angular velocity, that is, its angular acceleration α=0, then the tangential component of its linear acceleration becomes zero, but radial component remains non-zero. This condition is found in the uniform circular motion

Solved Numerical

Q7) Grind stone is rotating about axis have radius 0.24 m its angular acceleration is 3.2 rad/s^2. (a)Starting from rest what will be angular velocity after 2.7 s (b) Linear tangential speed of point of the rim (c) the tangential acceleration of a point on the rim and (d) the radial acceleration of point on the rim at the end of 2.7 s.

Solution

(a) Angular speed after 2.7s

$$\omega = \omega_0 + \alpha t$$

ω =0 + 3.2 × 2.7 = 8.6 rad/s = 1.4/s

(b) Linear tangential speed

V_t = ωr

V_t = 8.6 × 0.24 = 2.1 m/s

(c) Tangential acceleration

a_t = αr

$a_t = 3.2 \times 0.24 = 0.77\ m/s^2$

(d) $a_r = \omega^2 r$

$a_r = (8.6)^2 \times 0.24 = 18\ m/s^2$

7.12 Moment of force

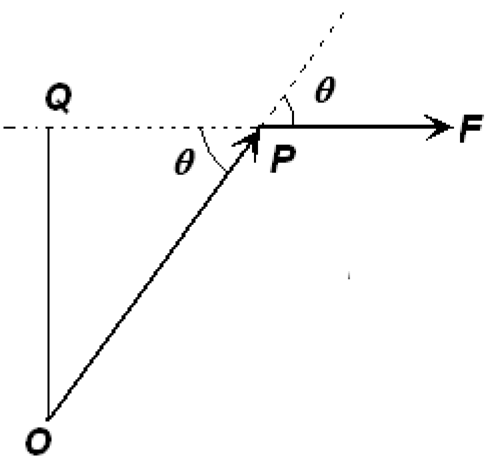
To translate a body, we need to apply a force on a body. i.e. cause of translation is force and it is related with linear acceleration of the body as **F** = M**a**

But for rotation, not only the magnitude of the force but its line of action and point of application is also important. Turning effect of the force depends on

(i) Magnitude of the force

(ii) Direction of the force

(iii) The distance of force from the axis of rotation



Taking consideration of all these we define the torque of a force which gives measure of a turning effect of a force Consider a force F acting on a body at point P, then turning effect of this force, torque, about point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where r is vector joining O to any point on the line of action of force

$$\tau = rFsin\theta\hat{n}$$

Here $\hat{n}$ is a unit vector perpendicular to plane formed by r and F and direction is given by right hand screw rule

Magnitude is given by $\tau = rF\sin\theta$

$\tau = F(OQ)$

Hence torque about a point can also be calculated by multiplying force with the perpendicular distance from the point on the line of action of the force. Direction of torque can be obtained by the definition of cross product.

To calculate torque of a force about an axis, we consider a point on the axis and then we define $\vec{\tau} = \vec{r} \times \vec{F}$ about point O

The component of vector **τ along** the axis gives the torque about the axis. If force is parallel to the axis or intersects the axis, its torque about the axis becomes zero. If a force is perpendicular to the axis, we calculate torque as product of magnitude of force and perpendicular distance of line of action of force from axis.

Torque acting on the system of particles:

The mutual internal forces between the particles of system are equal and opposite, the resultant torques produced due to them becomes zero. Hence we will not consider the internal force in our discussion

Suppose for a system of particles the position vectors of different particles are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, … $\mathbf{r}_n$. and respective forces acting on them are $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$,……, $\mathbf{F}_n$

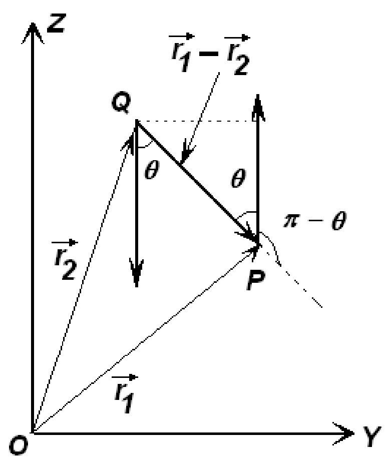
The resultant torque on the system means the vector sum of the torque acting on every particle of the system

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_n$$

Resultant torque

$$\tau = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$$

7.13 Couple



Two forces of equal magnitude and opposite directions which are not collinear form couple. A shown in figure F_1 and F_2 act on two particles P and Q of the rigid body having position vectors r_1 and r_2 respectively. Here $|F_1|= |F_2|$ and the directions are mutually opposite. The resultant torques τ_1 and τ_2 produce moment of couple τ

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$$

$$\vec{\tau} = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2)$$

$$\vec{\tau} = (\vec{r}_1 \times \vec{F}_1) - (\vec{r}_2 \times \vec{F}_1)$$

As F_2 =- F_2

$$\vec{\tau} = (\vec{r}_1 - \vec{r}_1) \times \vec{F}_1$$

$$\vec{\tau} = |\vec{r}_1 - \vec{r}_1|(F_1) sin(\pi - \theta)$$

Where (п – θ) is the angle between ($\mathbf{r}_1$ – $\mathbf{r}_2$) and F_1

$$\vec{\tau} = |\vec{r}_1 - \vec{r}_1|(F_1) sin\theta$$

From figure $|r_1 - r_2|\sin\theta$ = perpendicular distance between the two forces

∴ Moment of couple = (magnitude of any one of the two forces) (perpendicular distance between the forces)

Equilibrium

If the external forces acting on a rigid body are $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$… $\mathbf{F}_n$. and if resultant force

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 +$ ……. $+\mathbf{F}_n = 0$, then rigid body remains in translational equilibrium.

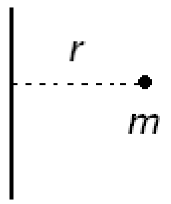
If the torques produced by the above mentioned forces are $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$… $\boldsymbol{\tau}_n$. Then the rigid body remains in rotational equilibrium when $\tau = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 +$ ……$+ \boldsymbol{\tau}_n = 0$

That is if rigid body is stationary, it will remain stationary and if it is performing rotational motion, it will continue rotational motion with constant angular velocity.

7.14 Moment of inertia

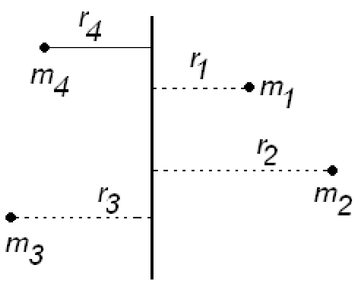
The state of motion of a body can undergo change in rotation if torque is applied. The resulting angular acceleration depends partly on the magnitude of the applied torque, however the same torque applied different bodies produce different angular acceleration, indicating that each body has an individual amount of rotational inertia is called moment of inertia and it is represented by I.

The moment of inertia of body is a function of the mass of the body, the distribution of the mass and the position of the axis of rotation



Consider a particle of mass m situated at a distance r from the axis as shown in the figure. Its moment of inertia I is defined as

I = mr^2



If a system of particles is made of number of particles of masses m_1, m_2, m_3, ……, m_n at distances r_1, r_2, r_3, ….. , r_n from the axis of rotation its momentum of inertia is defined as

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2$$

$$I = \sum_{i=1}^{n} m_ir_i^2$$

7.15 Moment of inertia of continuous body

For calculating moment of inertia of continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice depends on symmetry of the body. Consider an element of the body at a distance r from the axis of rotation. The moment of inertia of this element about the axis can be defined as $(dm)r^2$ and the discrete sum over particles becomes integral over the body $I = \int (dm)r^2$

Radius of gyration:

Suppose a rigid body has mass M. It is made up of n particles each having mass m

∴ $m_1 = m_2 = \ldots.. = m_n = m$

∴ M = nm

As shown in figure, the moment of inertia of the body the given axis

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2$$

Here r_1, r_2, r_3 , ….,r_n are perpendicular distances of the respective particles of the body from the axis

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$

$$I = \frac{nm(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2)}{n}$$

$$I = MK^2$$

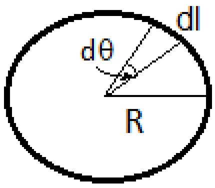
Where

$$K^2 = \frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}}$$

K^2 show the mean (average) value of the squares of the perpendicular distance of the particles of the body from the axis. K is called the radius of gyration of the body about the given axis. SI unit is 'm'

7.15.01 Moment of inertia of circular ring about axis passing through Centre of mass and perpendicular to plane of ring



Let M be mass of ring having radius R, then mass per unit length $\lambda = M/2\pi R$

Length of small segment $dl = Rd\theta$.

Mass of small segment $dm = \lambda\, dl$

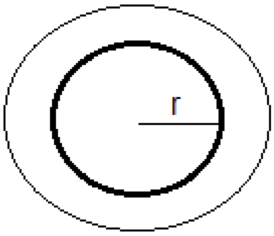
$$dm = \frac{M}{2\pi R}Rd\theta = \frac{M}{2\pi}d\theta$$

Momentum of inertia of small element about center $dI = dmR^2$

$$dI = \frac{MR^2}{2\pi}d\theta$$

$$I = \frac{MR^2}{2\pi}\int_0^{2\pi} d\theta = MR^2$$

7.15.02 Moment of inertia of disc about axis passing through Centre of mass and perpendicular to plane of disc



Let M be the mass of disc having radius R.

Mass per unit area $\sigma = \frac{M}{\pi R^2}$

Disc can be imagined as composed of concentric rings of thickness dr.

One such ring is show in figure. Let its radius be r and mass dm.

Volume of ring = 2πrdr

Mass of ring = Volume × density

Mass of ring $dm = \frac{M}{\pi R^2} 2\pi\mathrm{rdr} = \frac{2M}{R^2} rdr$

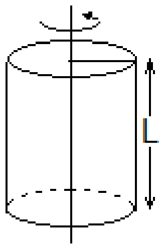
We know that moment of inertia of ring dmr^2

Thus moment of inertia of disc

$$I = \frac{2M}{R^2}\int_0^R rdrr^2 = \frac{2M}{R^2}\int_0^R r^3 dr$$

$$I = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}MR^2$$

7.15.03 Moment of inertia of hollow cylinder about geometric axis



Consider a hollow cylinder of radius R and length L, and mass M

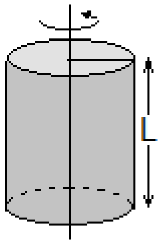
Hollow cylinder may be consider as stack of rings. If mass of each ring is dm then moment of inertia is dmR^2

If we integrate over the length as R^2 is constant integration of dm is M

Thus moment of inertia of hollow cylinder is same as ring

$$I = MR^2$$

7.15.04 Moment of inertia of Solid cylinder about geometric axis



Consider a solid cylinder of radius R and length L, and mass M

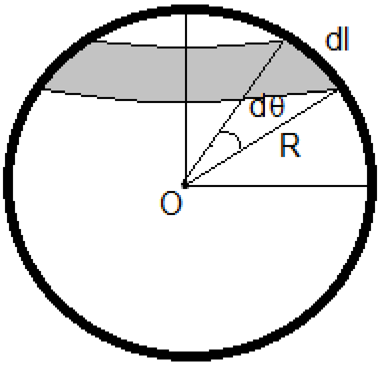
Solid cylinder may be consider as stack of rings. If mass of each ring is dm then moment of inertia is $\frac{1}{2}dmR^2$

If we integrate over the length as R^2 is constant integration of dm is M

Thus moment of inertia of solid cylinder is same as ring

$$I = \frac{1}{2}MR^2$$

7.15.05 Moment of inertia of hollow sphere about diameter



Let M be the mass of hollow sphere having radius R, rotating around the diameter. In figure axis is passing through O and is perpendicular to plane of paper. Surface density of mass $\frac{M}{4\pi R^2}$

Consider a very small element of length dl will trace a ring of radius r shown in figure have area A = dl×2πr , have mass $dm = \frac{M}{4\pi R^2}dl \times 2\pi r = \frac{M}{4\pi R^2}Rd\theta \times 2\pi r$

Moment of inertia of ring dmr^2

$$I = \int_0^{\pi/2} \frac{M}{4\pi R^2}Rd\theta \times 2\pi r \times r^2$$

$$I = \int_0^{\pi/2} \frac{M}{2R}d\theta \times r^3$$

Ring makes an angle of θ with O, then radius r= Rsinθ

Now ring makes angle of 0 to π/2 with center for upper hemisphere. For full sphere it will be twice of integration over 0 to π/2

$$I = 2\int_0^{\pi/2} \frac{M}{2R} d\theta d\theta R^3 \sin^3\theta$$

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} \sin^3\theta d\theta$$

As $\sin^3\theta = (1-\cos^2\theta)\sin\theta$

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} (1-\cos^2\theta) sin d\theta$$

Let cosθ = t

dt =sinθdθ

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} (1-t^2)dt$$

$$I = 2 \times \frac{MR^2}{2}\left[t - \frac{t^3}{3}\right]_0^{\pi/2}$$

$$I = 2 \times \frac{MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

As cos(π/2)=0

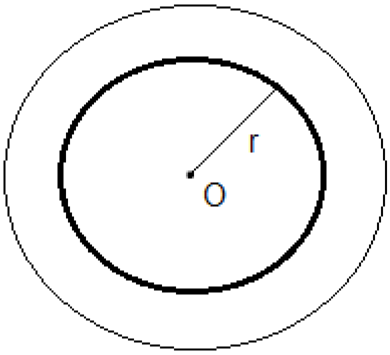
$$I = 2 \times \frac{MR^2}{2}\left[\cos 0 - \frac{\cos^3 0}{3}\right]$$

$$I = 2 \times \frac{MR^2}{2}\left[1 - \frac{1}{3}\right]$$

$$I = 2 \times \frac{MR^2}{2}\left[\frac{2}{3}\right] = \frac{2}{3}MR^2$$

$$I = \frac{2}{3}MR^2$$

7.15.06 Moment of inertia of solid sphere about diameter



Let M be the mass of solid sphere having radius R, rotating around the diameter. In figure axis is passing through O and is perpendicular to plane of paper. Volume density of sphere is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Solid sphere may be considered as composition of concentric small hollow sphere. Consider a hollow sphere as shown in figure of radius r and thickness dr. mass of sphere dm= volume × density.= $4\pi r^2 \times dr \times \rho$

$$dm = 4\pi r^2 dr \times \frac{3M}{4\pi R^3} = \frac{3M}{R^3} r^2 dr$$

Moment of inertia of hollow sphere =

$$dI = \frac{2}{3} dm r^2$$

By integrating over radius 0 to R we get I of solid sphere

$$I = \int_0^R \frac{2}{3} dm r^2$$

$$I = \frac{2}{3}\int_0^R r^2 dm = \frac{2}{3}\int_0^R r^2 \frac{3M}{R^3} r^2 dr$$

$$I = \frac{2M}{R^3}\int_0^R r^4 dr$$

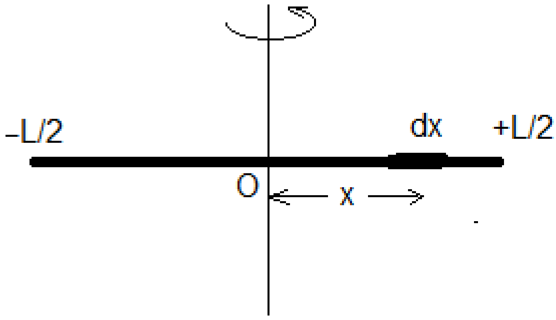
$$I = \frac{2M}{R^3}\int_0^R r^4 dr$$

$$I = \frac{2M}{R^3}\left[\frac{r^5}{5}\right]_0^R$$

$$I = \frac{2M}{R^3}\frac{R^5}{5}$$

$$I = \frac{2}{5}MR^2$$

7.15.07 Moment of inertia of rod about axis passing through center and perpendicular to length



Let length of thin rod be L and mass be M, rotating around axis perpendicular to length L and passing through center O. Liner mass density of rod λ= M/L

Consider a small segment of length dx , Mass of segment dm =Mdx/L

Moment of inertia of segment $dI \ = \frac{M}{L}x^2dx$

By integrating dm over −L/2 to +L/2 we get

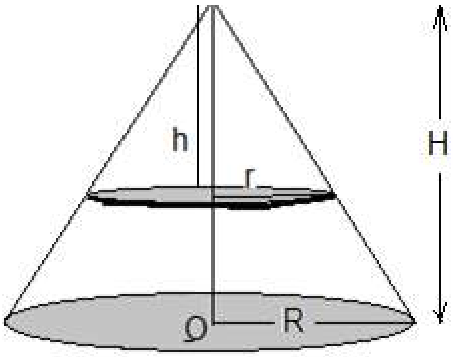
$$I = \int_{-L/2}^{+L/2}\frac{M}{L}x^2dx \ = \frac{M}{L}\int_{-L/2}^{+L/2}x^2dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

$$I = \frac{M}{3L}\left[\frac{L^3}{8} + \frac{L^3}{8}\right]$$

$$I = \frac{1}{12}ML^2$$

7.15.08 Moment of inertia of solid cone about axis passing through vertex and perpendicular to base of radius R and height H



Let M be the mass of cone and H and R bet the height and radius of cone.

Volume of cone

$$V = \pi R^2 \frac{H}{3}$$

Volume density of cone ρ

$$\rho = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

Consider a segmental disc of radius r and thickness dh mass of disc

$$dm = V \times \rho = \pi r^2 dh \times \rho$$

$$dm = \pi r^2 dh \times \frac{3M}{\pi R^2 H} = r^2 dh \times \frac{3M}{R^2 H} \ldots (i)$$

Moment of inertia of disc about axis perpendicular to plane

$$dI = \frac{1}{2} dm \times r^2 \ldots . (ii)$$

Substituting (i) in (ii)

$$dI = \frac{1}{2} r^2 dh \times \frac{3M}{R^2 H} \times r^2$$

$$dI = \frac{3}{2} \frac{M}{R^2 H} r^4 dh \ldots (iii)$$

Now

$$\frac{h}{r} = \frac{H}{R}$$

$$r = \frac{R}{H} h \ldots . (iv)$$

Substituting (iii) in (iv)

$$dI = \frac{3}{2} \frac{M}{R^2 H} \left(\frac{R}{H} h\right)^4 dh$$

$$dI = \frac{3}{2}\frac{MR^2}{H^5}(h)^4 dh$$

Integrating from 0 to H

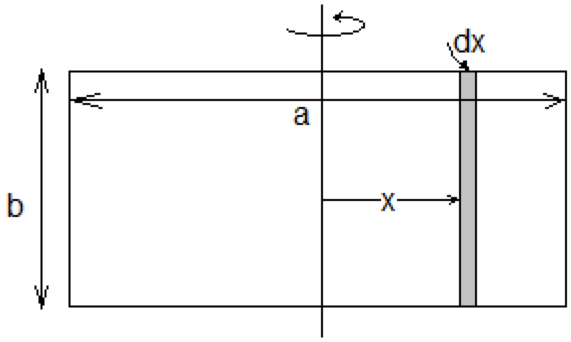
$$I = \frac{3}{2}\frac{MR^2}{H^5}\int_0^H (h)^4 dh$$

$$I = \frac{3}{2}\frac{MR^2}{H^5}\left[\frac{h^5}{5}\right]_0^H$$

$$I = \frac{3}{2}\frac{MR^2}{H^5}\frac{H^5}{5}$$

$$I = \frac{3}{10}MR^2$$

7.15.09 Moment of inertia of rectangular plate about axis passing through center of mass and parallel to plane



Let mass of the plate be M, sides are a and b as shown in figure.

Surface mass density $\sigma = \frac{M}{ab}$

Consider a strip element of thickness dx and height b, at position of x from axis

Mass of the strip dm= area ×density

$$dm = bdx \times \frac{M}{ab} = \frac{M}{a}dx$$

Moment of inertia dI = dmx^2 (as every point of strip is at distance x)

On integrating from −a/2 to a/2 moment of inertia along axis parallel to b

$$I_b = \int_{-a/2}^{a/2} \frac{M}{a}x^2 dx$$

$$I_b = \frac{M}{a}\int_{-a/2}^{a/2} x^2 dx$$

$$I_b = \frac{M}{a}\left[\frac{x^3}{3}\right]_{-a/2}^{a/2}$$

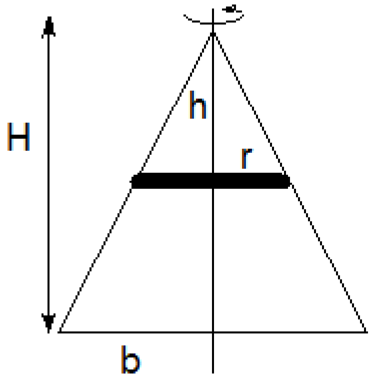
$$I_b = \frac{M}{3a}\left[\frac{a^3}{8} + \frac{a^3}{8}\right]$$

$$I_b = \frac{M}{12}a^2$$

Similarly

$$I_a = \frac{M}{12}b^2$$

7.15.10 Moment of inertia of triangle axis of rotation passing through vertices, and perpendicular to base



Let M be the mass of triangular plate having base b and height H.

Surface mass density of triangle is σ= M/A

$$\sigma = \frac{M}{\frac{1}{2}bH} = \frac{2M}{bH}$$

Consider elemental tin rod of length r and thickness dh as shown in figure area of element =2rdh

Mass of element $dm = 2rdh \times \frac{2M}{bH}$

Momentum of inertia of thin road

$$dI = \frac{1}{12}dm(2r)^2 = \frac{1}{3}dmr^2$$

$$dI = \frac{1}{3}2rdh \times \frac{2M}{bH}r^2$$

Integrating from 0 to H

$$I = \int_0^H \frac{4}{3}\frac{M}{bH} r^3 dh \dots (i)$$

Now from figure

$$\frac{h}{r} = \frac{H}{b}$$

$$r = \frac{b}{H} h \dots (ii)$$

Substituting (ii) in (i)

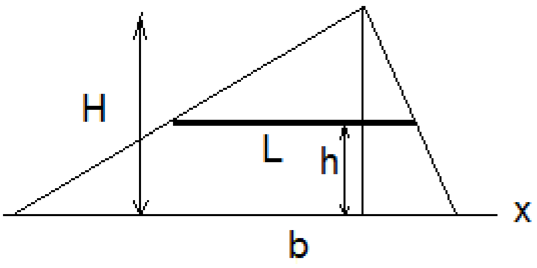
$$I = \int_0^H \frac{4}{3}\frac{M}{bH}\left(\frac{b}{H}h\right)^3 dh$$

$$I = \frac{4}{3}\frac{Mb^2}{H^4}\int_0^H h^3\, dh$$

$$I = \frac{4}{3}\frac{Mb^2}{H^4}\left[\frac{h^4}{4}\right]_0^H$$

$$I = \frac{1}{3}Mb^2$$

7.15.11 Moment of inertia of triangle axis of rotation passing through base



Let M be the mass of triangle and height H.

Surface mass density of triangle is σ= M/A

$$\sigma = \frac{M}{\frac{1}{2}bH} = \frac{2M}{bH}$$

Consider a elemental thin rod of length L and thickness dh. Area of rod =Ldh

Mass of rod $dm = Ldh \times \frac{2M}{bH}$

As every small mass of thin rod is at equal distance of (H-h) thus moment of inertia of rod $dI = dm(H-h)^2$

Integrating over o to H

$$I = \int_0^H dmh^2$$

$$I = \int_0^H Ldh \times \frac{2M}{bH} h^2 \dots (i)$$

Now

$$\frac{L}{b} = \frac{H-h}{H}$$

$$L = b\frac{H-h}{H} \dots (ii)$$

Substituting (ii) in (i)

$$I = \int_0^H b\frac{H-h}{H} dh \times \frac{2M}{bH}(h)^2$$

$$I = \frac{2M}{H^2}\int_0^H (H-h)dh \times (h)^2$$

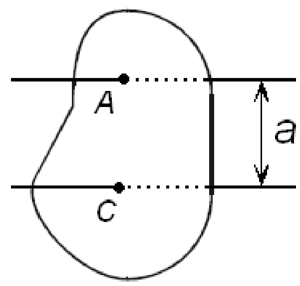
$$I = \frac{2M}{H^2}\int_0^H (Hh^2 - h^3)dh$$

$$I = \frac{2M}{H^2}\left[\frac{H^4}{3} - \frac{H^4}{4}\right]$$

$$I = \frac{MH^2}{6}$$

7.16 CHANGE OF AXIS

(i)The parallel axis theorem

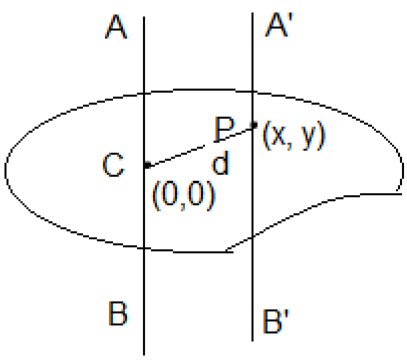


If the moment of inertia of uniform body of mass m about an axis through c, its centre of mass is I_C and I_A is the moment of inertia about a parallel axis through a point A, then

$I_A = I_C + Ma^2$

Where a is the distance between the parallel axes

Proof of theorem



Consider a planar body as shown in figure. Let C be center of mass and AB is axis passing through C which is along z-axis.

Now moment of inertia of point Q about AB is dI= dm×r^2

But perpendicular distance is $r^2 = x^2 + y^2$

Thus dI =dm (x^2+y^2)

Body is composed of many such particles

Thus moment of inertia of body along axis AB = I_C

$$I_C = \int dm(x^2 + y^2)$$

Now if we consider the moment of inertia along axis A'B'

Consider now point P at position (x',y') from origin. Then perpendicular distance of point Q from A'B' = $d^2 = (x' - x)^2 + (y' - y)^2$

Thus moment of inertia about axis A'B'

$$I_{AB} = \int dm[(x' - x)^2 + (y' - y)^2]$$

$$I_{AB} = \int dm(x^2 + y^2 + x'^2 + y'^2 - 2xx' - 2yy')$$

As x' and y' are constant

$$I_{AB} = \int dm(x^2 + y^2) + \int dm(x'^2 + y'^2) - 2x' \int dmx \; - 2y' \int dmy$$

Now first term

$$\int dm(x^2 + y^2)$$

Represent moment of inertia along AB = I_C

Second term

$$\int dm(x'^2 + y'^2)$$

Here x' and y' are the coordinates of point P through which axis of rotation AB pass and is fix

We know that d^2 = x'^2 + y'^2 thus integration will give Md^2

Third term

$$2x' \int dmx$$

Here x is the distance from center of mass, and coordinates of center of mass is (0,0) thus integration will be zero as x_{CM} =0

Fourth term

$$2y' \int dmy$$

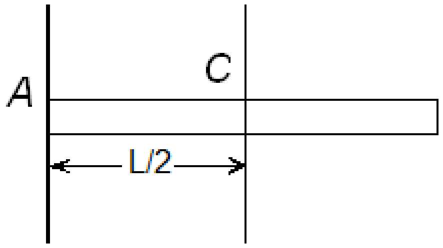
Here y is the distance from center of mass, and coordinates of center of mass is (0,0) thus integration will be zero as y_{CM} =0

From above discussions

$I_{AB} = I_C + Md^2$

Hence proved

7.16.01 Moment of inertia of rod about axis passing through one end perpendicular to length



M be a mass of thin rod and length L. Axis of rotation is passing through point A.

Point C is center of mass of rod. And moment of inertia about axis passing through C.M is

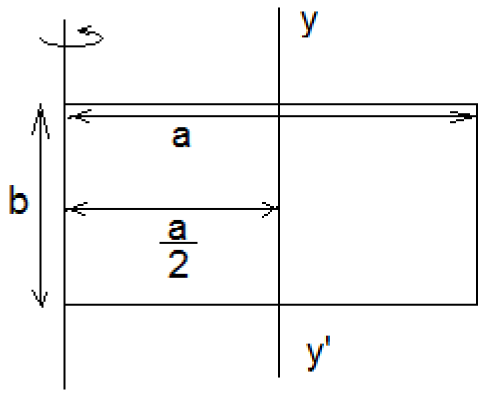
$$I_C = \frac{ML^2}{12}$$

Now by parallel axis theorem $I_A = I_C + M(L/2)^2$

$$I_A = \frac{ML^2}{12} + \frac{L^2}{4}$$

$$I_A = \frac{1}{3}ML^2$$

7.16.02 Moment of inertia of rectangle about axis passing through one side



As shown in figure axis of rotation is passing through side b, Mass of plate is M

Then moment of inertia of plate along axis passing through center of mass as proved

$$I_b = \frac{M}{12}a^2$$

Note that subscript b indicate axis is parallel to side b

Distance between yy' and b is a/2

Thus moment of inertia about axis passing through side b is

$$I = I_b + M\left(\frac{a}{2}\right)^2$$

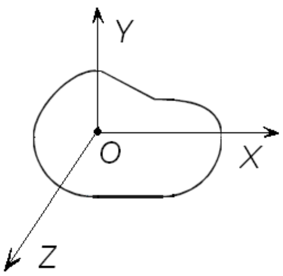
$$I = \frac{M}{12}a^2 + M\left(\frac{a}{2}\right)^2$$

$$I = \frac{1}{3}Ma^2$$

Similarly if axis of rotation is passing through side a the

$$I = \frac{1}{3}Mb^2$$

(ii)The perpendicular axis theorem

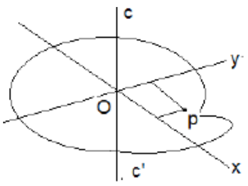


If a plane body has moments of inertia I_{OX} and I_{OY} about two perpendicular axes, OX and OY, in the plane of the body then its moment of inertia about an axis OZ perpendicular to the plane is $I_{OZ} = I_{OX} + I_{OY}$

The three axes under consideration must be mutually perpendicular and concurrent, although they need to pass through the centre of mass of the body

This theorem cannot be applied to three dimensional bodies

Proof



Perpendicular distance of point p of mass dm from x axis is x then moment of inertia about point p about x axis is dmx^2 since planer object consisting of many point total moment of inertia about x axis is $I_X = \int dmx^2$

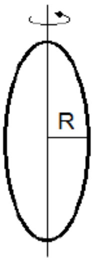
Similarly moment of inertia about y axis is $I_Y = \int dmy^2$

Now perpendicular distance of point p from z axis cc' is $x^2 + y^2$

Thus moment of inertia about Z axis $I_Z = \int dm(x^2+y^2) = \int dmx^2 + \int dmy^2$

$I_Z = I_x + I_y$ hence proved

7.16.03 Moment of inertia of ring about axis passing through any diameter



Let mass of ring is M and radius, rotate about the axis passing through any diameter.

For any two axis passing through two diameter, moment of inertia will be same

Thus $I_X = I_Y = I$

We know that moment of inertia about axis perpendicular to plane of ring is MR^2

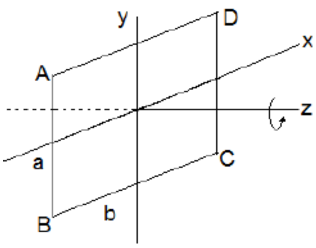
Now by perpendicular axis theorem

$I_Z = I_x + I_Y$

$MR^2 = 2I$

$$I = \frac{1}{2}MR^2$$

7.16.04 Moment of inertia of rectangular plate about axis perpendicular to plane and passing through any diameter



Let M be the mass, a and b sides of rectangular plate

Moment of inertia about x axis which is parallel to side of length b is

$$I_x = \frac{M}{12}a^2$$

Moment of inertia about y axis which is parallel to side of length a is

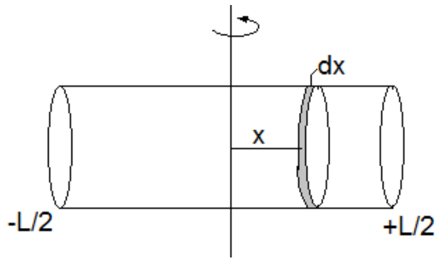
$$I_x = \frac{M}{12} b^2$$

Now according to perpendicular axis theorem

$$I_Z = \frac{M}{12} a^2 + \frac{M}{12} b^2$$

$$I_Z = \frac{M}{12}(a^2 + b^2)$$

7.16.05 Moment of inertia axis passing through its center and perpendicular to its length



Let M be the mass of cylinder , L is length and R is radius

Volume mass density

$$\rho = \frac{M}{\pi R^2 L}$$

Consider a disc of thickness dx at a distance of x from the center of mass

Mass of disc dm= volume × ρ

$$dm = \pi R^2 dx \frac{M}{\pi R^2 L} = \frac{M}{L} dx$$

Now

Moment of inertia of disc along diameter

$$dI = \frac{1}{4}\left(\frac{M}{L} dx\right) R^2$$

By paralle axis theorem moment of inertia about axis of rotation is

$$dI' = \frac{1}{4}\left(\frac{M}{L} dx\right) R^2 + \left(\frac{M}{L} dx\right)\left(\frac{x}{2}\right)^2$$

Now integrating fro –L/2 to +L/2

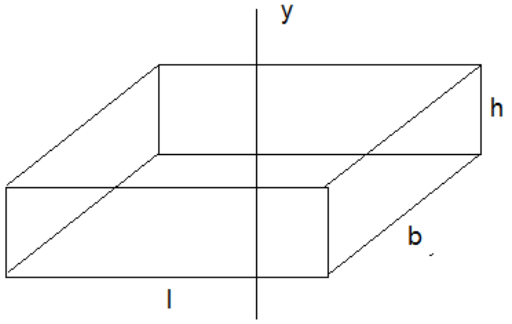
$$I = \int_{-L/2}^{L/2} \left\{\frac{1}{4}\left(\frac{M}{L} dx\right) R^2 + \left(\frac{M}{L} dx\right)\left(\frac{x}{2}\right)^2\right\}$$

$$I = \int_{-L/2}^{L/2} \frac{1}{4}\left(\frac{M}{L} dx\right) R^2 + \int_{-L/2}^{L/2} \left(\frac{M}{L} dx\right)\left(\frac{x}{2}\right)^2$$

$$I = \frac{MR^2}{4L}[x]_{-L/2}^{L/2} + \frac{M}{4L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

7.16.06 Moment of inertia of block



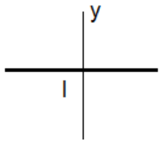
Let M be mass of block of length l, breadth b and height h

Axis Y is passing through the center of mass.

We can use a simple trick to solve this problem **(Note it is applicable to block only, not other shape like cone, prism)**

Momentum of inertia can be added algebraically

If we make h = 0 and b=0, but mass is same then we get is simple thin rod as shown in figure

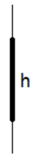


axis is perpendicular passing through center of mass and moment of inertia is

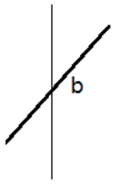
$$I = \frac{M}{12}(l^2) \ldots (i)$$

If we make l=0 ,b=0 then object will be like rod along z moment of inertia =0

As shown in figure



If we make h = 0 and l =0, we get rod with length b moment of inertia as shown in figure

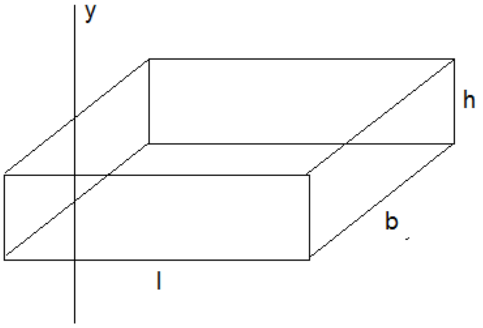


$$I = \frac{M}{12}(b^2) \ldots (ii)$$

Thus total is (i)+(ii)

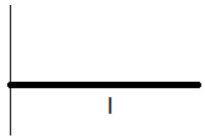
$$I = \frac{M}{12}(l^2 + b^2)$$

Now consider axis is passing through center of side b



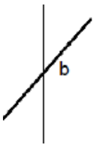
As state earlier we will compress the shape

If b= 0 and h =0, we get thin rod of length l and axis of rotation is at end as shown in figure below



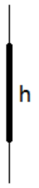
Moment of inertia is $I = \frac{1}{3}Ml^2$

If l=0 and h =0, we get thin rod of length b and axis of rotation is passing through its center as shown in figure



Moment of inertia is $I = \frac{1}{12}Mb^2$

If we make l= 0 and b=0 then we get rod along zxis of rotation having length =h as shown in figure



Moment of inertia is zero

Total momentum (i)+(ii)

$$I = \frac{1}{3}Ml^2 + \frac{1}{12}Mb^2$$

$$I = \frac{M}{12}(4l^2 + b^2)$$

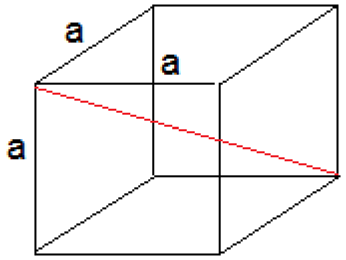
7.17 TABLE OF MOMENT OF INERTIA

Body	Dimension	Axis	Moment of inertia
Circular ring	Radius r	Through its centre and perpendicular to its plane	Mr^2
Circular disc	Radius r	Through its centre and perpendicular to its plane	$\frac{Mr^2}{2}$
Right circular solid cylinder	Radius r and length l	About geometrical axis	$\frac{Mr^2}{2}$
Solid cylinder	Radius r and length l	Through its centre and perpendicular to its length	$M\left[\frac{r^2}{4}+\frac{l^2}{12}\right]$
Uniform solid sphere	Radius R	About a diameter	$\frac{2}{5}MR^2$
Hollow sphere	Radius R	About diameter	$\frac{2}{3}MR^2$
Thin uniform rod	Length 2l	Through its centre and perpendicular to its length	$\frac{Ml^2}{3}$
Thin rectangular sheet	Side a and b	Through its cenre and perpendicular to its plane	$M\left[\frac{a^2}{12}+\frac{b^2}{12}\right]$
Circular disc	Radius R	Through any diameter	$\frac{1}{4}MR^2$
Hollow cylinder	Radius R	Geometrical axis	MR^2

Solved Numerical

Q7) A cube is fixed in sphere of radius R, such that all vertices of cube are on the surface of sphere of mass M. Find moment of inertia of cube around the axis passing through its center of mass and perpendicular to its face

Solution



As shown in figure diagonal shown in red is =2R.

Now diagonal of cube 2R =a√3

Thus $a = \frac{2}{\sqrt{3}}R$

Now mass volume density of sphere = $\frac{M}{V}$

$$\rho = \frac{3M}{4\pi R^3}$$

Mass of cube = Volume ×ρ

$$M' = a^3 \times \frac{3M}{4\pi R^3}$$

$$M' = \left(\frac{2}{\sqrt{3}}R\right)^3 \times \frac{3M}{4\pi R^3}$$

$$M' = \frac{2}{\sqrt{3}} \times \frac{M}{\pi}$$

From the formula

$$I = M'\left[\frac{a^2}{12} + \frac{a^2}{12}\right]$$

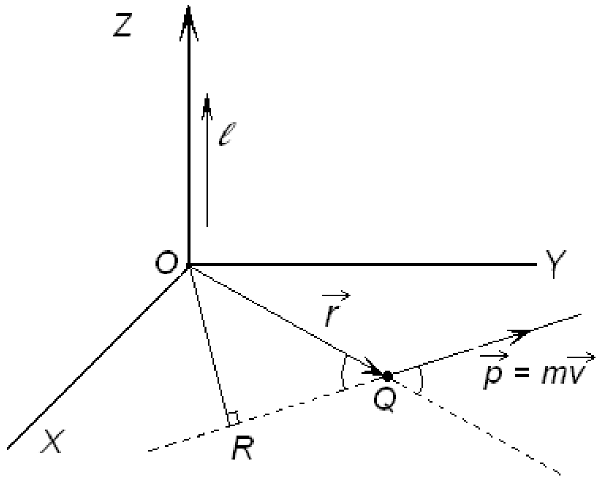
$$I = \frac{2}{\sqrt{3}} \times \frac{M}{\pi}\left[\frac{a^2}{6}\right]$$

$$I = \frac{a^2}{3\sqrt{3}} \times \frac{M}{\pi}$$

7.18 Angular momentum of a particle

The angular momentum in a rotational motion is similar to the linear momentum in translational motion. The linear momentum of a particle moving along a straight line is the product of its mass and linear velocity (i.e) $p = mv$.

The angular momentum of a particle is defined as the moment of linear momentum of



the particle. Let us consider a particle of mass m moving in the *XY* plane with a velocity *v* and linear momentum *p* =*m v* at a distance *r* from the origin The angular momentum L of the particle about an axis passing through O perpendicular to XY plane is defined as the cross product of *r* and *p*

(i.e) *L* =*r* ×*p*

Its magnitude is given by *L = r p sin* θ

where θ is the angle between *r* and *p* and L is along a direction perpendicular to the plane containing *r* and *p*

The unit of angular momentum is

kg m^2 s^{-1} or Js

Its dimensional formula is, M L^2 T^{-1}.

Angular momentum of a rigid body

Let us consider a system of n particles of masses m_1, m_2 ….. m_n situated at distances r_1, r_2, …..r_n respectively from the axis of rotation . Let v_1,v_2, v_3 ….. be the linear velocities of the particles respectively, then linear momentum of first particle = m_1v_1.

Since v_1= $r_1\omega$ the linear momentum of first particle = $m_1(r_1\ \omega)$

The moment of linear momentum of first particle = linear momentum × perpendicular distance = $(m_1r_1\omega) \times r_1$

angular momentum of first particle = $m_1r_1^2\omega$

Similarly,

angular momentum of second particle = $m_2r_2^2\omega$

angular momentum of third particle = $m_3r_3^2\omega$ and so on.

The sum of the moment of the linear momentum of all the particles of a rotating rigid body taken together about the axis of rotation is known as angular momentum of the rigid body.

∴ Angular momentum of the rotating rigid body = sum of the angular momentum of all the particles.

$$L = m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + \cdots + m_nr_n^2\omega$$

$$L = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2)\omega$$

$$L = \omega\sum_{i=1}^{n} m_ir_i^2$$

$$L = I\omega$$

7.19 Relation between angular momentum of a particle and torque on it

By definition angular momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating equation with time, we get

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But

$$\frac{d\vec{p}}{dt} = \vec{F} \quad and \quad \frac{d\vec{r}}{dt} = \vec{v}$$

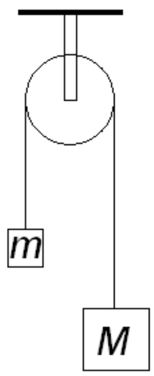
$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

But v and p are the same direction thus $\mathbf{v} \times \mathbf{p} = 0$

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Thus the time rate of change of angular momentum is equal to torque.

If total torque on a particle is zero the angular momentum of the particle is conserved. This is known as principle of conservation of angular momentum. Angular momentum of a rotating rigid body about an axis having pure rotation can be written as L = Iω



Solved numerical

Q8) The pulley shown in figure has moment of inertia I about axis and radius R. Find the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley

Solution

Suppose the tension in right string is T_1 and tension in left side string is T_2 . Suppose block M is going down with acceleration a and the other block is going up with same acceleration . This is also tangential acceleration of the rim of the wheel as the string does not slip over the rim of the wheel as the string does not slip over the rim

Tangential acceleration of pulley

a = αR

Thus α = a/R -----eq(1)

The equations for mass M

$Ma = Mg - T_1$

$T_1 = M(g-a)$ ------eq(2)

Equation for mass m

$ma = T_2 - mg$ --------eq(3)

$T_2 = m(a-g)$

Torque τ = Iα

Also $\tau = (T_1 - T_2)R$

Thus $(T_1 - T_2)R = I\alpha$

Substituting values of T_1 T_2 and α from equation 1, 2, 3 we get

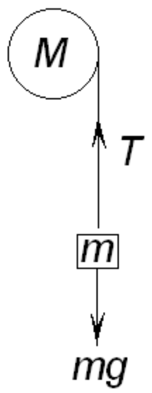
$$[M(g-a) - m(g+a)]R = \frac{Ia}{R}$$

Solving we get

$$a = \frac{(M-m)gR^2}{I + (M+m)R^2}$$

Q9) A string is wound around a disc of radius r and mass M and m is suspended. The body is then allowed to descend. Show that the angular acceleration of disc is

$$\alpha = \frac{mg}{R\left(m + \frac{M}{2}\right)}$$



Solution:

The equation for suspended body is ma = mg – T

T = m(g-a)

Acceleration of body is the tangential acceleration of disc thus

a = αR

Thus

T = m(g-a)

$$T = m(g - \alpha R)$$

This tension produces torque

τ = F×R

τ = m(g- αR)R

Now torque τ = Iα

Thus

$$I\alpha = m(g - \alpha R)R$$

$$I\alpha + m\alpha R^2 = mgR$$

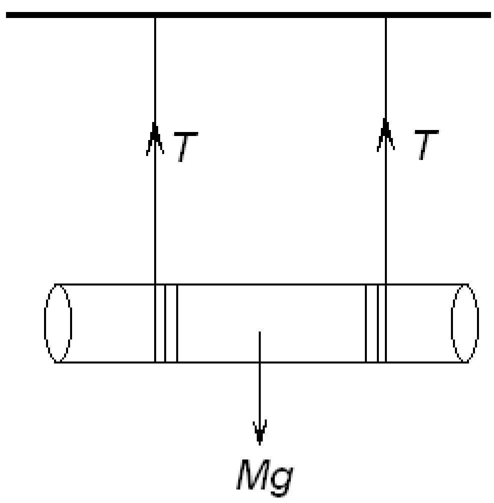
$$\alpha = \frac{mgR}{(I + mR^2)}$$

But I for disc = $MR^2/2$

$$\alpha = \frac{mgR}{\left(\frac{MR^2}{2} + mR^2\right)}$$

$$\alpha = \frac{gm}{\left(\frac{M}{2} + m\right)R}$$

Q10) A cylinder of M is suspended through two string wrapped around it as shown in figure. Fin the tension in the string and the speed of the cylinder as it falls through a distance h



Solution

The portion of the string between the ceiling and the cylinder are at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its own axis given by

$\alpha = a/R$

As the cylinder does not slip over the string, the equation of motion for the centre of mass of cylinder is

Mg- 2T = Ma -------eq(1)

And for the notion about the centre of mass is

$\tau = I\alpha$

$$2TR = \left(\frac{MR^2}{2}\right)\alpha$$

$$2TR = \left(\frac{MR^2}{2}\right)\frac{a}{R}$$

$$2T = \frac{Ma}{2} \quad ------eq(2)$$

From equation (1) and (2) we get

$$Mg = \frac{Ma}{2} + Ma$$

$$a = \frac{2g}{3}$$

From eq(2)

$$2T = \frac{M}{2}\frac{2g}{3} = \frac{2Mg}{6}$$

$$T = \frac{Mg}{6}$$

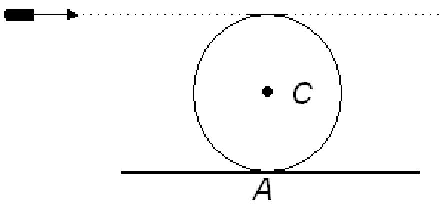
As the centre of the cylinder starts moving from rest, the velocity after it has fallen a height h is given by

$$v^2 = 2\left[\frac{2g}{3}\right]h$$

$$v = \sqrt{\frac{4gh}{3}}$$

Q11) A bullet of mass m moving with velocity u just grazes the top of a solid cylinder of mass M and radius R resting on a rough horizontal surface as shown. Assume that the cylinder rolls without slipping and at the instant the bullet leaves three is no relative motion between the bullet and the cylinder. Find the angular velocity of the cylinder and the velocity of the bullet.

Solution



At the instant the bullet leaves there is no relative motion thus angular momentum is conserved

Let v be the velocity of bullet when it leaves the cylinder and ω be the angular velocity of cylinder

Applying law of conservation of angular momentum at a point A on floor

$$Mu(2R) = mv(2R) + (I_{CM} + MR^2)\,\omega$$

Here $v = 2R\omega$ and $I_{CM} = (1/2)MR^2$

Substituting we get

$$2muR = m(2R\omega)(2R) + \frac{1}{2}MR^2\omega + MR^2\omega$$

$$2mu = 4mR\omega + \frac{3}{2}MR\omega$$

$$4mu = 8mR\omega + 3MR\omega$$

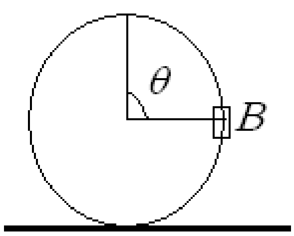
$$\omega = \frac{4mu}{(8m + 3M)R}$$

Solving for v we get as $\omega = v/2R$ angular velocity of bullet with respect to point A

$$\frac{v}{2R} = \frac{4mu}{(8m + 3M)R}$$

$$v = \frac{8muR}{(8m + 3M)}$$

Q12) A small body of mass m is attached at B to a hoop of mass 3m and radius r. the system is released from rest with $\theta = 90^{O}$ and rolls without sliding. Determine



(a) the angular acceleration of the hoop

(b)the horizontal and vertical components of the acceleration of B

(c)normal reaction and frictional force just after the release

Solution

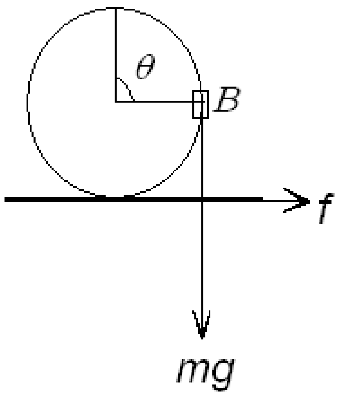
Mass at B will produce torque and will be opposed by torque produced due to friction. Thus direction of frictional force is as shown in figure

Thus resultant torque τ = mgR – fR -----eq(1)

Also τ = Iα ----eq(2)

From eq(1) and (2)

mgR – fR = Iα



Now I = ($3mR^2 + mR^2$)

∴mgR – fR = ($3mR^2 + mR^2$) α

$$\alpha = \frac{mg - f}{4mR} \qquad ------eq(3)$$

Linear acceleration = Friction / total mass

$$a = \frac{f}{4m} \qquad -----eq(4)$$

Also for pure rolling a = Rα. Thus from eq(3) and (4)

$$\frac{f}{4m} = \frac{mg - f}{4mR}R$$

$$f = \frac{mg}{2}$$

Therefore

(a) the angular acceleration of the hoop

$$\alpha = \frac{g}{8r}$$

(b) the horizontal and vertical components of the acceleration of B

$$a_h = R\alpha = \frac{g}{8R}R = \frac{g}{8} \qquad a_v = R\alpha = \frac{g}{8R}R = \frac{g}{8}$$

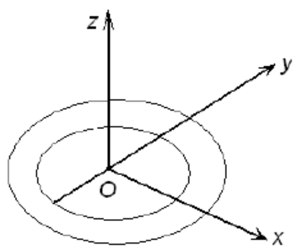
(c) Normal force = gravitational downwards – vertical upward

$$N = 4mg - \frac{mg}{8} = \frac{31mg}{8}$$

Frictional force = mg/2

Solved numerical

Q13) Find the moment of inertia, about a diameter, of a uniform ring of mass M and radius a



Solution:

WE know that moment of inertia I_{OZ}. of the ring about OZ is MA^2. We also know that, from symmetry, the moment of inertia about any one diameter is the same that about any other diameter i.e $I_{OX} = I_{OY}$

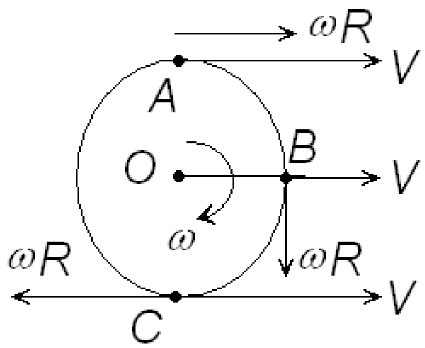
Using the perpendicular axes theorem gives

$I_{OZ} = I_{OX} + I_{OY}$

$Ma^2 = 2I_{OX} = 2I_{OY}$

The moment of inertia of the ring about any diameter = (1/2) Ma^2

7.20 Combined rotational and translational motion of a rigid body: Rolling motion



Translation motion caused by a force and rotational motion about fixed axis is caused by a torque. Rolling motion can be considered as combination of rotational and translational motion. For the analysis of rolling motion we deal translation separately and rotation separately and then we combine the resultant to analyze the overall motion.

Consider a uniform disc rolling on a horizontal surface.

Velocity of centre of mass is V and its angular speed is ω as shown

A, B, C are three points on the disc. Due to the translational motion each point A, B and C will move with centre of mass in horizontal direction with velocity V. Due to pure rotational motion each point will have tangential velocity ωR, R is the radius of the disc. When the two motions are combined, resultant velocities of different points are given by

$$V_A = V + \omega R$$

$$V_B = \sqrt{V^2 + \omega^2 R^2}$$

$$V_C = V - \omega R$$

Similarly, if disc rolls with angular acceleration α and its centre of mass moves with acceleration 'a' different points will have accelerations given by (for ω = 0)

$$a_A = a + \alpha R$$

$$a_B = \sqrt{a^2 + \alpha^2 R^2}$$

$$a_C = a - \alpha R$$

To write equation of motion for rolling motion. We can apply for translation motion

$$\vec{F}_{ext} = M\vec{a}_{CM}$$

And for rotational motion τ = I α

Rolling motions is possible in two ways

Rolling with slipping

Relative motion takes place between contact points

$V_C \neq 0$ and $a_C \neq 0$

Frictional force is μN

Rolling without slipping

$V_C = 0 \Rightarrow V = \omega R$

And $a_C = 0 \Rightarrow a = \alpha R$

Frictional force is unknown magnitude it may any value between zero to μN

Kinetic energy of rolling body

If a body of mass m is rolling on a plane such that velocity of its centre of mass is V and the angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

I is moment of inertia of the body about an axis passing through the centre of mass.

In case of rolling without slipping

$$KE = \frac{1}{2}m\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\because V = \omega R]$$

$$KE = \frac{1}{2}[MR^2 + I]\omega^2$$

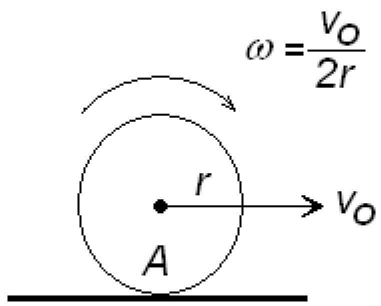
$$KE = \frac{1}{2}I_G\omega^2$$

I_G is the moment of inertia of the body about the axis passing through the point of contact on surface

Solved numerical

Q14) A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and the rotational velocity $v_0/2r$. Find the translational velocity after the sphere starts pure rolling.

Solution



Let us consider the torque about the initial point of contact A. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about A is zero. Hence the angular momentum about A is conserved. Initial angular momentum

$$L = L_{CM} + Mrv_0$$

$$L = \left(\frac{2}{5}Mr^2\right)\frac{v_0}{2r} + Mrv_0 = \frac{6}{5}Mrv_0$$

Suppose the translational velocity of the sphere, after it starts rolling, is v. The angular velocity is v/r. The angular momentum about A is

$$L = L_{CM} + Mrv$$

$$L = \left(\frac{2}{5}Mr^2\right)\frac{v}{r} + Mrv = \frac{7}{5}Mrv$$

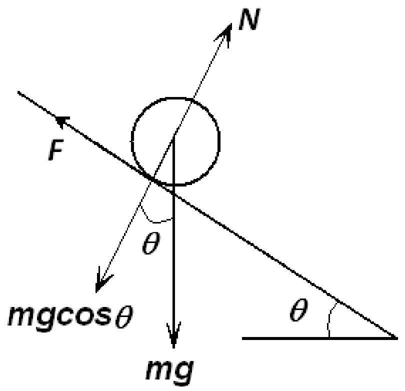
Thus

$$\frac{7}{5}Mrv = \frac{6}{5}Mrv_0$$

$$v = \frac{6}{7}v_0$$

7.21 Rigid bodies rolling without slipping on inclined plane

As stated earlier rolling without slipping is a combination of two motion translator and rotational motion



When object roll on incliner plane its centre of mass undergoes translator motion while other points on the body undergoes rotational motion

As shown in figure, suppose a rigid body rolls down without slipping along an inclined plane of height h and angle θ. Here, the mass of the body is m, moment of inertia I, geometrical radius R and the radius of gyration is K. When body reaches the bottom of the inclined plane, its potential energy decreases by mgh. According to law of conservation of mechanical energy , decrease in potential energy is converted n kinetic energy of the body

Thus

Potential energy = translation kinetic energy + rotational kinetic energy

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

Now ω=V/R and I = mK^2

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}mK^2\left(\frac{V}{R}\right)^2$$

$$2gh = V^2 + K^2\left(\frac{V}{R}\right)^2$$

$$2gh = V^2\left(1 + \frac{K^2}{R^2}\right)$$

$$V^2 = \frac{2gh}{1 + \frac{K^2}{R^2}} \qquad ---eq(1)$$

If the length of the slope is d, and the body started from rest, moves with linear acceleration a to reach bottom

$$V^2 = 2ad \qquad --------eq(2)$$

From geometry of figure

$$d = \frac{h}{\sin\theta}$$

$$\therefore V^2 = 2a\frac{h}{\sin\theta}$$

Combining equations

$$2a\frac{h}{\sin\theta} = \frac{2gh}{1+\frac{K^2}{R^2}}$$

$$a = \frac{gsin\theta}{1+\frac{K^2}{R^2}}$$

Here linear acceleration should be gsinθ. Loss in acceleration is due to frictional force

Decrease in linear acceleration

$$g\sin\theta - \frac{gsin\theta}{1+\frac{K^2}{R^2}} = g\sin\theta\left[\frac{K^2}{K^2+R^2}\right]$$

Thus frictional force

$$F = mg\sin\theta\left[\frac{K^2}{K^2+R^2}\right]$$

From figure normal reaction and mgcosθ balance each other hence N = mgcosθ

$$\therefore \frac{F}{N} = \left[\frac{K^2}{K^2+R^2}\right]\tan\theta$$

But F/N = μ_S

$$\therefore \mu_S = \left[\frac{K^2}{K^2+R^2}\right]\tan\theta$$

$$\mu_S = \left[\frac{1}{1+\frac{R^2}{K^2}}\right]\tan\theta$$

The work done against the frictional force, resultants in the rotational kinetic energy and hence even in presence of frictional force we have been able to use the law of conservation of mechanical energy.

Now if

$$\mu_S \geq \left[\frac{1}{1+\frac{R^2}{K^2}}\right]\tan\theta \quad ----eq(3)$$

Condition is satisfied object will roll down the slope without slipping.

Cases

(i)Thin ring

For thin ring K = R substituting in above equation(3) we get

$$\mu_S \geq \frac{1}{2}\tan\theta$$

(ii) Circular disc

For circular disc K = R/√2 substituting in above equation(3) we get

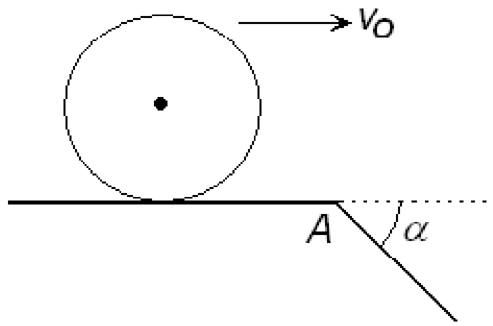
$$\mu_S \geq \frac{1}{3}\tan\theta$$

(iii)Solid sphere $K = \sqrt{\frac{2}{5}}R$

$$\mu_S \geq \frac{2}{7}\tan\theta$$

Solved numerical

Q15) A uniform solid cylinder of radius R = 15cm, rolls over a horizontal plane passing into an inclined plane forming an angle α = 30^{O} with the horizontal as shown in figure. Find the maximum value of the velocity v_o which still permits the cylinder to roll on the inclined plane section without a jump. The sliding is assumed to be zero.



Solution:

Since the cylinder have velocity along x axis. Will follow a projectile motion when it leave the horizontal surface. If projectile motion is prevented then the cylinder is acted upon by centripetal force at the corner , takes a turn and comes on inclined plane. Thus point in contact at A is the centre for circular motion. With radius R. Let v_1 be the velocity at point A . thus centripetal force

$$\frac{mv_1^2}{R}$$

Thus force is provided by gravitational force thus

$$mgcos\alpha \ = \frac{mv_1^2}{R}$$

$$v_1^2 = Rgcos\alpha \qquad ------eq(1)$$

With reference to point A potential energy of cylinder is mgR and when it comes on the inclined plane at A potential energy = mgRcosα (as gravitational acceleration is gcosα)

Let ω_1 be the angular acceleration on inclined plane.

Now according to law of conservation of energy

Potential energy on horizontal + Liner kinetic energy + rotational kinetic energy

= Potential energy at inclined + linear kinetic energy + rotational kinetic energy

$$mgR + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgRcos\alpha + \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega_1^2$$

Now

$$I = \frac{mR^2}{2}, \ \omega = \frac{v}{R} \ , \ \omega_1 = \frac{v_1}{R}$$

Substituting above values

$$mgR + \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v}{R}\right)^2 = mgRcos\alpha + \frac{1}{2}mv_1^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v_1}{R}\right)^2$$

From equation(1)

$$mgR + \frac{3}{4}mv^2 = mgRcos\alpha + \frac{3}{4}mRgcos\alpha$$

$$gR + \frac{3}{4}v^2 = gRcos\alpha + \frac{3}{4}Rgcos\alpha$$

$$gR + \frac{3}{4}v^2 = gRcos\alpha\left(1 + \frac{3}{4}\right)$$

$$\frac{3}{4}v^2 = gRcos\alpha\left(\frac{7}{4}\right) - gR$$

$$v^2 = \frac{4}{3}\left[gR\left(\frac{7}{4}cos\alpha - 1\right)\right]$$

$$v^2 = \frac{4}{3}\left[9.8 \times 0.15 \times \left(\frac{7}{4}cos30 - 1\right)\right] = 4m/s$$

v = 2m/s

Q))Two identical trains are moving on rails along the equator on the earth in opposite directions with the same speed. They will exert the same pressure on the rails.

Answer :The question is not so simple as it appears. Actually the entire concept is related to the angular veolcity of the earth and centrifugal forces. Since the earth rotates frm west to east. the train moving along the equatior from west to east will experience a lesser centrifugal force and hence applies greater pressure. On the other hand the one moving from east to west will experiences a larger centriugal force and applies smaller pressures on the track.

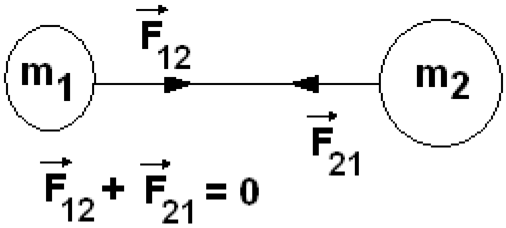
GRAVITATION

Newton's law of gravitation

The law states that every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the line of particles

Gravitational force is always attractive.

Consider two bodies of masses m_1 and m_2 with their centres separated by a distance r. The gravitational force between them is



Therefore from Newton's law of gravitation in vector form is

$$\vec{F} = \frac{Gm_1m_2}{r^2}\hat{r}$$

Here F_{12} represents force on mass 1 due to mass 2

And F_{21} is force of mass 2 due to mass 1

Note F_{12} and F_{21} are equal and opposite. The gravitational force forms between two particles form an action reaction pair.

Solved Numerical

Q) How a mass M be divided so that gravitational force is maximum between the parts

Solution:

Let r be the distance between two parts m and M-m, the gravitational force between them is

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}[Mm - m^2]$$

For F to be maximum

$$\frac{dF}{dm} = 0$$

$$\frac{d}{dm}\left(\frac{G}{r^2}[Mm - m^2]\right) = 0$$

Or M -2m = 0 [as $G/r^2 \neq 0$]

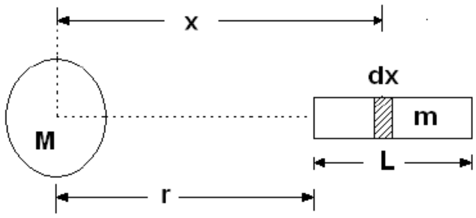
Or m/M = ½

The force is maximum when two parts are equal

Q) Find the gravitational force of attraction between a uniform sphere of mass M and a uniform rod of length L and mass m oriented as shown in figure. (given in solution)

Solution

Since the sphere is uniform its entire mass may be considered at the centre. The force on the elementary mass dm is



$$dF = \frac{GM\,dm}{x^2}$$

$$F = \int_r^{r+L} \frac{GM\,dm}{x^2}$$

$$but\ dm = \frac{m}{L}dx$$

$$F = \int_r^{r+L} \frac{GM}{x^2}\frac{m}{L}dx$$

$$F = -\frac{GMm}{L}\left[\frac{1}{x}\right]_r^{r+L}$$

$$F = -\frac{GMm}{L}\left[\frac{1}{r+L} - \frac{1}{r}\right]$$

$$F = -\frac{GMm}{L}\frac{L}{r(r+L)}$$

$$F = -\frac{GMm}{r(r+L)}$$

Gravitational Field or Intensity (I)

Process of action at a distance in which gravitational force is exerted mutually on two bodies separated by some distance is explained through the field

(i) Every object produces a gravitational field around it, due to mass

(ii) This field exerts a force on another body brought (or lying) in this field

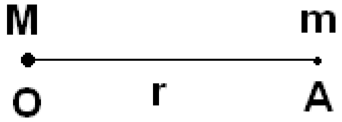
The gravitational force exerted by the given body on a body of unit mass at a given point is called the intesity of gravitational field (I) at that point" It is also known as the gravitational field or gravitational intensity

The gravitational field intensity is a vector quanity and its direction is the direction along which the unit mass ha a tendensy to move. The unit of gravitational field intesity is N/Kg and its dimensions are [LT^{-2}]

Calculation of gravitational field

(a)Gravitational field intensity due to a point mass.

Consider a point mass M at O and let us calculate gravitational intesity at A due to this point mass.



Suppose a test mass is placed at A

By Newton's law of gravitation, force on test mass

$$F = \frac{GMm}{r^2} \quad along\ \overrightarrow{AO}$$

$$I = \frac{F}{m} = -\frac{GM}{r^2}\hat{e}_r \quad ---eq(1)$$

(b) Gravitational firld intensity due to a uniform circular ring at a point on its axis

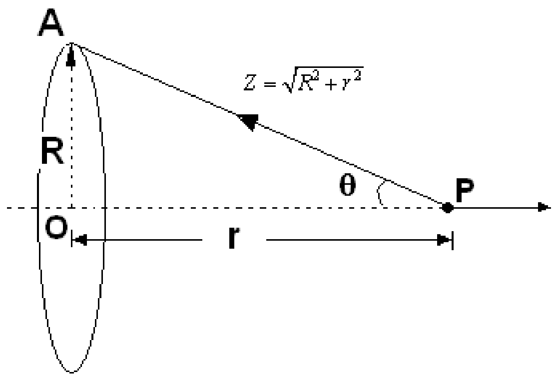


Figure shows a ring of mass M and radius R. Let P is the point at a distance r from the centre of the ring. By symmetry the field must be towards the centre that is along PO
Let us assume that a particle of mass dm on the ring say, at point A. Now the distance APis

$$\sqrt{R^2+r^2}$$

Again the gravitational field at P due to dm is along PA and the magnitude is

$$dI = \frac{G\ dm}{Z^2}$$

$$\therefore dEcos\theta = \frac{G\ dm}{Z^2}cos\theta$$

Sine components will be canceled when we consider magnetic field due to entire ring and only cos components will be added
Net gravitational field I

$$I = \frac{Gcos\theta}{Z^2}\int dm$$

$$I = \frac{GM}{Z^2}cos\theta$$

But cosθ = r/Z

$$I = \frac{GM}{Z^2}\frac{r}{Z} = \frac{GMr}{Z^3}$$

$$I = \frac{GMr}{(R^2+r^2)^{3/2}} \quad along\ PO$$

Cases

(i) If r >>R, $r^2 + R^2 = r^2$

$$\therefore I = -\frac{GMr}{r^3} = -\frac{GM}{r^2}[\,negative\ sign\ indicates\ attraction]$$

(ii) If r <<R, $r^2 + R^2 = R^2$

$$\therefore I = -\frac{GMr}{R^3}$$

$$\therefore I \propto r$$

(iii) For maximum I

$$\frac{\partial I}{\partial r} = 0$$

$$\frac{GM\left[(r^2+R^2)^{3/2}-\frac{3}{2}(r^2+R^2)^{1/2}\times 2r^2\right]}{[r^2+R^2]^3}=0$$

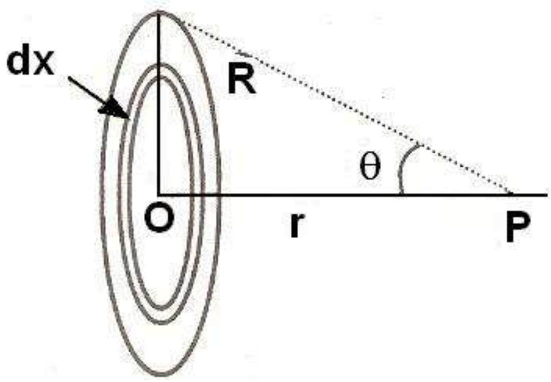
$$\left[(r^2+R^2)^{3/2}-\frac{3}{2}(r^2+R^2)^{1/2}\times 2r^2\right]=0$$

$$\left[(r^2+R^2)-\frac{3}{2}\times 2r^2\right]=0$$

$$[(r^2+R^2)-3r^2]=0$$

$$r=\pm\frac{R}{\sqrt{2}}$$

(c) Gravitational field intensity due to a uniform disc at a point on its axis



Let the mass of disc be M and its radius is R and P is the point on its axis where gravitational field is to be calculated

Let us draw a ring of radius x and thickness dx

O is the centre of circle. Area of ring is 2πxdx

The mass of ring

$$dm=\frac{M}{\pi R^2}2\pi x dx=\frac{2Mxdx}{R^2}$$

Gravitational field at p due to ring is

$$dI=\frac{G\left(\frac{2Mxdx}{R^2}\right)r}{(r^2+x^2)^{3/2}}$$

$$\int dI=\frac{2GMr}{R^2}\int_0^R\frac{xdx}{(r^2+x^2)^{3/2}}$$

$$I=\frac{2GMr}{R^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^R$$

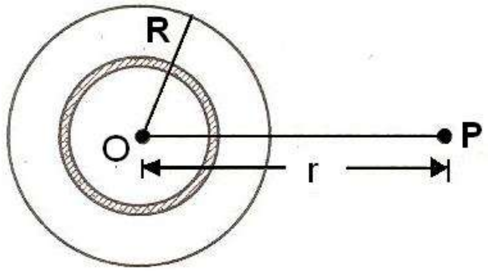
$$I=\frac{2GMr}{R^2}\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+x^2}}\right]$$

In terms of θ

$$I=\frac{2GM}{R^2}\left[\frac{r}{r}-\frac{r}{\sqrt{r^2+x^2}}\right]$$

$$I = \frac{2GM}{R^2}(1 - cos\theta)$$

(d) Gravitational field due to a uniform solid sphere



Case I

Field at an external point

Let the mass of the sphere be M and its radius be R. We have to calculate the gravitational field at P

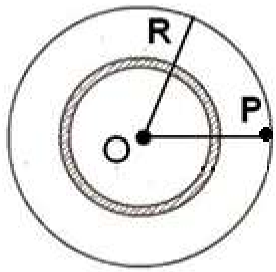
$$\int dI = \int \frac{Gdm}{r^2}$$

$$\int dI = \frac{G}{r^2}\int dm = \frac{Gm}{r^2}$$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point

Case II

Find at an internal point



Suppose the point P is inside the solid sphere, in this case r<R the sphere may be divided into thin spherical shells all centered at O. Suppose the mass of such a shell is dm. then gravitational field due to this spherical shell

$$dI = \frac{Gdm}{r^2} \quad along\ PO$$

$$\int dI = \int \frac{Gdm}{r^2}$$

$$\int dI = \frac{G}{r^2}\int dm$$

But dm = density × volume

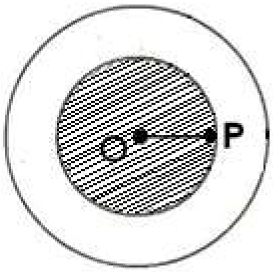
$$\int dm = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$$

$$\therefore I = \frac{GM}{R^3}r$$

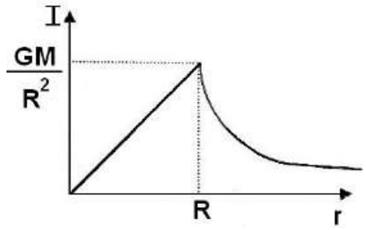
Therefore gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre r = 0 the field is zero. At the surface of the sphere r = R

$$I = \frac{GM}{R^2}$$

Note : (I)Gravitational field due to solid sphere is continuous but it is not differentiable function



(ii) Gravitational field at point inside the sphere is only due to the mass enclosed by the surface passing through the point, volume enclosed is shown by shaded portion in diagram and field due to outer volume is zero .



(e) Field due to uniform thin spherical shell

Case I

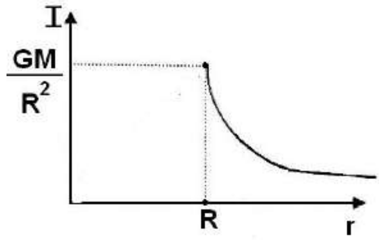
When point lies inside the spherical shell surface passing through point do not enclose any mass thus I = 0

Case II

Point P lies outside the spherical shell

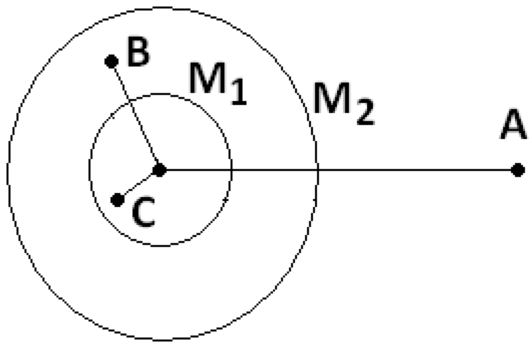
$$I = \int dI = \frac{G}{r^2}\int dm = \frac{GM}{r^2}$$

Note : Gravitational field due to thin spherical shell is both discontinuous and non-differentiable function



Solved Numerical

Q) Two concentric shells of masses M_1 and M_2 are situated as shown in figure. Find the force on a particle of mass m when the particle is located at (a) point A (b) point B (c) point C. The distance r is measured from the centre of the shell



Solution:
We know that attraction at an external point due to spherical shell of mass M is

$$\frac{GM}{r^2}$$

While at an internal point is zero. So

(a) At point A let r = a, the external point for both shells so field intensity

$$I_A = \frac{G(M_1 + M_2)}{a^2}$$
$$\therefore F_A = mI_A = \frac{mG(M_1 + M_2)}{a^2}$$

(b) For point B, let r = b, the point is external to shell of mass M_2 and internal to the shell of mass M_1, so

$$I_B = \frac{GM_2}{b^2} + 0$$
$$\therefore F_B = mI_B = \frac{GmM_2}{b^2}$$

(c) For point C, let r = c, the point is internal to both the shells; so

$$I_C = 0 + 0 = 0$$
$$\therefore F_C = mI_C = 0$$

Gravitational potential

Gravitational potential (V) at a point is defined as the amount of work done in moving unit mass from the point to infinity against the gravitational field. It is a scalar quantity. Its unit is N m kg^{-1}. Or J kg^{-1} dimensional formula $M^0L^2T^{-2}$

Mathematically

V = W/m

By the definition of potential energy U = W

So V = U/m

Or U = mV

Thus gravitational potential at a point represents potential energy of unit point mass at that point

Work is done against gravitational force thus

$$W = -\int \vec{F}_{gravitation} \cdot \overrightarrow{dr}$$

$$\therefore V = \frac{W}{m} = -\int \frac{\vec{F}_{gravitation}}{m} \cdot \overrightarrow{dr}$$

$$But \quad \frac{\vec{F}_{gravitation}}{m} = I$$

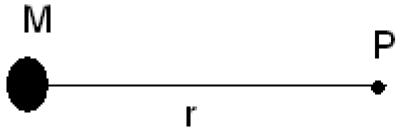
$$\therefore V = -\int \vec{I} \cdot \overrightarrow{dr}$$

i.e dV = _Idr

$$or \quad E = -\frac{dV}{dr}$$

Calculation of Gravitational potential

(a) Gravitational potential at a point (P) due to a point mass(M)



We have gravitational field due to a point mass

$$I = -\frac{GM}{r^2}$$

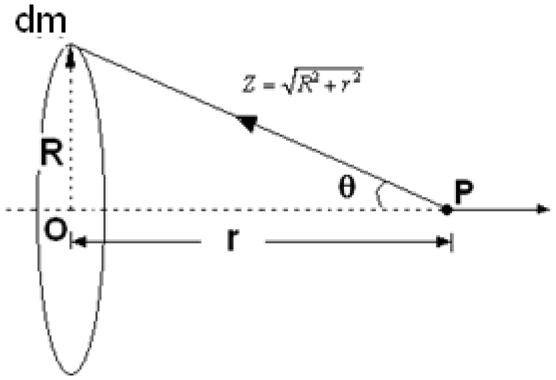
The negative sign is used as gravitational force is attractive

$$\therefore V = -\int_{\infty}^{r} -\frac{GM}{r^2}dr$$

$$V = GM\int_{\infty}^{r}\frac{dr}{r^2}$$

$$V = GM\left[\frac{-1}{r}\right]_{\infty}^{r} = -GM\left[\frac{1}{r}-\frac{1}{\infty}\right] = -\frac{GM}{r}$$

(b) Gravitational potential at a point due to a ring



Let M be the mass and R be the radius of thin ring.
Considering a small element of the ring and treating it as a point mass, the potential at the point P is

$$dV = \frac{-G\ dm}{Z} = \frac{-G\ dm}{\sqrt{R^2+r^2}}$$

Hence, the total potential at the point P is given by

$$V = -\int\frac{G\ dm}{\sqrt{R^2+r^2}} = \frac{GM}{\sqrt{R^2+r^2}}$$

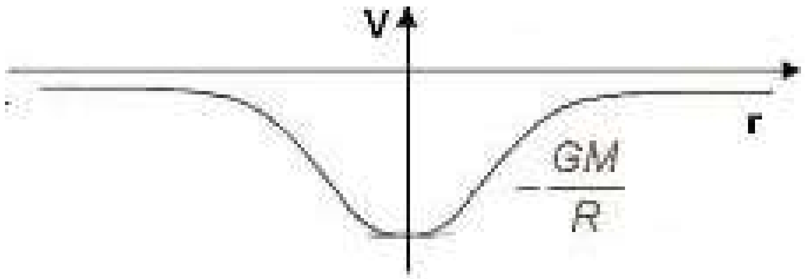
At r = 0

$$V = \frac{-GM}{R}\ \ and\ \ \frac{dV}{dr} = 0$$

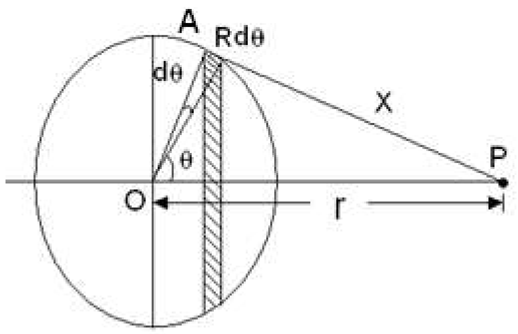
Thus at centre of ring gravitational field is zero but potential is not zero
Also

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{-GM}{\sqrt{R^2+r^2}}\right)$$

$$\frac{dV}{dr} = \frac{GM\times 2r}{R^2+r^2} = 0 \Longrightarrow V\ is\ minimum\ at\ r = 0$$



(c) Gravitational potential at a point due to a spherical shell (hollow sphere)



Consider a spherical shell of mass M and radius R. P is a point at a distance 'r' from the centre O of the shell.
Consider a ring at angle to OP. Let θ be the angular position of the ring from the line OP.
The radius of the ring = Rsinθ
The width of the ring = Rdθ
Surface area of ring = (2πRsinθ)Rdθ
Surface area of ring = $2\pi R^2 \sin\theta d\theta$
The mass of the ring =

$$(2\pi R^2 sin\theta d\theta)\frac{M}{4\pi R^2} = \frac{M sin\theta d\theta}{2}$$

If 'x' is the distance of the point P from a point on the ring, then the potential at P due to the ring

$$dV = -\frac{GM sin\theta d\theta}{2x} \quad ----eq(1)$$

From cosine property of triangle OAP
$x^2 = R^2 + r^2 - 2Rr\cos\theta d\theta$
Differentiating
$2xdx = 2Rr \sin\theta d\theta$

$$\therefore\ sin\theta d\theta = \frac{x\,dx}{Rr}$$

On substituting above value of sinθdθ

$$dV = -\frac{GM}{2x}\times\frac{x\,dx}{Rr}$$

$$dV = -\frac{GM}{2Rr}dx$$

Case I
When point P lies outside the spherical shell

$$V = -\frac{GM}{2Rr}\int_{r-R}^{r+R} dx = -\frac{GM}{2Rr}[x]_{r-R}^{r+R}$$

$$V = -\frac{GM}{2Rr}[(r+R)-(r-R)] = -\frac{GM}{r}$$

This is the potential at P due to a point mass M at O

For an external point, a spherical shell behaves as a point mass supposed to be placed at its center

Case II
When the point P lies inside the spherical shell

$$V = -\frac{GM}{2Rr}\int_{R-r}^{R+r} dx = -\frac{GM}{2Rr}[x]_{R-r}^{R+r}$$
$$V = -\frac{GM}{R}$$

This expression is independent of r. Thus, the potential at every point inside the spherical shell is the same and is equal to the potential of the surface of the shell

(d) Gravitational potential due to a homogeneous solid sphere
Case (I)
When the point P lies outside the sphere.
For external point, a solid sphere behaves as if its entire mass is concentrated at the centre.
Case(II)
When the point O lies inside the sphere
Let us consider a concentric spherical surface through the point O. The potential at P arises out of the inner sphere and the outer thick spherical shell
$V = V_1 + V_2$, where V_1 = potential due to the inner sphere and V_2 = potential due to outer thick shell
The mass of the inner sphere =

$$\frac{4}{3}\pi r^3 \rho$$

ρ = density of the sphere =

$$\frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The potential at P due to this sphere

$$V_1 = -\frac{G\left[\frac{4\pi r^3}{3}\right]\rho}{r} = -\frac{4\pi G\rho}{3}r^2$$

To find V_2, consider a thin concentric shell of radius x and thickness dx
The volume of the shell = $4\pi x^2$ dx
The mass of the shell = $4\pi x^2$ dx ρ
The potential at P due to this shell

$$V_2 = -\int_r^R 4\pi G\rho x dx$$

$$V_2 = -4\pi G\rho\left[\frac{x^2}{2}\right]_r^R$$

$$V_2 = -4\pi G\rho\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$V_2 = -2\pi G\rho[R^2 - r^2]$$

$$V = V_1 + V_2 = -\frac{4\pi G\rho}{3}r^2 = -2\pi G\rho[R^2 - r^2]$$

$$V = -\frac{4\pi G\rho}{3}\left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right]$$

$$V = -\frac{4\pi G\rho}{3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

$$V = -\frac{4\pi G}{3}\frac{3M}{4\pi R^3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

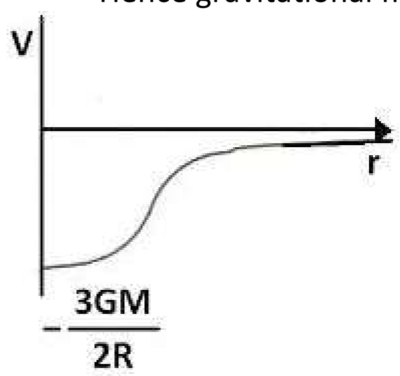
$$V = -\frac{GM}{2R^3}[3R^2 - r^2]$$

At r = 0

$$V = \frac{-3GM}{2R}$$

$$\frac{dV}{dr} = 0$$

Hence gravitational field is 0 at the centre of a solid sphere



Gravitational potential energy

The gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done against gravitational force in moving the mass m from infinity to a distance r

$$U_{(r)} = -\int_{\infty}^{r}\vec{F}\cdot\overrightarrow{dr}$$

Work is done against gravitational force so negative sign

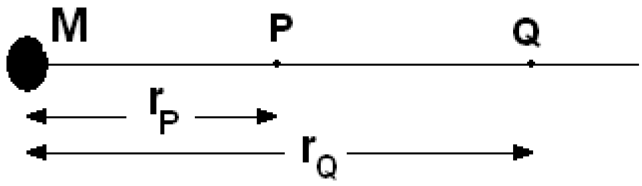
$$U_{(r)} = -\int_{\infty}^{r} \frac{-GMm}{r^2} dr$$

Gravitational force is attractive hence negative sign taken

$$U_{(r)} = GMm\left[\frac{-1}{r}\right]_{\infty}^{r}$$

$$U_{(r)} = -\frac{GMm}{r}$$

Gravitational potential difference



If we take point at P at a distance r_p and other point Q at a distance r_Q . Object of mass m is moved from P to Q then, Work done

$$U = -\int_{r_P}^{r_Q} \frac{-GMm}{r^2} dr$$

$$U = GMm\left[\frac{-1}{r}\right]_{r_P}^{r_Q}$$

$$U = U_P - U_Q = -GMm\left[\frac{1}{r_Q} - \frac{1}{r_P}\right]$$

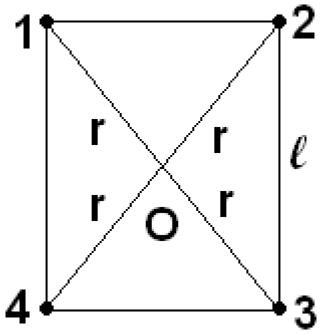
Or

$$U = U_Q - U_P = GMm\left[\frac{1}{r_P} - \frac{1}{r_Q}\right]$$

Solved Numerical

Q) A particle of mass m is placed on each vertex of a square of side l . Calculate the gravitational potential energy of this system of four particles. Also calculate the gravitational potential at the centre of the square

Solution:



Here we can write energy due to every pair of particles as

$$U_{ij} = \frac{-Gm_i m_j}{r_{ij}}$$

Where m_i and m_j respectively are the masses of the particles I and j respectively and r_{ij} is the distance between them. $m_i = m_j = m$

Therefore potential energy

$$U = -Gm^2\left[\sum_{i<j}\frac{1}{r_{ij}}\right]$$

$$U = -Gm^2\left[\frac{1}{r_{12}}+\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}+\frac{1}{r_{24}}+\frac{1}{r_{34}}\right]$$

$$U = -Gm^2\left[\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}+\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}\right]$$

$$U = -Gm^2\left[\frac{4+\sqrt{2}}{l}\right]$$

Gravitational potential at the centre, due to each particle is same
The total gravitational potential at the centre of the square is

V= 4(potential due to every particle)

$$V = 4\left(\frac{-Gm}{r}\right)$$

Where $r = \frac{\sqrt{2}l}{2}$

$$V = \frac{-4\sqrt{2}Gm}{l}$$

Q) Two objects of masses 1kg and 2kg respectively are released from rest when their separation is 10m. Assuming that on it mutual gravitational force act on them, find the velocity of each of them when separation becomes 5m (Take G = 6.66×10^{-11} Nm^2/kg^2)
Solution:
Let v_1 and v_2 be the final velocity of masses, m_1 = 1kg, m_2 = 2 initial velocity is zero
From law of conservation of momentum

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

$$\vec{v}_1 = -\frac{m_2}{m_1}\vec{v}_2$$

$$|\vec{v}_1| = 2|\vec{v}_2| \qquad ----eq(1)$$

Initial potential energy

$$U_i = \frac{-Gm_1m_2}{r_i} = \frac{-(6.67\times10^{-11})(1\times2)}{10}$$

$$U_i = -13.32\times10^{-12}J$$

Final potential energy

$$U_f = \frac{-Gm_1m_2}{r_f} = \frac{-(6.67\times10^{-11})(1\times2)}{5}$$

$$U_f = -26.64\times10^{-12}J$$

Change in Potential energy = -13.32×10^{-12} J
According to law of conservation of energy

ΔK = -ΔU

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_1v_2^2 = 13.32 \times 10^{-12}$$

From equation (1)

$$\frac{4}{2}v_2^2 + \frac{1}{2}(2)v_2^2 = 13.32 \times 10^{-12}$$

$$3v_2^2 = 13.32 \times 10^{-12}$$

$$v_2 = 21.07 \times 10^{-5} m/s$$

$$v_1 = 42.14 \times 10^{-5} m/s$$

Variation in acceleration due to gravity

(a) With altitude

At the surface of earth

$$g = \frac{GM_e}{R_e^2}$$

At height 'h' above the surface of earth

$$g' = \frac{GM_e}{(R_e + h)^2}$$

$$\frac{g'}{g} = \frac{R_e^2}{(R_e + h)^2} = \frac{1}{\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{g}{\left(1 + \frac{h}{R_e}\right)^2}$$

So, with increase in height, g decreases. If h <<R, then for binomial theorem

$$g' = g\left[1 + \frac{h}{R_e}\right]^{-2} = g\left[1 - \frac{2h}{R_e}\right]$$

(b) With depth

At the surface of the earth

$$g = \frac{GM_e}{R_e^2}$$

For a point at the depth 'd' below the surface

Mass the earth enclosed by the surface passing through point P as shown in figure be m then

$$g' = \frac{Gm}{(R_e - d)^2}$$

We know that gravitational at point P due to shaded portion is zero thus

$$m = \frac{4}{3}\pi(R_e - d)^3 \times \frac{M_e}{\frac{4}{3}\pi R_e^3}$$

$$m = \frac{M_e}{R_e^3}(R_e - d)^3$$

$$g' = \frac{G}{(R_e - d)^2}\frac{M_e}{R_e^3}(R_e - d)^3$$

$$g' = \frac{GM_e}{R_e^3}(R_e - d)$$

Thus

$$\frac{g'}{g} = \frac{\frac{GM_e}{R_e^3}(R_e - d)}{\frac{GM_e}{R_e^2}} = \frac{R_e - d}{R_e}$$

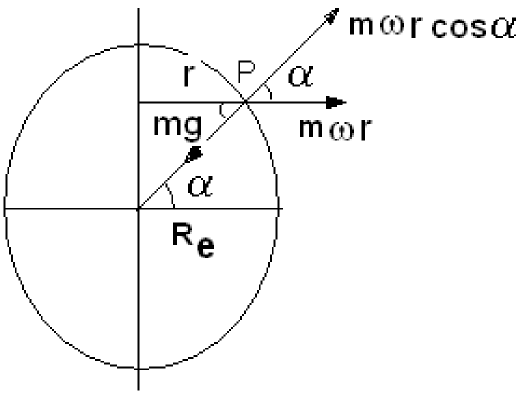
$$g' = g\left[1 - \frac{d}{R_e}\right]$$

So with increase in depth below the surface of the earth, g decreases and at the center of the earth it becomes zero

It should be noted that value of g decreases, if we move above the surface or below the surface of the earth

(C) Due to rotation of the earth

The earth is rotating about its axis from west to east. So, the earth is a non-inertial frame of reference. Everybody on its surface experiences a centrifugal force. Consider a point P . Perpendicular distance from point with axis of rotation is r . Then centrifugal force at point is $m\omega^2 r \cos\alpha$, going ouward where α is the latitude of the place.



Here α is the angle made by the line joining a given place on the Earth's surface to the centre of the Earth with the equatorial line is called latitude of the place. Hence for equator latitude is and for poles latitude is 90^O

Gravitational force mg is acting towards the centre of earth.

Thus resultant force

Is $mg' = mg - m\omega^2 r \cos\alpha$ ----eq(1)

From the geometry of figure $r = R_e\cos\alpha$ theus from equation (1)

$g' = g - \omega^2 R_e\cos^2\alpha$

Cases

(i) At equator α = 0 ∴ cos α = 1

∴ $g' = g - \omega^2 R_e$

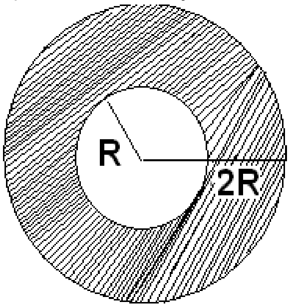
Which shows minimum value of the effective gravitational acceleration

(ii) At poles, α = 90 ∴ cos α = 0

∴ g' = g , which shows the maximum value of the effective gravitational acceleration

Solved Numerical

Q) The density of the core of planet is ρ_1 and that of the outer shell is ρ_2. The radii of the core and that of the planet are R and 2R respectively. Gravitational acceleration at the surface of the planet is same as at a depth R. Find the ratio ρ_1/ρ_2



Solution:

Mass of inner sphere M_1

$$M_1 = \frac{4}{3}\pi R^3 \rho_1$$

Volume of outer shell

$$\frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(7R^3)$$

Mass of outer shell M_2

$$\frac{4}{3}\pi(7R^3)\rho_2$$

Gravitational acceleration at surface of planet:

$$a = \frac{G}{(2R)^2}(M_1 + M_2)$$

$$a = \frac{G}{(2R)^2}\left(\frac{4}{3}\pi R^3\rho_1 + \frac{4}{3}\pi 7R^3\rho_2\right)$$

$$a = \frac{GR\pi}{3}(\rho_1 + 7\rho_2) \quad ----- eq(1)$$

Gravitational acceleration at depth R

$$a' = \frac{G\frac{4}{3}\pi R^3\rho_1}{R^2} = \frac{4G\pi R\rho_1}{3} \quad ----eq(2)$$

Given a = a' thus

$$\frac{GR\pi}{3}(\rho_1 + 7\rho_2) = \frac{4G\pi R\rho_1}{3}$$

$$(\rho_1 + 7\rho_2) = 4\rho_1, \qquad \frac{\rho_1}{\rho_2} = \frac{7}{3}$$

Satellite

(a) Orbital speed of satellite

The velocity of a satellite in its orbit is called orbital velocity. Let v_o be the orbital velocity Gravitational force provides necessary centripetal acceleration

$$\therefore \frac{GM_e m}{r^2} = \frac{mv_o^2}{r}$$

$$v_0 = \sqrt{\frac{GM_e}{r}}$$

As $r = R_e + h$

$$v_0 = \sqrt{\frac{GM_e}{R_e + h}}$$

Notes

Orbital velocity is independent of the mass of the body and is always along the tangent to the orbit

Close to the surface of the earth, r = R as h=0

$$v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \sqrt{10 \times 6.4 \times 10^6} = 8\ km/s$$

(b) Time period of a Satellite

The time taken by a satellite to complete one revolution is called the time period (T) of the satellite

It is given by

$$T = \frac{2\pi}{v_o} = 2\pi r\sqrt{\frac{r}{GM}}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$\Longrightarrow T^2 \propto r^3$$

(c) Angular momentum of a satellite (L)

In case of satellite motion, angular momentum will be given by

$$L = mvr = mr\sqrt{\frac{GM}{r}}$$

Or

$$L = (m^2 GMr)^{1/2}$$

In the case of satellite motion, the net force on the satellite is centripetal force. The torque of this force about the centre of the orbit is zero. Hence, angular momentum of the satellite is conserved. i.e L is constant

(d) Energy of satellite

The P.E. of a satellite is

$$U = -\frac{GMm}{r}$$

The kinetic energy of the satellite is

$$K = \frac{1}{2}mv_0^2$$

$$But\ v_0 = \sqrt{\frac{GM}{r}}$$

$$K = \frac{GMm}{2r}$$

Total mechanical energy of the satellite

$$E = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r}$$

Note

We have K = -E

Also U = 2E

Total energy of a satellite in its orbit is negative. Negative energy means that the satellite is bounded to the central body (earth) by an attractive force and energy must be supplied to remove it from the orbit to infinity.

(e) Binding energy of the satellite

The energy required to remove the satellite from its orbit to infinity is called binding energy of the satellite. i.e.

$$Binding\ energy = -E = \frac{GMm}{2r}$$

Solved Numerical

Q) An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth

a) Determine the height of satellite above the earth's surface

b) If the satellite is suddenly stopped, find the speed with which the satellite will hit the earth's surface after falling down

Solution:

Escape velocity = √(2gR), where g is the acceleration due to gravity on the surface of earth and R is the radius

Orbital velocity =

$$\frac{1}{2}v_e = \frac{1}{2}\sqrt{2gR} = \sqrt{\frac{gR}{2}} \qquad ----eq(1)$$

(1/2)V_e = (1/2) √(2gR) =√(gR/2)

a) If h is the height of satellite above earth's surface, the gravitational force provides the centripetal force for circular motion

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$\Rightarrow v_0^2 = \frac{GM}{R+h} = \frac{gR^2}{(R+h)}$$

$$\therefore \left(\frac{1}{2}v_0\right)^2 = \frac{gR^2}{R+h}$$

From equation (1)

$$\frac{gR}{2} = \frac{gR^2}{R+h}$$

R+h = 2R

H = R

b) If the satellite is stopped in orbit, the kinetic energy is zero and its potential energy is

$$\frac{-GMm}{2R}$$

When it reaches the earth, let v be its velocity

Hence kinetic energy = (1/2) mv^2

Potential energy =

$$\frac{-GMm}{R}$$

By the law of conservation of energy

$$\therefore \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2R}$$

$$v^2 = 2GM\left(\frac{1}{R} - \frac{1}{2R}\right) = \frac{2gR^2}{2R} = gR$$

Velocity with which the satellite will hit the earth's surface after falling down is

$$v = \sqrt{gR}$$

Q) Two satellites of same masses are launched in the same orbit around the earth so as to rotate opposite to each other. They collide inelastically and stick together as wreckage. Obtain the total energy of the system before and after collisions. Describe the subsequent motion of wreckage.

Solution

Potential energy of satellite in orbit

$$-\frac{GMn}{r}$$

If v is the velocity in orbit, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

Kinetic energy

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Total energy

$$\frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

For the two satellites, the total energy before collision

$$2\left(-\frac{GMm}{2r}\right) = -\frac{GMm}{r}$$

After collision, let v' be the velocity of the wreckage. By the law of conservation of momentum, since they are approaching each other

$$m\vec{v} - m\vec{v} = 2mv'$$
$$\therefore v' = 0$$

The wreckage has no kinetic energy after collision but has potential energy

$$P.E. = \frac{-GM(2m)}{r}$$

Total energy after collision = $\frac{-2GMm}{r}$

After collision, the centripetal force disappears and the wreckage falls down under the action of gravity.

Geostationary satellite

If there is a satellite rotating in the direction of earth's rotation. i.e. from west to east, then for an observer on the earth the angular velocity of the satellite will be same as that of earth $\omega_S = \omega_E$

However, if $\omega_S = \omega_E = 0$, satellite will appear stationary relative to the earth. Such a satellite is called 'Geostationary satellite' and is used for communication purposes

The orbit of geostationary satellite is called 'Parking Orbit'

We know that

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$For\ geostationary\ satellite, T = 24\ Hours$$

Putting this value of T in the above equation, we get

R = 42000Km

Or h= 3600.0 km

Where h is height of the satellite from the surface of the earth

Weightlessness in a satellite

When the astronaut is in an orbiting satellite, both the satellite and astronaut have the same acceleration towards the centre of the Earth. Hence, the astronaut does not exert any force on the floor of the satellite. So, the floor of the satellite also does not exert any force of reaction on the astronaut. As there is no reaction, the astronaut has a feeling of weightlessness.

The radial acceleration of the satellite is given by

$$a_r = \frac{F_r}{m} = \frac{GMm}{r^2} \times \frac{1}{m} = \frac{GM}{r^2}$$

For an astronaut of mass m_a inside the satellite, we have following forces

$$\text{Downward force} = \frac{GMm_a}{r^2}$$

Upward pseudo force as motion of satellite is accelerated motion

$$\text{Upward force} = \frac{GMm_a}{r^2}$$

Thus resultant force on Astronaut is zero, or normal force is zero
Hence, the astronaut feels weightlessness

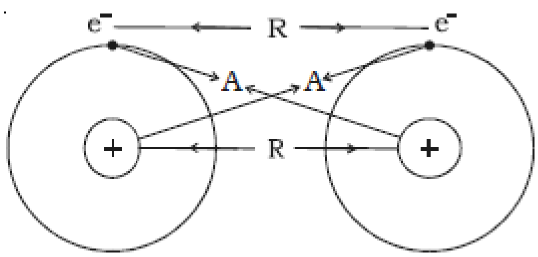
SOLIDS AND FLUIDS

ELASTICITY

In solids, the atoms and molecules are free to vibrate about their mean positions. If this vibration increases sufficiently, molecules will shake apart and start vibrating in random directions. At this stage, the shape of the material is no longer fixed, but takes the shape of its container. This is liquid state. Due to increase in their energy, if the molecules vibrate at even greater rates, they may break away from one another and assume gaseous state. Water is the best example for this changing of states. Ice is the solid form of water. With increase in temperature, ice melts into water due to increase in molecular vibration.
If water is heated, a stage is reached where continued molecular vibration results in a separation among the water molecules and therefore steam is produced. Further continued heating causes the molecules to break into atoms.

Intermolecular or inter atomic forces

Consider two isolated hydrogen atoms moving towards each other as shown in Fig As they approach each other, the following interactions are observed.



(i) Attractive force A between the nucleus of one atom and electron of the other. This attractive force tends to decrease the potential energy of the atomic system.
(ii) Repulsive force R between the nucleus of one atom and the nucleus of the other atom and electron of one atom with the electron of the other atom. These repulsive forces always tend to increase the energy of the atomic system. There is a universal tendency of all systems to acquire a state of minimum potential energy. This stage of minimum potential energy corresponds to maximum stability. If the net effect of the forces of attraction and repulsion leads to decrease in the energy of the system, the two atoms come closer to each other and form a covalent bond by sharing of electrons. On the other hand, if the repulsive forces are more and there is increase in the energy of the system, the atoms will repel each other and do not form a bond. The forces acting between the atoms due to electrostatic interaction between the charges of the atoms are called inter atomic forces. Thus, inter atomic forces are electrical in nature. The inter atomic forces are active if the distance between the two atoms is of the order of atomic size ≈ 10^{-10} m. In the case of molecules, the range of the force is of the order of 10^{-9} m.

Elasticity

When an external force is applied on a body, which is not free to move, there will be a relative displacement of the particles. Due to the property of elasticity, the particles tend to regain their original position. The external forces may produce change in length, volume and shape of the body.

This external force which produces these changes in the body is called deforming force. A body which experiences such a force is called deformed body.
When the deforming force is removed, the body regains its original state due to the force developed within the body. This force is called restoring force.
The property of a material to regain its original state when the deforming force is removed is called elasticity.
The bodies which possess this property are called elastic bodies. Bodies which do not exhibit the property of elasticity are called plastic. The study of mechanical properties helps us to select the material for specific purposes. For example, springs are made of steel because steel is highly elastic

Stress and strain

In a deformed body, restoring force is set up within the body which tends to bring the body back to the normal position. The magnitude of these restoring force depends upon the deformation caused. *This restoring force per unit area of a deformed body is known as stress.* This is measured by the magnitude of the deforming force acting per unit area of the body when equilibrium is established.

$$Stree = \frac{restoring\ force}{Area}$$

Unit of stress in S.I. system is N/m^2. When the stress is normal to the surface, it is called Normal Stress. The normal stress produces a achange in length or a change in volume of the body. The normal stress to a wire or a body may be compressive or tensile (expansive) according as it produces a decrease or increase in length of a wire or volume of the body. When the stress is tangential to the surface, it is called tangential (shearing) stress

Solved Numerical

Q) A rectangular bar having a cross-sectional area of 28 mm^2 has a tensile force of a 7KN applied to it. Determine the stress in the bar
Solution
Cross-sectional area A = 25mm^2 = $28\times(10^{-3})^2 = 28\times 10^{-6}$ m^2
Tensile force F = 7KN = 7×10^3N

$$Stree = \frac{7\times 10^3}{28\times 10^{-6}} = 0.25\times 10^9 N/m^2$$

Strain

The external force acting on a body cause a relative displacement of its various parts. A change in length volume or shape takes place. The body is then said to be strained. The relative change produced in the body under a system of force is called strain

$$Strain\ (\varepsilon) = \frac{Change\ in\ dimension}{original\ dimension}$$

Strain has no dimensions as it is a pure number. The change in length per unit length is called linear strain. The change in volume per unit volume is called Volume stain. If there is a change in shape the strain is called shearing strain. This is measured by the angle through which a line originally normal to the fixed surface is turned

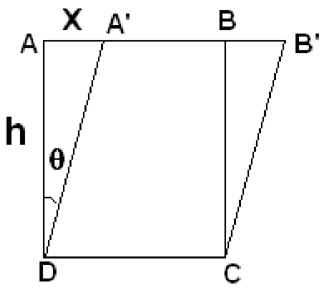
Longitudinal Strain: The ratio of change in length to original length

$$\varepsilon_l = \frac{\Delta l}{l}$$

Volume strain

$$\varepsilon_v = \frac{\Delta v}{v}$$

Shearing strain

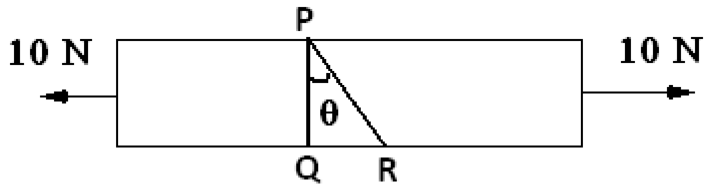


In figure a body with square cross section is shown a tangential force acts on the top surface AB causes shift of Surface by 'X' units shown as surface A'B' ,thus side DA' now mates an angle of θ with original side DA of height h

$$\varepsilon_S = \frac{x}{h} = tan\theta$$

Solved Numerical

Q) As shown in figure 10N force is applied at two ends of a rod. Calculate tensile stress and shearing stress for section PR. Area of cross-section PQ is 10 cm^2 , θ=30^O



Solution

Given cross-section area of PQ = 10 cm^2

Now PQ = PRcosθ

10 = PRcos30

10= PR (√3 /2)

PR = 20/√3 cm^2 or 2/√3 m^2

Now normal force to area PR will be Fcos30 = 10 ×(√3 /2) = 5√3 N

Tangential force to area PR will be Fsin30 = 10×(1/2) = 5 N

∴ Tensile stress for section PR

$$\sigma_l = \frac{normal\ force}{area\ of\ PR} = \frac{5\sqrt{3}}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 7.5 \times 10^3 \frac{N}{m^2}$$

Shearing stress for section PR

$$\sigma_t = \frac{tangential\ force}{area\ of\ PR} = \frac{5}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 2.5\sqrt{3} \times 10^3 \frac{N}{m^2}$$

Hooke's Law and types of moduli

According to Hooke's law, *within the elastic limit, strain produced in a body is directly proportional to the stress that produces it.*

$$\frac{stress}{strain} = constant = \lambda$$

Where λ is called modulus of elasticity.

Its unit is N m^{-2} and its dimensional formula is $ML^{-1}T^{-2}$.

Depending upon different types of strain, the following three moduli of elasticity are possible

(i) Young's modulus: When a wire or rod is stretched by a longitudinal force the ratio of the longitudinal stress to the longitudinal strain within the elastic limits is called Young's modulus

$$Young'smodulus\ (Y) = \frac{Longitudinal\ stress}{linear\ strain}$$

Consider a wire or rod of length L and radius r under the action of a stretching force applied normal to its face. Suppose the wire suffers a change in length l then

$$Longitudinal\ stress = \frac{F}{\pi r^2}$$

$$Linear\ strain = \frac{l}{L}$$

$$Young'smodulus\ (Y) = \frac{\frac{F}{\pi r^2}}{\frac{l}{L}} = \frac{FL}{\pi r^2 l}$$

(ii) Bulk modulus: When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, the shape remains the same, but there is a change in volume. The force perunit area applied normally and uniformly over the surface is called normal stress. The change in volume per unit volume is called volume or bulk strain.

$$Bulk\ modulus\ (B) = \frac{Volume\ stress}{Volume\ strain}$$

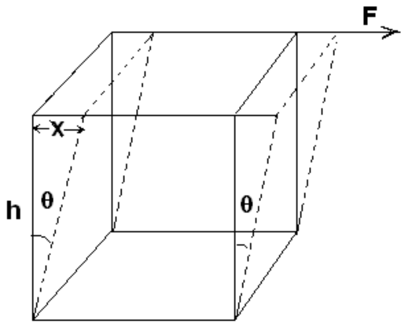
$$B = \frac{-\frac{F}{A}}{\frac{\Delta V}{V}} = -\frac{FV}{A\Delta V}$$

Negative sign indicate reduction in volume

The reciprocal of bulk modulus is called compressibility

$$compressibility = \frac{1}{bulk\ modulus}$$

(iii) Modulus of rigidity: According to the definition, the ratio of shearing stress to shearing strain is called modulus of rigidity (η) . in this case the sphape of the body changes but its volume remains unchanged. Consider the case of a cube fixed at its lower face and acted upon by a tangential force F on its upper surface of area A as shown in figure



$$shearing\ stress = \frac{F}{A}$$

$$Shearing\ strain = \theta = \frac{x}{h}$$

$$\eta = \frac{F}{A\theta} = \frac{Fh}{Ax}$$

Solved Numerical

Q) A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in cylindrical container. A massless piston of area A flots on the surface of the liquid. Find the fractional change in the radius of the sphere (dR/R) when a mass M is placed on the piston to compress the liquid

Solution

From the formula of Bulk modulus

$$B = -\frac{FV}{A\Delta V}$$

$$V = \frac{4}{3}\pi R^3$$

$$dV = 4\pi R^2 dR$$

$$B = -\frac{F\frac{4}{3}\pi R^3}{A4\pi R^2 dR}$$

$$\frac{dR}{R} = \frac{Mg}{3AB}$$

Q) Find the natural length of rod if its length is L_1 under tension T_1 and L_2 under tension T_2 within the limits of elasticity

Solution

From the formula of Young's modulus

$$Young'smodulus\ (Y) = \frac{\frac{F}{A}}{\frac{l}{L}}$$

Let increase in length for tension T_1 be x and that for tension T_2 be y then

$$\frac{\frac{T_1}{A}}{\frac{x}{L}} = \frac{\frac{T_2}{A}}{\frac{y}{L}}$$

$$\frac{T_1}{x} = \frac{T_2}{y}$$

$$T_1 y = T_2 x$$

But x = L_1 – L and y = L_2 – L

$$T_1(L_2 - L) = T_2(L_1 - L)$$

On simplification we get

$$L = \frac{(L_1T_2 - L_2T_1)}{(T_2 - T_1)}$$

Q) A copper wire of negligible mass, 1 m length and cross-sectional area $10^{-6}m^2$ is kept on a smooth horizontal table with one end fixed. A ball of mass 1kg is attached to the other end. The wire and the ball are rotated with an angular velocity of 20 rad/s. if the elongation in the wire is 10^{-3}m, obtain the Young's modulus. If on increasing the angular velocity to 100 rad/s the wire breaks down, obtain the breaking stress.

Solution

Given m = 1kg, ω= 20 rad/s, L = 1m ΔL = 10^{-3} m, A = 10^{-6} m^2

Tension in the thread

T = $m\omega^2 L = 1\times(20)^2\times1$ = 400N

$$Y = \frac{TL}{A\Delta L} = \frac{400 \times 1}{10^{-6} \times 10^{-3}} = 4 \times 10^{11} N/m^2$$

On increasing the angular velocity to 100 rad/s, the wire breaks down then

$$breaking\ stress = \frac{T'}{A} = \frac{m(\omega')^2 L}{A}$$

$$breaking\ stress = \frac{1 \times (100)^2 \times 1}{10^{-6}} = 10^{10} N/m^2$$

Q) A cube is subjected to pressure of 5×10^5 N/m^2. Each side of the cubic is shorteed by 1%. Find volumetric strain and bulk modulus of elasticity of cube

Solution

V = l^3

Now dV = $3l^2$ dl

Thus

$$\frac{dV}{V} = \frac{3l^2 dl}{l^3} = 3\frac{dl}{l}$$

Sides are reduced by 1% thus dl/l =- 0.01

Thus reduction in volume = -0.03

Normal stress = Increase in pressure

$$B = -\frac{P}{\frac{\Delta V}{V}} = \frac{5 \times 10^5}{0.03} = 1.67 \times 10^7 N/m^2$$

Q) A rubber cube of each side 7cm has one side fixed, while a tangential force equal to the weight of 300kg f is applied to the opposite face. Find the shearing strain produced and the distance through which the strained site moves. The modulus of rigidity for rubber is 2×10^7 dyne/cm^2 g = 10m/s^2

Solution

Here L = 7cm = 7×10^{-2} m

F = 300 kg f = 300×10 N

Modulus of rigidity η = 2×10^7 dynes/cm^2 = 2×10^6 N/m^2

As

$$\eta = \frac{F}{A\theta}$$

$$\theta = \frac{F}{A\eta} = \frac{F}{h^2\eta}$$

$$\theta = \frac{3000}{(7\times10^{-2})^2\times2\times10^6} = 0.3\ rad$$

$$\theta = \frac{x}{h}$$

X = hθ

X = 7×0.3 = 2.1 cm

Poisson's Ratio:

It is the ratio of lateral strain to the longitudinal strain. For example, consider a force F applied along the length of the wire which elongates the wire along the length and it contracts radially. Then the longitudinal strain =Δl/l and lateral strain = Δr/r , where r is the radius of the wire

$$Poisson's\ ratio\ (\sigma) = -\frac{\frac{\Delta r}{r}}{\frac{\Delta l}{l}}$$

$$\frac{\Delta r}{r} = -\sigma\frac{\Delta l}{l}$$

For rectangular bar: let b be breadth and h be thickness then

$$\frac{\Delta b}{b} = -\sigma\frac{\Delta l}{l}$$

$$\frac{\Delta h}{h} = -\sigma\frac{\Delta l}{l}$$

The negative sign indicates that change in length and radius is of opposite sign. Change in volume due to longitudinal force

Due to application of tensile force, lateral dimension decreases and length increases. As a result there is a change in volume (usually volume increases). Let us consider the case of a cylindrical rod of length l and radius r.

Since V=πr^{2}L

$$\therefore \frac{\Delta V}{V} = 2\frac{\Delta r}{r} + \frac{\Delta l}{l} \ (for\ very\ small\ change)$$

From above equations or radius and Length

$$\therefore \frac{\Delta V}{V} = -2\sigma\frac{\Delta l}{l} + \frac{\Delta l}{l}$$

$$\therefore \frac{\Delta V}{V} = \frac{\Delta l}{l}(1-2\sigma)$$

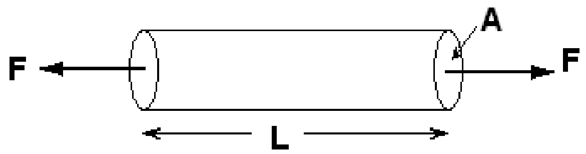
Longitudinal Strain: $\varepsilon_l = \frac{\Delta l}{l}$

$$\therefore \frac{\Delta V}{V} = \varepsilon_l(1-2\sigma)$$

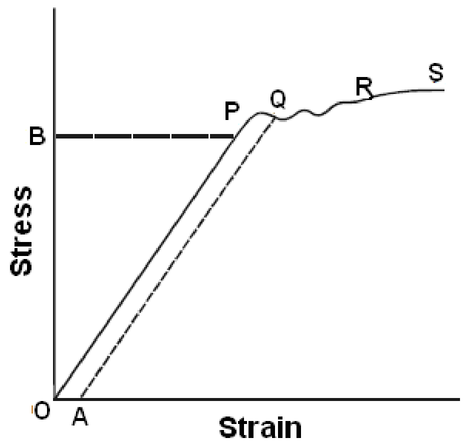
Above equation suggest that since Δv > 0, value of σ cannot exceed 0.5

Stress –Strain relationship for a wire subjected to longitudinal stress

Consider a long wire (made of steel) of cross-sectional area A and original length L in equilibrium under the action of two equal and opposite variable force F as shown in figure. Due to the application of force, the length gets changed to L +l. Then, longitudinal stress = F/A and Longitudinal strain = l/L



The extension of the wire is suitably measured and a stress – strain graph is plotted



(i) In the figure the region OP is linear. Within a normal stress, strain is proportional to the applied stress. This is
Hooke’s law. Up to P, when the load is removed the wire
regains its original length along PO. The point P represents the **elastic limit**, PO represents the elastic range of the material and OB is the **elastic strength**.
(ii) Beyond P, the graph is not linear. In the region PQ the material is partly elastic and partly plastic. From Q, if we start decreasing the load, the graph does not come to O via P, but traces a straight line QA.
Thus a permanent strain OA is caused in the wire. This is called **permanent set**.
(iii) Beyond Q addition of even a very small load causes enormous strain. This point Q is called the yield point. The region QR is the **plastic range**.
(iv) Beyond R, the wire loses its shape and becomes thinner and thinner in diameter and ultimately breaks, say at S. Therefore S is the **breaking point**. The stress corresponding to S is called **breaking stress**.

Elastic potential energy or Elastic energy stored in a deformed body

The elastic energy is measured in terms of work done in straining the body within its elastic limit

Let F be the force applied across the cross-section A of a wire of length L. Let l be the increase in length. Then

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al}$$

$$F = \frac{YAl}{L}$$

If the wire is stretched further through a distance of dl, the work done dw

$$dW = F \times dl = \frac{YAl}{L} dl$$

Total work done in stretching the wire from original length L to a length L +l (i.e. from l = 0 to l = l)

$$W = \int_0^l \frac{YAl}{L} dl$$

$$W = \frac{YA}{L}\frac{l^2}{2} = \frac{1}{2}(AL)\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)$$

$$W = \frac{1}{2} \times volume \times stress \times strain$$

Solved Numerical

Q) The rubber cord of catapult has a cross-section area 1mm^2 and total unstrtched length 10 cm. It is stretched to 12cm and then released to project a body of mass 5g. taking the Young's modulus of rubber as 5×10^8 N/m^2, calculate the velocity of projection

Solution

It can be assumed that the total elastic energy of catapult is converted into kinetic energy of the body without any heat loss

L = 12cm = 12×10^{-2} m , l = 2cm = 2×10^{-3} m , A = 1mm^2 = 10^{-6}m

$$U = \frac{YA}{L}\frac{l^2}{2} = \frac{5 \times 10^8 \times (1 \times 10^{-6}) \times (2 \times 10^{-2})^2}{2 \times 10 \times 10^{-2}} = 1$$

Now K.E of projectile = elastic energy of catapult

$$\frac{1}{2}mv^2 = U$$

$$\frac{1}{2} \times 5 \times 10^{-3} \times v^2 = 1$$

V = 20 m/s

FLUID STATICS

Thrust and Pressure

A perfect fluid resists force normal to its surface and offers no resistance to force acting tangential to it surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the surface of water. Thus fluids are capable of exerting normal stress on the surface with it is in contact

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure

Variation of pressure with height

Let *h* be the height of the liquid column in a cylinder of cross sectional area *A*. If ρ is the density of the liquid, then weight of the liquid column *W* is given by

W = mass of liquid column × g = $Ah\rho g$

By definition, pressure is the force acting per unit area.

$$Pressure = \frac{weight\ of\ liquid\ column}{area\ of\ cross\ -\text{section}}$$

$$P = \frac{Ah\rho g}{A} = h\rho g$$

$dP = \rho g\,(dh)$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases. Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of liquid. With ρ as constant, equation may be integrated as it stands, and the result is

$P = P_0 + \rho gh$

The pressure P_0 is the pressure at the surface of the liquid where h = 0

Force due to fluid on a plane submerged surface

The pressure at different points on the submerged surface varies so to calculate the resultant force, we divide the surface into a number of elementary areas and we calculate the force on it first by treating pressure as constant then we integrate it to get the net force i.e $F_R = \int P\,(dA)$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis

Solved Numerical

Q) Water is filled upto the top in a rectangular tank of square cross-section. The sides of cross-section is a and height of the tank is H. If density of water is ρ, find force on the bottom of the tank and on one of its wall. Also calculate the position of the point of application of the force on the wall

Solution

Force on the bottom of thank

Area of bottom of tank = a^2

Force = pressure × Area

Force = $H\rho g a^2$

Force on the wall and its point of application

Force on the wall of the tank is different at different heights so consider a segment at depth h of thickness dh

Pressure at depth h = $h\rho g$

Area of strip = a dh

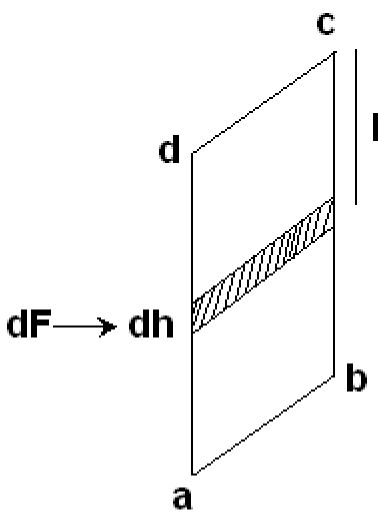
Force on strip dF = $h\rho g$ a dh

Total force at on the wall

$$F = \int_0^H \rho g a h \, dh$$

$$F = \rho g a \left[\frac{h^2}{2}\right]_0^H$$

$$F = \rho g a \frac{H^2}{2}$$



The point of application of the force on the wall can be calculated by equating the moment of resultant force about any line, say dc to the moment of distributioed force about the same line dc

Moment of dF about line cd = dF (h) = ($h\rho g a dh$) h = $\rho g a h^2$ dh

∴ Net moment of distributed forces

$$\rho g a \int_0^H h^2 dh = \rho g a \frac{H^3}{3}$$

Let the point of application of the net force is at a depth 'x' from the line cd

Then the torque of the resultant force about the line cd =

$Fx = \rho g a \frac{H^2}{2} x$

Now Net moment of distribution of force = Torque

$$\rho g a \frac{H^3}{3} = \rho g a \frac{H^2}{2} x$$

$$x = \frac{2H}{3}$$

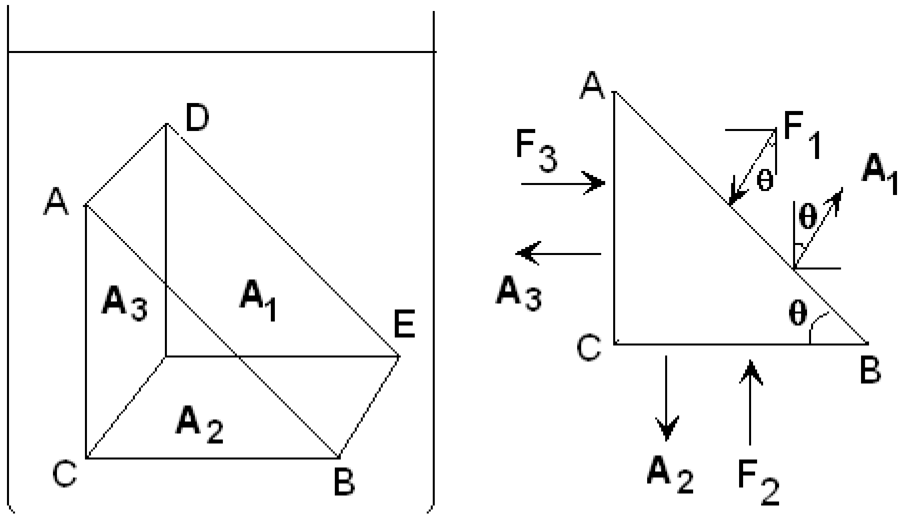
Hence, the resultant force on the vertical wall of the tank will act at a depth 2H/3 from the free surface of water or at the height of H/3 from bottom of tank

Pascal’s Law

Pascal’s law states that if the effect of gravity can be neglected then the pressure in an incompressible fluid in equilibrium is the same everywhere..

This statement can be verified as follows

Consider a small element of liquid in the interior of the liquid at rest. The liquid element is in the shape of prism consisting of two right angled triangle surfaces



Let the areas of surface ADEB, CFEB, ADFC be A_1, A_2, A_3

It is clear from figure that

$A_2 = A_1\cos\theta$ and $A_3 = A_1\sin\theta$

Also, since liquid element is in equilibrium $F_3 = F_1\cos\theta$ and $F_3 = F_1\sin\theta$

now pressure on surface ADEB is $P_1 = F_1 / A_1$

Pressure on the surface CFED is

$$P_2 = \frac{F_2}{A_2} = \frac{F_1 cos\theta}{A_1 cos\theta} = \frac{F_1}{A_1}$$

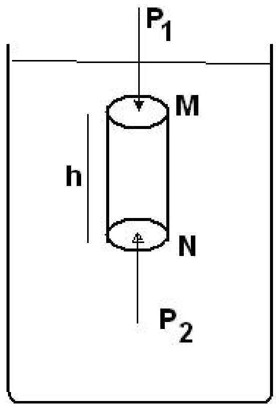
And pressure on the surface ADFC is

$$P_3 = \frac{F_3}{A_3} = \frac{F_1 sin\theta}{A_1 sin\theta} = \frac{F_1}{A_1}$$

So, $P_1 = P_2 = P_3$

Since θ is arbitrary this result holds for any surface. Thus Pascal’s law is verified

Pascal’s law and effect of gravity



When gravity is taken into account, Pascal’s law is to be modified. Consider a cylindrical liquid column of height *h* and density ρ in a vessel as shown in the Fig.

If the effect of gravity is neglected, then pressure at *M* will be equal to pressure at *N*.

But, if force due to gravity is taken into account, then they are not equal.

As the liquid column is in equilibrium, the forces acting on it are balanced. The vertical forces acting are

(i) Force P_1A acting vertically down on the top surface.
(ii) Weight mg of the liquid column acting vertically downwards.
(iii) Force P_2A at the bottom surface acting vertically upwards. where P_1 and P_2 are the pressures at the top and bottom faces, A is the area of cross section of the circular face and m is the mass of the
cylindrical liquid column.

$$\text{At equilibrium, } P_1A + mg - P_2A = 0 \text{ or } P_1A + mg = P_2A$$
$$P_2 = P_1 + mg\ A$$
$$\text{But } m = Ah\rho$$
$$\therefore P_2 = P_1 + Ah\rho gA$$
$$(i.e)\ P_2 = P_1 + h\rho g$$

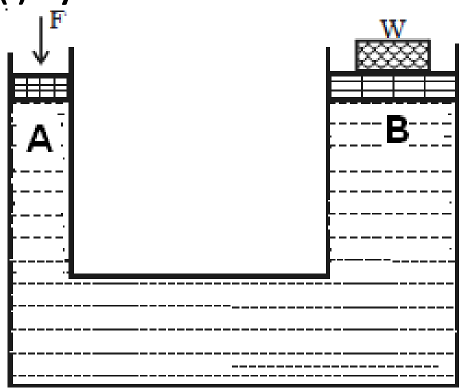
This equation proves that the pressure is the same at all points at the same depth. This results in another statement of *Pascal's law* which can be stated as *change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid and act in all directions.*

Characteristics of the fluid pressure

(i) Pressure at a point acts equally in all directions
(ii) Liquids at rest exerts lateral pressure, which increases with depth
(iii) Pressure acts normally on any area in whatever orientation the area may be held
(iv) Free surface of a liquid at rest remains horizontal
(v) pressure at every point in the same horizontal line is the same inside a liquid at rest
(vi) liquid at rest stands at the same height in communicating vessels

Application of Pascal's law

(i) Hydraulic lift



An important application of Pascal's law is the hydraulic lift used to lift heavy objects. A schematic diagram of a hydraulic lift is shown in the Fig.. It consists of a liquid container which has pistons fitted into the small and large opening cylinders. If a_1 and a_2 are the
areas of the pistons A and B respectively, F is the force applied on A and W is the load on B, then

$$\frac{F}{a_1} = \frac{W}{a_2}$$
$$F = W\frac{a_1}{a_2}$$

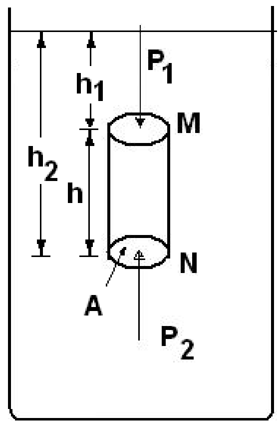
This is the load that can be lifted by applying a force F on A. In the above equation a_2/a_1 is called mechanical advantage of the hydraulic lift. One can see such a lift in many automobile service stations.

Buoyancy and Archimedes principle

If an object is immersed in or floating on the surface of a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as Buoyant force or force of Buoyancy and it acts from the centre of gravity of the displaced liquid. According to Archimedes principle, "the magnitude of force of buoyancy is equal to the weight of the displaced liquid"

To prove Archimedes principle, consider a body totally immersed in a liquid as shown in the figure.

The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to that one shown in figure



The net vertical force on the element is

dF = (P_2- P_1) A

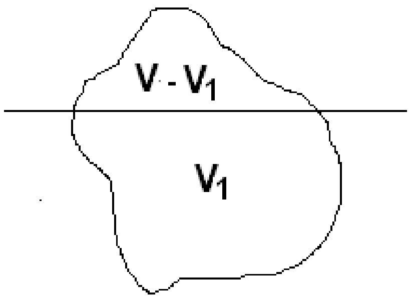
$$F = [(P_0 + h_2\rho g) - (P_0 + h_1\rho g)]A$$
$$F = (h_2 - h_1)\rho g A$$
$$F = h\rho g A$$

But volme V = hA

Thus F = Vρg

∴ force of Buoyancy = Vρg = Weight of liquid displaced

Expression for immersed volume of a floating Body



Let a solid of volume V and density ρ floats in liquid of density ρ_0. Volume V_1 of the body is immersed inside the liquid

The weight of floating body = Vρg

The weight of the displaced liquid = $V_1\rho_0 g$

For the body to float

Weight of body = Weight of liquid displaced

$V\rho g = V_1\rho_0 g$

$$\frac{V_1}{V} = \frac{\rho}{\rho_0}$$
$$V_1 = \frac{\rho V}{\rho_0}$$

∴ Immersed volume = mass of solid / density of liquid

From above it is clear that density of the solid volume must be less than density of the liquid to enable it to float freely in the liquid. How ever a metal vessel floats in water though the density of metal is much higher that the that of eater because floating bodies are hollow inside and hence displaces large volume. When thy float on water, the weight of the displaced water is equal to the weight of the body

Laws of floatation

The principle of Archimedes may be applied to floating bodies to give the laws of flotation

(i) When a body floats freely in a liquid the weight of the floating body is equal to the weight of the liquid displaced

(ii) The centre of gravity of the displaced liquid B (called the centre of buoyancy) lies vertically above or below the centre of gravity of the floating body G

Solved numerical

Q) A stone of mass 0.3kg and relative density 2.5 is immersed in a liquid of relative density 1.2. Calculate the resultant up thrust exerted on the stone by the liquid and the weight of stone in liquid

Solution

Volume of stone V = mass/density

V = 0.3/2.5 = 0.12 m^3

Upward thrust =buoyant force = Vρ_0g =0.12×1.2×9.8 = 1.41 N

Weight of stone in liquid = Gravitational force – buoyant force

=0.3×9.8 – 1.41 = 1.53 N or 0.156 kg wt

Q) A metal cube floats on mercury with (1/8) th of its volume under mercury. What portion of the cube will remain under mercury if sufficient water is added hust to cover the cube. Assume that the top surface of the cube remains horizontal in both cases. Relative density of mercury = 13.6

From the formula

$$V_1 = \frac{\rho V}{\rho_0}$$

Here V_1 is volume immersed in mercury = V/8 given and ρ_0 density of mercury, ρ density of metal

$$\frac{V}{8} = \frac{\rho V}{13.5}$$

ρ=1.725 is density of metal

Now let V' be the volume immersed in mercury then V-V' is volume immersed in water then

Weight of metal = Buoyant force due to Water + Buoyant force due to mercury

V(1.725) g= (V-V')×1×g +V'×13.6g

V(1.725) = (V-V')×1 + (V'×13.6)

V(0.725) = 12.6V'

$$\frac{V'}{V} = \frac{0.725}{12.6} = \frac{1}{8}$$

Thus in the second case only (1/18)th of the volume of the cube is under mercury

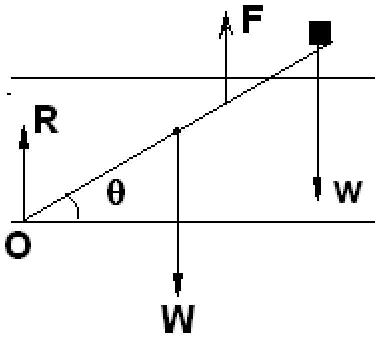
Q) A rod of length 6 m has a mass of 12kg. If it is hinged at one end at a distance of 3m below a water surface

(i) What weight should be attached to the other end so that 5 m of rod be submerged?

(ii) find the magnitude and direction of the final force exerted on the rod exerted by hinge. Specific gravity of the material of the rod is 0.5

Solution

Since one end is fixed in water we have to calculate moment of force



Moment of force due to weight of rod about point O= Wg(L/2)cosθ

Moment of force due to additional weight about point O =wL cosθ

Moment of force due to Buoyant force(F) about point O = F(l/2)cosθ

Here l is the length of rod immersed in water = 5m And L is total length of rod

Since rod is at rotational equilibrium at equilibrium

$F(l/2)\cos\theta = wL\cos\theta + Wg(L/2)\cos\theta$

$F(l/2) = wL + W(L/2)g$ --- eq(1)

But F = Vρg

Since 5m is immersed in water thus (5/6) of volume of rod is immersed

Volume of rod = mass/density = 12/0.5 = 24 m^3

Thus F = (5/6)×24×1×g = 20g N

Substituting values in eq(1) we get

∴ 20(2.5)g = w(6) + (12)(3)g

50 = 6w + 36

w = 14g/6 = 2.33g N

w = 2.33 kg wt

Now R = W+w-F

R = 12g +2.33g – 20g

R = -5.67g N

R = -5.67 kg wt

The negative sign indicates that the reaction (vertical) at the hinges acts downwards

Liquid in accelerated Vessel

Variation of pressure and force of buoyancy in a liquid kept in accelerated vessel

Consider a liquid of density ρ kept in a vessel moving with acceleration **a in upward** direction. Let height of liquid column be h

Then effective gravitational acceleration on liquid = g +a

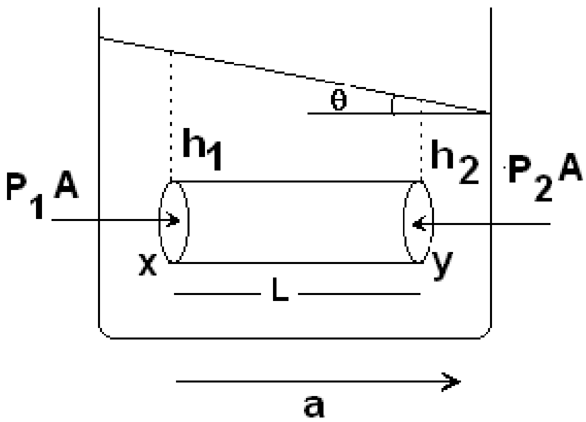
Thus pressured exerted at depth h $P = P_0 + \rho(g+a)h$

Similarly if liquid in container moves **down with acceleration a**

Then effective gravitational acceleration on liquid = g –a
Thus pressure exerted at depth h, $P = P_0 + \rho(g-a)\,h$

Also Buoyant force on immersed body when liquid is **moving up**
$F_B = V\rho(g+a)$
Buoyant force on immersed body when liquid is **moving down**
$F_B = V\rho(g+a)$
V is volume of the liquid displaced

Shape of free surface of a liquid in horizontal accelerated vessel



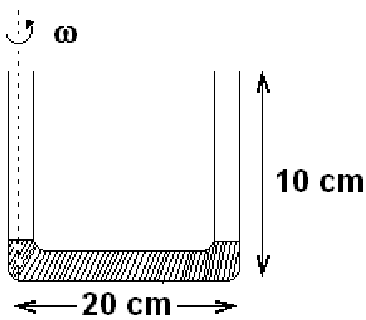
When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal, consider a horizontal liquid column including two points x and y at the depth of h_1 and h_2 from the inclined free surface of liquid as shown in figure

Force on area at x = $P_1A = h_1\rho g$
Pseudo force at y= mass of liquid tube of length L and cross sectional area A ×acceleration
Pseudo force at y=ρ(LA)
Total force at y = P_1A + Pseudo force
Force on area at y = $h_1\rho g + \rho(LA)$
Since liquid is in equilibrium
Force on area at x = Force on area at y
$h_1\rho g = h_1\rho g + \rho(LA)$
$(h_1 - h_2)\, g = La$
From geometry of figure

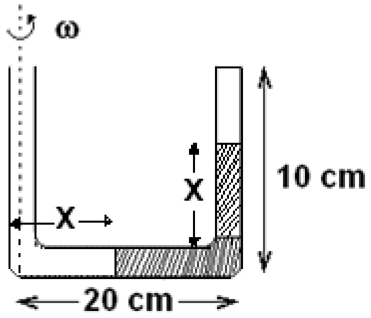
$$\frac{h_1 - h_2}{L} = \frac{a}{g}$$

$$tan\theta = \frac{a}{g}$$

Q) Length of a horizontal arm of a U-tube is 20cm and end of both the vertical arms are open to a pressure 1.01×10^3 N/m^2. Water is poured into the tube such that liquid just fills horizontal part of the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity ω. If length of water in sealed tube rises to 5cm, calculate ω. Take density of water = 10^3 kg/m^3 and g = 10 m/s^2. Assume temperature to be constant.



Solution

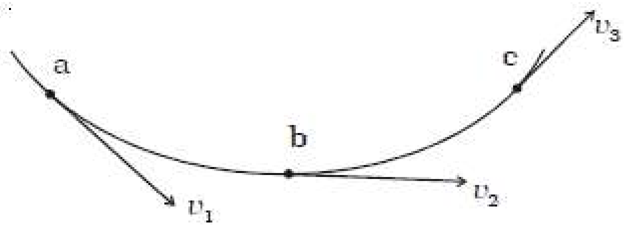


when tube is rotated liquid will experience a centrifugal force thus water moves up in second arm of the U tube.
When centrifugal force + pressure in first arm = force due to pressure in second closed arm +force due to liquid column then equilibrium condition is established ---eq(1)

Calculation of force due to pressure in closed tube
Before closing pressure $P_i = 1.01\times10^3$ N/m^2
Volume before closing V_i= 0.1A (A is area of cross-section)
After closing the other arm Pressure P_f and volume V_f = 0.05A
From equation $P_iV_i = P_fV_f$
$(1.01\times10^3)\times 0.1A = P_f \times(0.05A)$
$P_f = 2.02\times10^3$
Force due to pressure = $(2.02\times10^3)\times A$
Pressure in first arm = 1.01×10^3
Calculation of force due to liquid column in second arm
Height of liquid column = 0.05 m
Thus pressure due to column = hρg = $0.05\times10^3\times10$ =500 N/m^2
Force due to liquid column PA= 0.5A
Calculation of centrifugal force
Mass of the liquid in horizontal part = volume ×density = $(0.2\text{-}0.05)A\times10^3$=150A
Centre of mass of horizontal liquid from first arm 'r' $=0.05+\frac{0.2-0.05}{2}=0.125\ m$
Centrifugal force = $m\omega^2 r = 150A\times\omega^2\times0.125 = (18.75A)\omega^2$

Now substituting values in equation 1 we get
$(18.75A)\times\omega^2 + (1.01\times10^3)\times A = (2.02\times10^3)\times A + 500A$
$(18.75)\times\omega^2 + 1.01\times10^3 = (2.02\times10^3) + 500$
ω = 8.97 rad/s

Fluid dynamics
Streamline flow
The flow of a liquid is said to be steady, streamline or laminar if every particle of the liquid follows exactly the path of its preceding particle and has the same velocity of its preceding particle at every point.

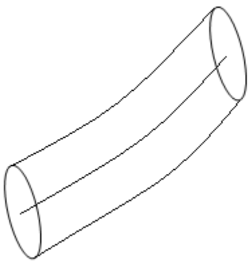


Let abc be the path of flow of a liquid and v_1, v_2 and v_3 be the velocities of the liquid at the points a, b and c respectively. During a streamline flow, all the particles arriving at 'a' will have the same velocity v_1 which is directed along the tangent at the point 'a'. A particle arriving at b will always have the same velocity v_2. This velocity v_2 may or may not be equal to v_1.

Similarly all the particles arriving at the point 'c' will always have the same velocity v_3. In other words, in the streamline flow of a liquid, the velocity of every particle crossing a particular point is the same.

The streamline flow is possible only as long as the velocity of the fluid does not exceed a certain value. This limiting value of velocity is called critical velocity.

Tube of flow



In a fluid having a steady flow, if we select a finite number of streamlines to form a bundle like the streamline pattern shown in the figure, the tubular region is called a tube of flow.

The tube of flow is bounded by a streamlines so that by fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.

Turbulent flow

When the velocity of a liquid exceeds the critical velocity, the path and velocities of the liquid become disorderly. At this stage, the flow loses all its orderliness and is called turbulent flow. Some examples of turbulent flow are :

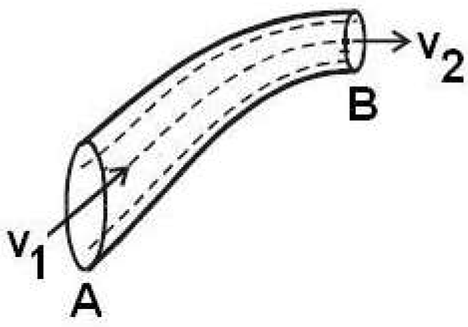
(i) After rising a short distance, the smooth column of smoke from an incense stick breaks up into irregular and random patterns.

(ii) The flash - flood after a heavy rain.

Critical velocity of a liquid can be defined as that velocity of liquid upto which the flow is streamlined and above which its flow becomes turbulent.

Equation of continuity

Consider a non-viscous liquid in streamline flow through a tube AB of varying cross section as shown in Fig. Let a_1 and a_2 be the area of cross section, v_1 and v_2 be the velocity of flow of the liquid at A and B respectively.



∴ Volume of liquid entering per second at A = a_1v_1.

If ρ is the density of the liquid, then mass of liquid entering per second at A = $a_1v_1\rho$.

Similarly, mass of liquid leaving per second at B = $a_2v_2\rho$

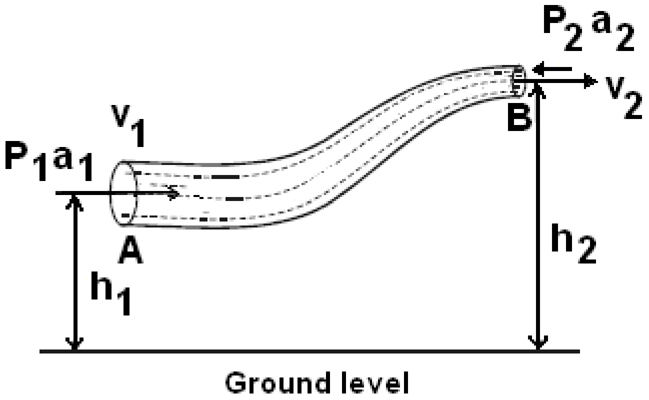
If there is no loss of liquid in the tube and the flow is steady, then mass of liquid entering per second at A = mass of liquid leaving per second at B

(i.e) $a_1v_1\rho = a_2v_2\rho$ or $a_1v_1 = a_2v_2$

i.e. av = constant
This is called as the equation of continuity. From this equation v is inversely proportional to area of cross-section along a tube of flow
i.e. the larger the area of cross section the smaller will be the velocity of flow of liquid and vice-versa.

Bernoulli's Equation

The theorem states that the work done by all forces acting on a system is equal to the change in kinetic energy of the system



Consider streamline flow of a liquid of density ρ through a pipe AB of varying cross section.
Let P_1 and P_2 be the pressures and a_1 and a_2, the cross sectional areas at A and B respectively. The liquid enters A normally with a velocity v_1 and leaves B normally with a velocity v_2. The liquid is accelerated against the force of gravity while flowing from A to B, because the height of B is greater than that of A from the ground level. Therefore P_1 is greater than P_2. This is maintained by an external force.
The mass m of the liquid crossing per second through any section of the tube in accordance with the equation of continuity is $a_1v_1\rho = a_2v_2\rho = m$
Or

$$a_1v_1 = a_2v_2 = \frac{m}{\rho}$$

As $a_1 > a_2$, $v_1 < v_2$
The force acting on the liquid at A = P_1a_1
The force acting on the liquid at B = $P_2 a_2$
Work done per second on the liquid at A = $P_1a_1 \times v_1 = P_1V$
Work done by the liquid at B = $P_2a_2 \times v_2 = P_2V$
∴ Net work done per second on the liquid by the pressure energy in moving the liquid from A to B is = $P_1V - P_2V$
If the mass of the liquid flowing in one second from A to B is m, then increase in potential energy per second of liquid from A to B is = $mgh_2 - mgh_1$
Increase in kinetic energy per second of the liquid.

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

According to work-energy principle,
work done per second by the
pressure energy = (Increase in potential energy + Increase in kinetic energy) per second

$$P_1V - P_2V = (mgh_2 - mgh_1) + \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)$$

$$P_1V + mgh_1 + \frac{1}{2}mv_1^2 = P_2V + mgh_2 + \frac{1}{2}mv_2^2$$

$$\frac{P_1V}{V} + \frac{m}{V}gh_1 + \frac{1}{2}\frac{m}{V}v_1^2 = \frac{P_2V}{V} + \frac{m}{V}gh_2 + \frac{1}{2}\frac{m}{V}v_2^2$$

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

Since subscripts 1 and 2 refer to any location on the pipeline, we can write in general

$$P + \rho gh + \frac{1}{2}mv^2 = constant$$

The above equation is called Bernoulli's equation for steady non-viscous incompressible flow. Dividing the above equation by gh we can rewrite the above equation as

$$h + \frac{v^2}{2g} + \frac{P}{\rho g} = constnat, which\ is\ called\ total\ head$$

Term h is called elevation head or gravitational head

$$\frac{v^2}{2g}\ is\ called\ velocity\ head$$

$$\frac{P}{\rho g}\ is\ called\ pressure\ head$$

Above equation indicates for **ideal liquid velocity increases when pressure decreases and vice-versa**

Q) A vertical tube of diameter 4mm at the bottom has a water passing through it. If the pressure be atmospheric at the bottom where the water emerges at the rate of 800gm per minute, what is the pressure at a point in the tube 5cm above the bottom where the diameter is 3mm

Solution

Rate of flow of water = 800 gm/min = (40/3)gm/sec

Now mass of water per sec = velocity ×area × density

$40/3 = V_1 \times[\pi\ (0.2)^2]\times 1$

$V_1 = (333.33/\pi)$ cm/sec

Now $A_1V_1 = A_2V_2$ Thus

$V_2 = (4/3)V_1$

$V_2 = (444.44/\pi)$ cm/sec is the velocity at height 25cm

Now P_1 = atmospheric pressure = 1.01×10^7 dyne

Now from Bernoulli's equation

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

$$1.01 \times 10^6 + 0 + \frac{1}{2}\left(\frac{333.33}{\pi}\right)^2 = P_2 + 1 \times 981 \times 25 + \frac{1}{2}\left(\frac{444.44}{\pi}\right)^2$$

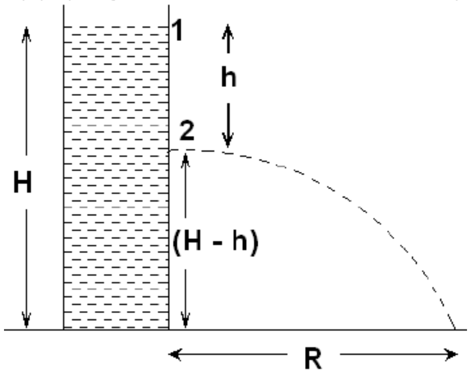
On solving

$P_2 = 0.98\times10^6$ dyne

Now pressure = $h\rho_0 g$ here ρ_0 is density of mercury = 13.6 in cgs system

$0.98\times10^6 = h\times980\times13.6$
H = 73.5 cm of Hg

Q) Water stands at a depth H in a tank whose side walls are vertical. A hole is made at one of the walls at depth h below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the flower. What is the maximum possible range?
Solution
Applying Bernoulli's theorem at point 1 and 2



$$P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2$$

$P_1 = P_2 = P$ (atmospheric pressure)
$V_1 = 0$, $h_2 = H-h$ and $h_1 = H$

$$\rho g H = \rho g(H-h) + \frac{1}{2}\rho v_2^2$$
$$v_2^2 = 2gh$$

The vertical component of velocity of water emerging from hole at 2 is zero. Therefore time taken (t) by the water to fall through a distance (H-h) is given bu

$$H - h = \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{2(H-h)}{g}}$$

Required horizontal range $R = v_2t$

$$R = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$
$$R = 2\sqrt{h(H-h)}$$

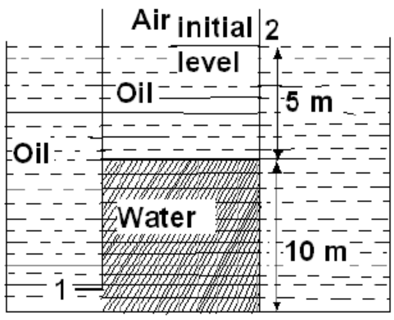
the range is maximu when dR/dh = 0

$$2\times\frac{1}{2}(Hh-h^2)^{\frac{-1}{2}}(H-2h) = 0$$

This gives h = H/2
Therefore Maximu range =

$$R = 2\sqrt{\frac{H}{2}\left(H-\frac{H}{2}\right)} = H$$

Q) A tank with a small circular hole contains oil on top of water. It is immersed in a large tank of same oil. Water flows through the hole. What is the velocity of the flow initially? When the flow stops, what would be the position of the oil-water interface in the tank? The ratio of the cross-section area tank to the that of hole is 50, determine the time at which the flow stops, density of oil = 800 kg/m^3



Solution:

Pressure at hole and pressure at point on the bottom of water is different thus water flows through the hole

Pressure at point 1 $P_1 = P_0 + h\rho_0 g$ here h = 15m and ρ_0 = 800 kg/m^3

Pressure at point 2 is $P_2 = P_0$

And potential = $5\rho_0 g + 10\rho g$ here ρ is density of water

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$$P_0 + h\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + 10\rho g + \frac{1}{2}\rho v_2^2$$

For continuity equation $A_1V_1 = A_2 V_2$

$$V_2 = \frac{A_1}{A_2}V_1 = \frac{1}{50}V_1$$

$$15 \times 800 \times 10 + \frac{1}{2}1000v_1^2 = 5 \times 800 \times 10 + 10 \times 1000 \times 10 + \frac{1}{2}1000 \times \left(\frac{v_1}{50}\right)^2$$

$$120000 + 500v_1^2 = 140000 + 500 \times \left(\frac{v_1}{50}\right)^2$$

$$v_1^2\left(500 - \frac{1}{2500}\right) = 20000$$

$$v_1^2(500) = 20000$$

V_1 = 6.32 m/s

Let x be the height of water column when flow of water is stopped

Applying Bernoull's equation between point a and x we het

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + x\rho g + \frac{1}{2}\rho v_2^2$$

Since velocities are zero

$$15\rho_0 g = 5\rho_0 g + x\rho g$$

$$15 \times 800 = 5 \times 800 + x \times 1000$$

X = 8 m

Let at any moment of time height of water column be y then level of oil in samll tank is (15-y) accoding to bernolli's equation

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + (5)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$av_1 = Av_2$

$$v_2 = \frac{a}{A}v_1 = \frac{v_1}{50}$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

Neglecting term

$$\frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g$$

$$\frac{1}{2}1000 v_1^2 = (-10)\times 800\times 10 + y\times 1000\times 10$$

$$v_1^2 = -160 + 20y$$

Differentiating

$$2v_1\frac{dv_1}{dt} = 20\frac{dy}{dt}$$

But

$$\frac{dy}{dt} = v_2 = -\frac{v_1}{50}$$

Negative sign since vleocity is decreasing

$$v_1\frac{dv_1}{dt} = 10\times\frac{-v_1}{50}$$

$$\frac{dv_1}{dt} = \frac{-1}{5}$$

$$dv_1 = \frac{-1}{5}dt$$

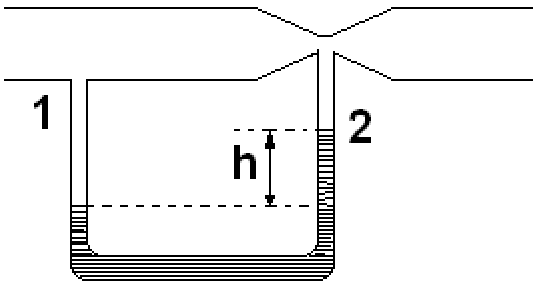
Integrating

$$\int_{6.32}^{0} dv_1 = \frac{-1}{5}\int_0^t dt$$

$$-6.23 = \frac{-t}{5}$$

t = 31.15 sec

Venturimeter:



This is a device based on Bernoulli's principle used for measuring the flow of a liquid in pipes. A liquid of density ρ flows through a pipe of cross-sectional area A. Let the constricted part of the cross-sectional area be 'a'. A manometer tube with a liquid say mercury having a densityρ_0 is attached to the tube as shown in figure

If P_1 is the pressure at point 1 and P_2 the pressure at point 2, we have

$$P_1 + \frac{1}{2}mv_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Where v_1 and v_2 are the velocities at these points respectively

$$\frac{1}{2}v_1^2 - \frac{1}{2}v_2^2 = \frac{P_2}{\rho} - \frac{P_1}{\rho}$$

We have $Av_1 = av_2$

$$v_2 = \frac{A}{a}v_1$$

$$\frac{1}{2}v_1^2 - \frac{1}{2}\left(\frac{A}{a}\right)^2 v_1^2 = \frac{P_2 - P_1}{\rho}$$

$$v_1^2 - \left(\frac{A}{a}\right)^2 v_1^2 = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2\left(1 - \frac{A^2}{a^2}\right) = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2 = \frac{\frac{2(P_2 - P_1)}{\rho}}{1 - \frac{A^2}{a^2}} = \frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}$$

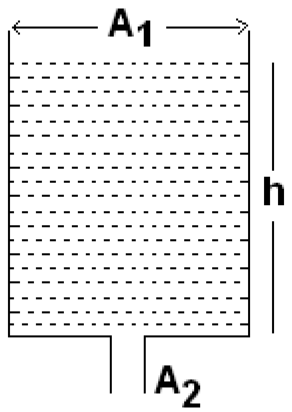
$$v_1 = \sqrt{\frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Volume of liquid flowing through the pipe per second $Q = Av_1$

$$Q = Aa\sqrt{\frac{2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Speed of Efflux

As shown in figure a tank of cross-sectional area A, filled to a depth h with a liquid of density ρ. There is a hole of cross-section area A_2 at the bottom and the liquid flows out of the tank through the hole $A_2 << A_1$



Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2. As both the cross sections are opened to the atmosphere, the pressure there equals to atmospheric pressure P_o. If the height of the free surface above the hole is h_1

Bernoulli's equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

By the equation of continuity,

$A_1v_1 = A_2 v_2$

$$P_0 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2$$

$$\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh$$

$$v_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

If $A_2 <<< A_1$, the equation reduces to $v_2 = \sqrt{(2gh)}$

The speed of efflux is the same as the speed a body that would acquire in falling freely through a height h. This is known as Torricelli's theorem.

SURFACE TENSION & VISCOSITY

SURFACE TENSION

Intermolecular forces

The force between two molecules of a substance is called intermolecular force. This intermolecular force is basically electric in nature. When the distance between two molecules is greater, the distribution of charges is such that the mean distance between opposite charges in the molecule is slightly less than the distance between their like charges. So a force of attraction exists. When the intermolecular distance is less, there is overlapping of the electron clouds of the molecules resulting in a strong repulsive force. The intermolecular forces are of two types. They are (i) cohesive force and (ii) adhesive force.

Cohesive force

Cohesive force is the force of attraction between the molecules of the same substance. This cohesive force is very strong in solids, weak in liquids and extremely weak in gases.

Adhesive force

Adhesive force is the force of attraction between the molecules of two different substances. For example due to the adhesive force, ink sticks to paper while writing. Fevicol, gum etc exhibit strong adhesive property.

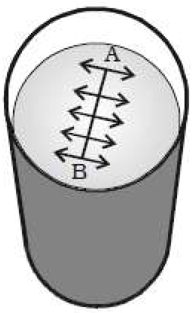
Water wets glass because the cohesive force between water molecules is less than the adhesive force between water and glass molecules. Whereas, mercury does not wet glass because the cohesive force between mercury molecules is greater than the adhesive force between mercury and glass molecules.

Molecular range and sphere of influence

Molecular range is the maximum distance upto which a molecule can exert force of attraction on another molecule. It is of the order of 10^{-9} m for solids and liquids.

Sphere of influence is a sphere drawn around a particular molecule as centre and molecular range as radius. The central molecule exerts a force of attraction on all the molecules lying within the sphere of influence.

Surface tension of a liquid



Surface tension is the property of the free surface of a liquid at rest to behave like a stretched membrane in order to acquire minimum surface area.

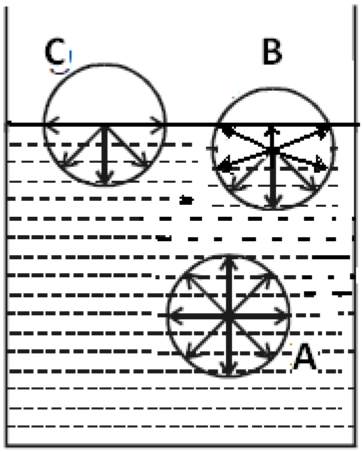
Imagine a line AB in the free surface of a liquid at rest (Fig. 5.20). The force of surface tension is measured as the force acting per unit length on either side of this imaginary line AB. The force is perpendicular to the line and tangential to the liquid surface. If F is the force acting on the length *l* of the line AB, then surface tension is given by

$$T = \frac{F}{L}.$$

Surface tension is defined as the force per unit length acting perpendicular on an imaginary line drawn on the liquid surface, tending to pull the surface apart along the line. Its unit is N m^{-1} and dimensional formula is MT^{-2}.

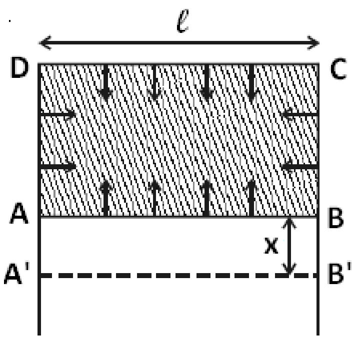
It depends on temperature. The surface tension of all liquids decreases linearly with temperature
It is a scalar quantity and become zero at critical temperature

Molecular theory of surface tension



The surface tension of liquid arises out of the attraction of its molecules. Molecules of fluid (liquid and gas) attract one another with a force. If any other molecule is within the sphere of influence of first molecule it will experience a force of attraction
Consider three molecules A, B, C having their spheres of influence as shown in the figure. The sphere of influence of A is well inside the liquid, that of B partly outside and that of C exactly half of total
Molecules like A do not experience any resultant force, as they are attracted equally in all directions. Molecules like B or C will experience a resultant force directed inward. Thus the molecules will inside the liquid will have only kinetic energy but the molecule near surface will have kinetic as well as potential energy which is equal o the work done in placing them near the surface against the force of attraction directed inward

Surface energy



Any Strained body possesses potential energy, which is equal to the work done in bringing it to the present state from its initial unstained state. The surface of liquid is also a strained system and hence the surface of a liquid also has potential energy, which is equal to the work done increasing the surface. This energy per unit area of the surface is called surface energy
To derive an expression for surface energy consider a wire frame equipped with a sliding wire AB as shown in figure. A film of soap solution is formed across ABCD of the frame. The side AB is pulled to the left due to surface tension. To keep the wire in position a force F has to be applied to the right. If T is the surface tension and l is the length of AB, then the force due to surface tension over AB is 2lT to the left because the film has two surfaces (upper and lower)
Since the film is in equilibrium F = 2lT

Now, if the wire AB is pulled down, energy will flow from the agent to the film and this energy is stored as potential energy of the surface created just now. Let the wire be pulled slowly through x.
Then the work done = energy added to the film from above agent
W=Fx = 2lTx
Potential energy per unit area (surface energy) of the film

$$U = \frac{2lTx}{2lx} = T$$

$$T = \frac{W}{area}$$

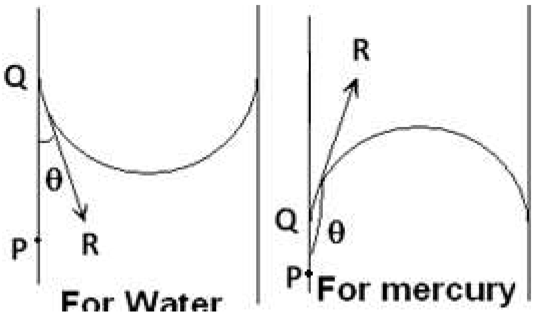
Thus surface energy numerically equal to its surface tension
It s unit is Joule per square metre (Jm^{-2})

Solved Numerical

Q) Calculate the work done in blowing a soap bubble of radius 10cm, surface tension being 0.08 Nm^{-1}. What additional work will be done in further blowing it so that its radius is doubled?
Solution
In case of a soap bubble, there are two free surfaces
Surface tension = Work done per unit area
∴ Work done in blowing a soap bubble of radius R is given by = Surface tension × Area
W = T ×(2×4πR^2)
W = (0.06)×(8×3.14×0.1^2)
W = 1.51 J
Similarly, work done in forming a bubble of radius 0.2 m is
W' = (0.06)×(8×3.14×0.2^2) = 60.3 J
Additional work done in doubling the radius of the bubble is given by
W' – W = 60.3 – 1.51 = 5.42J

Q) A mercury drop of radius 1cm is sprayed into 10^6 droplets of equal size. Calculate the energy expended if surface tension of mercury is 35×10^{-3} N/m
Solution
Since total volume of 10^6 droplet has remains same
If radius small droplet is r' and big drop is r then r = $(10^6)^{1/3}$ r'
1 = 10^2r' or r' = 0.01 cm = 10^{-4} m
Since surface area is increased energy should be supplied to make small small drops
Total energy of small droplet =[T (4πr'^2)] 10^6
Total energy of big droplet = [T(44πr^2)]
Spending of energy = Total energy of small droplets - Total energy of big droplet
Spending of energy = [T (4πr'^2)] 10^6 - [T(4πr^2)]
Spending of energy = T×4π [10^6× r'^2 – r^2]
Spending of energy = 35×10^{-3}×4×3.14 [10^6×$(10^{-4})^2$ –$(10^{-2})^2$]
Spending of energy =0.44[10^{-2} – 10^{-4}]
Spending of energy = 4.356×10^{-3} J

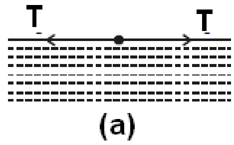
Angle of contact

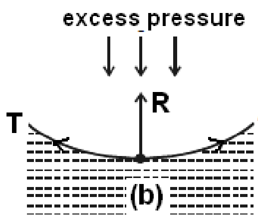


When the free surface of a liquid comes in contact with a solid, it becomes curved at the point of contact. *The angle between the tangent to the liquid surface at the point of contact of the liquid with the solid and the solid surface inside the liquid is called angle of contact.* In Fig., QR is the tangent drawn at the point of contact Q. The angle PQR is called the angle of contact. When a liquid has concave meniscus, the angle of contact is acute. When it has a convex meniscus, the angle of contact is obtuse. The angle of contact depends on the nature of liquid and solid in contact. For water and glass, θ lies between 8^o and 18^o. For pure water and clean glass, it is very small and hence it is taken as zero. The angle of contact of mercury with glass is 138^o.

Pressure difference across a liquid surface

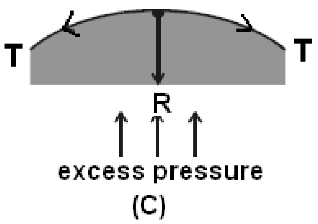
If the free surface of a liquid is plane, then the surface tension acts horizontally (Fig. a). It has no component perpendicular to the horizontal surface. As a result, there is no pressure difference between the liquid side and the vapour side.





If the surface of the liquid is concave (Fig. b), then the resultant force *R* due to surface tension on a molecule on the surface act vertically upwards. To balance this, an excess of pressure acting downward on the concave side is necessary.

On the other hand if the surface is convex (Fig.c), the resultant *R* acts downward and there must be an excess of pressure on the concave side acting in the upward direction.



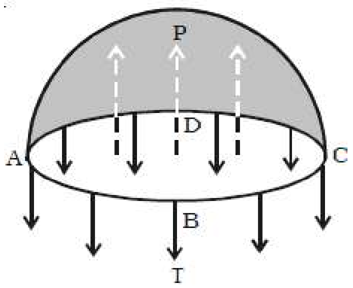
Thus, there is always an excess of pressure on the concave side of a curved liquid surface over the pressure on its convex side due to surface tension.

Excess pressure

The pressure inside a liquid drop or a soap bubble must be in excess of the pressure outside the bubble or drop because without such pressure difference a drop or a bubble cannot be in state of equilibrium. Due to surface tension the drop or bubble has got the tendency to contract and disappear altogether.

To balance this, there must be an excess of pressure inside the bubble.

To obtain a relation between the excess pressure and the surface tension, consider a water drop of radius r and surface tension T, The excess of pressure *P* inside the drop provides a force acting outwards perpendicular to the surface, to balance the resultant force due to surface tension.



Imagine the drop to be divided into two equal halves. Considering the equilibrium of the upper hemisphere of the drop, the upward force on the plane face ABCD due to excess pressure *P* is $P\pi r^2$

If *T* is the surface tension of the liquid, the force due to surface tension acting downward along the circumference of the circle ABCD is $T\,2\pi r$.

At equilibrium, $P\pi r^2 = T\,2\pi r$

$$P = \frac{2T}{r}$$

Here P is excess pressure $P = P_i - P_O$

$$P_i - P_O = \frac{2T}{r}$$

Excess pressure inside a soap bubble

A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. Therefore the force due to surface tension = $2 \times 2\pi rT$

∴ At equilibrium, $P\pi r^2 = 2 \times 2\pi rT$

$$P = \frac{4T}{r}$$

Thus the excess of pressure inside a drop is inversely proportional to its radius

the pressure needed to form a very small bubble is high. This explains why one needs to blow hard to start a balloon growing. Once the balloon has grown, less air pressure is needed to make it expand more.

Solved Numerical

Q) An air bubble of radius R is formed on a narrow tube having a radius r where R>>r. Air of density ρ is blown inside the tube with velocity V. The air molecules collide perpendicularly with the wall of bubble and stop. Find the radius at which the bubble separates from the tube. Take surface tension of bulb as T

Solution:

Air molecules collides at stops thus force exerted on the soap bubble

Mass of air = Volume ×ρ

Volume of air = velocity of air × area of hole = $v(\pi r^2)$

Mass of air = $v\rho(\pi r^2)$

Force exerted by the air = change in momentum of air molecules

Force due to air molecule = $(v\rho\pi r^2)v = \rho\pi r^2 v^2$

Pressure of blown air in side the bubble = ρv^2

Now Force due to surface tension of bubble of radius R

Pressure difference in bubble = 4T/R

Bubble gets separated when pressure difference in bubble = pressure of blown air

$$\frac{4T}{R} = \rho v^2$$
$$R = \frac{4T}{\rho v^2}$$

Q) Two spherical soap bubbles coalesce to form a single bubble. If V is the consequent change in volume of the contained air and S the change in the total surface area, show that 3PV+4ST = 0, where T is the surface tension of the soap bubble and P the atmospheric pressure

Solution:

$$P_1 = P + \frac{4T}{r_1}\ ;\ P_2 = P + \frac{4T}{r_2}$$

Since the total number of moles remains same

$N_1 + n_2 = n$

$P_1V_1 + P_2 V_2 = P_3 V_3$

$$\left(P + \frac{4T}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(P + \frac{4T}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right) = \left(P + \frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right)$$

$$\left(P + \frac{4T}{r_1}\right)(r_1^3) + \left(P + \frac{4T}{r_2}\right)(r_2^3) = \left(P + \frac{4T}{r}\right)(r^3)$$
$$Pr_1^3 + 4Tr_1^2 + Pr_2^3 + 4Tr_2^2 = Pr^3 + 4Tr^2$$
$$4Tr_1^2 + 4Tr_2^2 - 4Tr^2 = Pr^3 - Pr_1^3 - Pr_2^3$$
$$4T(r_1^2 + r_2^2 - r^2) = P(r^3 - r_1^3 - r_2^3)$$

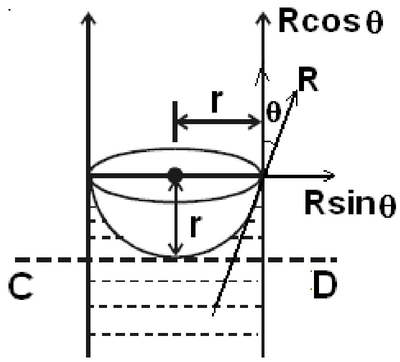
$$\frac{4}{3}\pi 4T(r_1^2 + r_2^2 - r^2) = \frac{4}{3}\pi P(r^3 - r_1^3 - r_2^3)$$
$$4T(S_1 + S_2 - S_3) = 3P(V_3 - V_1 - V_3)$$
$$4TS = -3PV$$

Negative V because $V_3 < V_1 + V_2$

$$4TS + 3PV = 0$$

Surface tension by capillary rise method

Let us consider a capillary tube of uniform bore dipped vertically in a beaker containing water. Due to surface tension, water rises to a height *h* in the capillary tube as shown in Fig.. The surface tension *T* of the water acts inwards and the reaction of the tube *R* outwards. *R* is equal to *T* in magnitude but opposite in direction. This reaction *R* can be resolved into two rectangular components.



(i) Horizontal component *R sin* θ acting radially outwards

(ii) Vertical component *R cos* θ acting upwards.

The horizontal component acting all along the circumference of the tube cancel each other whereas the vertical component balances the weight of water column in the tube.

Total upward force = *R cos* θ × circumference of the tube

$F = 2\pi r\, R \cos\theta$ or $F = 2\pi r\, T \cos\theta$ ...(1)

[∵ $R = T$]

This upward force is responsible for the capillary rise. As the water column is in equilibrium, this force acting upwards is equal to weight of the water column acting downwards.

(i.e) $F = W$ ...(2)

Now, volume of water in the tube is assumed to be made up of

(i) a cylindrical water column of height *h* and (ii) water in the meniscus above the plane CD.

Volume of cylindrical water column = $\pi r 2h$

Volume of water in the meniscus

= (Volume of cylinder of height *r* and radius *r)* – (Volume of hemisphere)

∴Volume of water in the meniscus=

$$\pi r^2 \times r - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

∴Total volume of water in the tube

$$\pi r^2 h + \frac{1}{3}\pi r^3 = \pi r^2\left(h + \frac{r}{3}\right)$$

If ρ is the density of water, then weight of water in the tube is

$$W = \pi r^2\left(h + \frac{r}{3}\right)\rho g \quad ---eq(3)$$

Substituting (1) and (3) in (2),

$$\pi r^2\left(h + \frac{r}{3}\right)\rho g = 2\pi r T cos\theta$$

$$T = \frac{\pi r^2\left(h + \frac{r}{3}\right)\rho g}{2\pi r cos\theta}$$

Since *r* is very small, r/3 can be neglected compared to *h*.

$$T = \frac{hr\rho g}{2cos\theta}$$

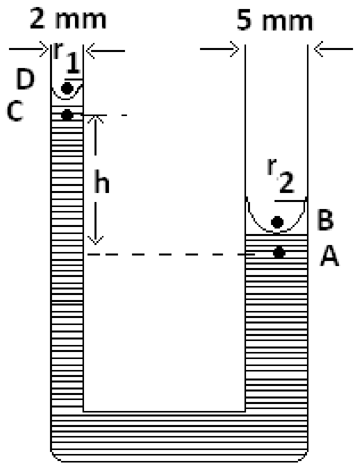
For water θ is very small cosθ =1

$$T = \frac{hr\rho g}{2}$$

Solved Numerical

Q) An U-tube with limbs of diameter 5mm and 2mm contains water of surface tension 7×10^{-2} N/m,angle of contact zero and density 1×10^{3} kg/m^3. Find the difference in levels (g= 10 m/s^2)

Solution: If the menisci are spherical, they will be hemispheres Since angle of contact is zero, their radii will then equal to radii of the limbs. The pressure on the concave side of



each surface exceeds that on the convex side by 2T/r, where T is surface tension and r is the radius of the limb concerned

Now r_1 = 2.5 mm = 2.5×10^{-3} m and r_2 = 1mm = 10^{-3}m

Hence

$$P_B - P_A = \frac{2T}{r_2} = \frac{2\times7\times10^{-2}}{2.5\times10^{-3}} = 56\ Pa$$

$P_A = P_B + 56 = P + 56$

Similarly

$$P_D - P_C = \frac{2T}{r_1} = \frac{2\times7\times10^{-2}}{10^{-3}} = 140\ Pa$$

$P_D = P_C + 140 = P + 140$

Since $P_D = P_B = P$

$\therefore P_A - P_C = (P-56) - (P-140)$

$P_A - P_C = 84$ Pa

But $P_A = P_C + h\rho g$

$H\rho g = 84$ Pa

$$\therefore h = \frac{84}{10^3\times10} = 8.4\ mm$$

Q) A mercury barometer has a glass tube with an inside diameter equal to 4mm. Since the contact angle of mercury with glass is 140^o, capiliary depresses the column. How many millimeters of mercury must be added to the reading to correct for capillarity (Assume surface tension of mercury T = 0.545 N/m, density of mercury = 13.6×10^3)

Solution:

The height difference due to capillarity give by

$$h = \frac{2T\cos\theta}{r\rho g}$$

$$h = \frac{2\times0.545\times\cos140}{(2\times10^{-3})(13.6\times10^3)(9.8)} = -0.0031m$$

Therefore 3.1mm must be added to the barometer reading

Factors affecting surface tension

Impurities present in a liquid appreciably affect surface tension. A highly soluble substance like salt increases the surface tension whereas sparingly soluble substances like soap decreases the surface tension.

The surface tension decreases with rise in temperature. The temperature at which the surface tension of a liquid becomes zero is called critical temperature of the liquid.

Applications of surface tension

(i) During stormy weather, oil is poured into the sea around the ship. As the surface tension of oil is less than that of water, it spreads on water surface. Due to the decrease in surface tension, the velocity of the waves decreases. This reduces the wrath of the waves on the ship.

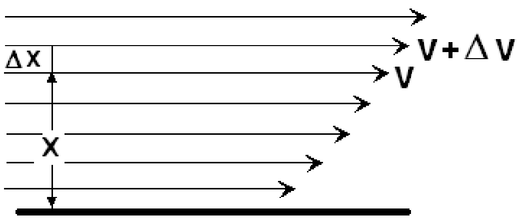
(ii) Lubricating oils spread easily to all parts because of their low surface tension.
(iii) Dirty clothes cannot be washed with water unless some detergent is added to water. When detergent is added to water, one end of the hairpin shaped molecules of the detergent get attracted to water and the other end, to molecules of the dirt. Thus the dirt is suspended surrounded by detergent molecules and this can be easily removed. This detergent action is due to the reduction of surface tension of water when soap or detergent is added to water.
(iv) Cotton dresses are preferred in summer because cotton dresses have fine pores which act as capillaries for the sweat.

VISCOSITY

If we pour equal amounts of water and castor oil in two identical funnels. It is observed that water flows out of the funnel very quickly whereas the flow of castor oil is very slow. This is because of the frictional force acting within the liquid. This force offered by the adjacent liquid layers is known as viscous force and the phenomenon is called viscosity. *Viscosity is the property of the fluid by virtue of which it opposes relative motion between its different layers. Both liquids and gases exhibit viscosity but liquids are much more viscous than gases.*

Co-efficient of viscosity

Consider the slow and steady flow of a fluid over a fixed horizontal surface as shown in the Fig. Let v be the velocity of thin layer of liquid at a distance x from the fixed solid surface.



Then according to Newton, the viscous force acting tangentially to the layer is proportional to the area of the layer and the velocity gradient at the layer. If F is the viscous force on the layer then ,
F ∝ A, where A is the area of the layer and

$$F \propto -\frac{\Delta v}{\Delta t}$$

The negative sign is put to account for the fact that the viscous force is opposite to the direction of motion Thus

$$F = -\eta A\frac{dv}{dt}$$

Where η is a constant depending upon the nature of the liquid and is called the coefficient of viscosity and

$$velocity\ gradiant = \frac{dv}{dt}$$

If A = 1 and dv/dx = 1. We have F=- η
Thus the coefficient of viscosity of a liquid may be defined as the viscous force per unit area of the layer where velocity gradient is unity
The coefficient of viscosity has the dimension $[ML^{-1}T^{-1}]$ and its unit is Newton second per square metre (Nsm^{-2}) or kilogram per metre per second ($kgm^{-1}s^{-}$). In CGS, the unit of viscosity is Poise, 1kilogram per metre per second = 10 Poise

Stroke’s Law

When a solid moves through a viscous medium, its motion is oppsed by a viscous force depending on the velocity and shape and size of the body. The energy of the body is continuously decreases in overcoming the viscous resistance of the medium. This is why cars, aeroplanes etc are shaped streamline to minimize the viscous resistance on them
The viscous drag on a spherical body of radius r, moving with velocity v, in a viscous medium of viscosity η is given by

$$F_{viscous} = 6\pi\eta rv$$

This relation is called **Stoke’s law**
This law can be deduced by the method of dimensions.

Terminal Velocity

Let the body be driven by a constant force. In the beginning velocity v = 0 and acceleration ‘a’ is max so the body experiences small viscous force. With increase in speed viscous force goes on increasing till resultant force acting on the body becomes zero, and body moves with constant speed , this speed is known as terminal velocity
Consider the downward movement of a spherical body through a viscous medium such as a ball falling through a viscous medium as a ball falling through a liquid. If r is the radius of the body, ρ the density of the material of the body and σ is the density of the liquid, then
(i)The weight of the body down ward force

$$\frac{4}{3}\pi r^3 \rho g$$

(ii) The buoyancy of the body upward force

$$\frac{4}{3}\pi r^3 \rho_0 g$$

Net down ward force

$$\frac{4}{3}\pi r^3 (\rho - \rho_0) g$$

If v is the terminal velocity of the body , then viscous force $F_{viscous} = 6\pi\eta rv$

When acceleration becomes zero
upward viscous force = resultant down ward force

$$6\pi\eta rv = \frac{4}{3}\pi r^3 (\rho - \rho_0) g$$

$$v = \frac{2}{9}\frac{r^2 g(\rho - \rho_0)}{\eta}$$

Solved Numerical

Q) A steel ball of diameter d = 3.0mm starts sinking with zero initial velocity in oil whose viscosity is 0.9P. How soon after the beginning of motion will the velocity of the ball differ from the steady state velocity by n = 1.0%? Density of steel = 7.8×10^3 kg/m^3
Solution: Initial acceleration is maximum and becomes zero thus acceleration is not constant:
Viscocity = 0.9P = 0.09 kgm^{-1}s^{-1}

Net force on ball = W – F_B - F_v

F_B = Buoyant force up ward F_v = viscous force upwards , W = weight of ball down wards

Force = ma thus

$$m\frac{dv}{dt} = mg - F_B - 6\eta'\pi r v$$

Let A = mg-F_B is constant and B = $6\eta\pi r$ is another constant

$$m\frac{dv}{dt} = A - Bv$$

$$m\frac{dv}{(A - Bv)} = dt$$

Velocity after time t differs from the steady state velocity by n = 1.0%

v = (1-n)v' here v' is terminal velocity

$$m\int_0^{(1-n)v\prime}\frac{dv}{(A - Bv)} = \int_0^t dt$$

$$-\frac{m}{B}ln\left[\frac{A - B(1-n)v'}{A}\right] = t$$

At steady state net force is zero

A - Bv' = 0 ∴ v_s = A/B

$$t = -\frac{m}{B}ln\left[\frac{A - B(1-n)\frac{A}{B}}{A}\right]$$

$$t = -\frac{m}{B}\ln n$$

$$t = -\frac{m}{6\eta\pi r}\ln n$$

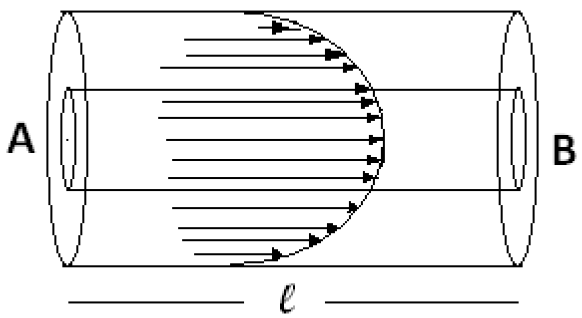
$$t = -\frac{\frac{4}{3}\pi r^3\rho}{6\eta\pi r}\ln n$$

$$t = -\frac{2r^2\rho}{9\eta}\ln n$$

$$t = -\frac{2\left(\frac{3\times 10^{-3}}{2}\right)^2 7.8\times 10^3}{9(0.09)}\ln(0.01)$$

t = 0.2 sec

Q) As shown in figure laminar flow is obtained in a tube of internal radius r and length l. To maintain such flow, the force balancing the viscous force obtained by producing the pressure difference(P) across the ends of the tube. Derive the equation of velocity of a layer situated at distance ‘x’ from the axis of the tube



Solution

Consider a cylindrical layer of radius x as shown in figure. The force acting on it are as follows

(1) At face A let pressure be P_1 Thus force $F_1 = \pi x^2 P_1$

(2) At face B let pressure be P_2 ($<P_1$) Thus force $F_2 = \pi x^2 P_2$ is against F_1

(3) Viscous force $F_3 = \eta A\left(-\frac{dv}{dx}\right)$

A is curved area of cylinder of radius x, thus $A = 2\pi xl$

Negative sign indicates as we go from axis of cylinder to walls of cylinder velocity decreases

Viscous force F_3

$$F_3 = -\eta(2\pi xl)\frac{dv}{dx}$$

For the motion of the cylinder layer with a constant velocity

$F_3 = F_1 - F_2$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 P_1 - \pi x^2 P_2$$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 (P_1 - P_2)$$

$$-\eta(2\pi xl)\frac{dv}{dx} = \pi x^2 (P) \quad [\because P_1 - P_2 = P]$$

$$-dv = \frac{P}{2\eta l}x\,dx$$

At x = r, v = 0 and at x = x , v = v , v so integrating the above equation in these limits we get

$$-\int_v^0 dv = \int_x^v \frac{P}{2\eta l}x\,dx$$

$$-[v]_v^0 = \frac{P}{4\eta l}[x^2]_x^r$$

$$-[0 - v] = \frac{P}{4\eta l}[r^2 - x^2]$$

$$v = \frac{P}{4\eta l}(r^2 - x^2)$$

If we want to fing the volume of liquid flowing the tube in one second
Then velocity at axis x= 0

$$v = \frac{Pr^2}{4\eta l}$$

At the wall (x = r) velocity is zero
∴ Average velocity

$$< v >= \frac{Pr^2}{8\eta l}$$

Now volume of liquid = (average velocity)(Area of cross-section)

$$V = \frac{Pr^2}{8\eta l}(\pi r^2)$$

$$V = \frac{P\pi r^2}{8\eta l}$$

Above equation is called Poiseiulle's Law

KINETIC THEORY OF GASES AND THERMODYNAMICS

SECTION I

Kinetic theory of gases

Some important terms in kinetic theory of gases

Macroscopic quantities:

Physical quantities like pressure, temperature, volume, internal energy are associated with gases. These quantities are obtained as an average combined effect of the process taking place at the microscopic level in a system known as macroscopic quantities. These quantities can be directly measured or calculated with help of other measurable macroscopic quantities

Macroscopic description:

The description of a system and events associated with it in context to its macroscopic quantities are known as macroscopic description.

Microscopic quantities:

Physical quantities like speed, momentum, kinetic energy etc. associated with the constituent particle at microscopic level, are known as microscopic quantities

Microscopic description:

When the system and events associated with it are described in context to microscopic quantities, this description is known as microscopic description

Postulates of Kinetic theory of gases

(1) A gas consists of a very large number of molecules. Each one is a perfectly identical elastic sphere.

(2) The molecules of a gas are in a state of continuous and random motion. They move in all directions with all possible velocities.

(3) The size of each molecule is very small as compared to the distance between them. Hence, the volume occupied by the molecule is negligible in comparison to the volume of the gas.

(4) There is no force of attraction or repulsion between the molecules and the walls of the container.

(5) The collisions of the molecules among themselves and with the walls of the container are perfectly elastic. Therefore, momentum and kinetic energy of the molecules are conserved during collisions.

(6) A molecule moves along a straight line between two successive collisions and the average distance travelled between two successive collisions is called the mean free path of the molecules.
(7) The collisions are almost instantaneous (i.e) the time of collision of two molecules is negligible as compared to the time interval between two successive collisions.

Behavior of gases

It has been observed from experiments that for very low densities, the pressure, volume and temperature of gas are interrelated by some simple relations.

Boyle's law

At constant temperature and low enough density, the pressure of a given quantity (mass) of gas is inversely proportional to its volume
Thus at constant mass and constant temperature

$$P \propto \frac{1}{V}$$

Or PV = Constant

Charles's law

At constant pressure and low enough density, the volume of a given quantity (mass) of a gas is proportional to its absolute temperature
Thus at constant mass and constant pressure

$$V \propto T$$

Or $\frac{V}{T} = constant$

Gay Lussac's law

For a given volume and low enough density the pressure of a given quantity of gas is proportional to its absolute temperature.
Thus at constant mass and constant volume

$$P \propto T$$

Or $\frac{P}{T} = constant$

Avogadro's Number

"For given constant temperature and pressure, the number of molecules per unit volume is the same for all gases"
At standard temperature (273K) and pressure (1 atm), the mass of 22.4 litres of **any** gas is equal to its molecular mass (in grams). This quantity of gas is called 1 mole.
The number of particles (atoms or molecules) in one mole of substance (gas) is called Aveogadro number, which has a magnitude $N_A = 6.023 \times 10^{23}$ mol^{-1}

If N is the number of gas molecules in a container , then the number of mole of given gas is

$$\mu = \frac{N}{N_A}$$

If M is the total mass of gas in a container, and mass of one mole of gas called molar mass M_O, then the number of moles of gas is

$$\mu = \frac{M}{M_O}$$

Other important laws of an ideal gas

Grahm's law of diffusion states that when two gases at the same pressure and temperature are allowed to diffuse into each other, the rate of diffusion of each gas is inversely proportional to the square root of the density of the gas

$$rate\ of\ diffusion \propto \sqrt{\frac{1}{density\ of\ gas}}$$

Dalton's law of partial pressure states that the pressure exerted by a mixture of several gases equals the sum of the pressure exerted by each gas occupying the same volume as that of the mixture

$P_1, P_2, \ldots, P_n$ are the pressure exerted by individual gases of the mixture, then pressure of the mixture of the gas is

$P = P_1 + P_2 + \ldots + P_n$

Ideal gas-state equation and it different forms

If we combine Boyle's law and Charle's law we get

$$\frac{PV}{T} = constant$$

For a given quantity of gas, which shows that for constant temperature and pressure, if quantity or mass of gas is varies, then volume of the gas is proportional to the quantity of gas.

Thus constant on the right hand side of the equation depends on the quantity of the gas. If quantity is represented in mole then

$$\frac{PV}{T} = \mu R$$

Equation is called an ideal gas-state equation

Here R is universal gas constant = 8.314 J $mole^{-1}$ K^{-1}

If gas completely obeys equation

$$PV = \mu RT \quad ---eq(1)$$

at all values of pressure and temperature, then such a (imaginary) gas is called an ideal gas. By putting

$$\mu = \frac{N}{N_A}$$

In above equation we get

$$PV = \frac{N}{N_A}RT = N\frac{R}{N_A}T$$

Putting $R/N_A = k_B$ (Boltzmann's constant = 1.38×10^{-23} J K^{-1})

$$PV = Nk_BT$$

$$\therefore P = \frac{N}{V}k_BT$$

If n = N/V number of molecules per unit volume of gas

$$\therefore P = nk_BT \quad --- eq(2)$$

Putting

$$\mu = \frac{M}{M_O}$$

In equation (1)

$$PV = \frac{M}{M_O}RT$$

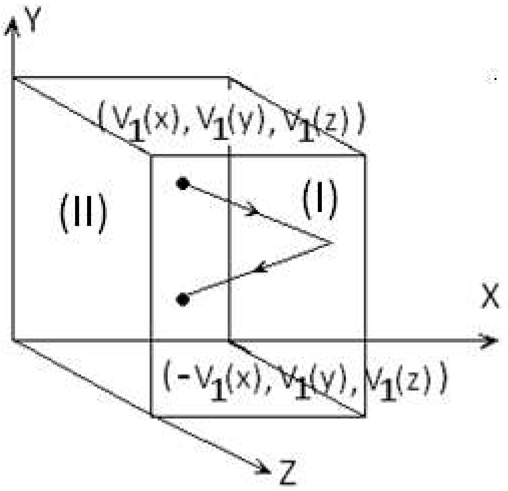
$$P = \frac{M}{V}\frac{RT}{M_O}$$

$$P = \frac{\rho RT}{M_O} \quad --- eq(3)$$

ρ is the density of gas

Pressure of an ideal gas and rms speed of gas molecules

The molecules of a gas are in a state of random motion. They continuously collide against the walls of the container. During each collision, momentum is transferred to the walls of the container.



The pressure exerted by the gas is due to the continuous collision of the molecules against the walls of the container. Due to this continuous collision, the walls experience a continuous force which is equal to the total momentum imparted to the walls per second. The force experienced per unit area of the walls of the container determines the pressure exerted by the gas.

Consider a cubic container of side L containing n molecules of perfect gas moving with velocities C_1, C_2, C_3 ... C_n

A molecule moving with a velocity v_1, will have velocities $C_1(x)$, $C_1(y)$ and $C_1(z)$ as components along the x, y and z axes respectively. Similarly $C_2(x)$, $C_2(y)$ and $C_2(z)$ are the velocity components of the second molecule and so on.

Let a molecule P shown in figure having velocity C_1 collide against the wall marked I perpendicular to the x-axis. Only the x-component of the velocity of the molecule is relevant for the wall . Hence momentum of the molecule before collision is *m* $C_1(x)$ where *m* is the mass of the molecule.
Since the collision is elastic, the molecule will rebound with the velocity $C_1(x)$ in the opposite direction. Hence momentum of the molecule after collision is $-mC_1(x)$

Change in the momentum of the molecule = Final momentum - Initial momentum
Change in the momentum of the molecule = $- mC_1(x) - mC_1(x) = -2mC_1(x)$
During each successive collision on face I the molecule must travel a distance 2L from face I to face II and back to face I.

Time taken between two successive collisions is = $2L/ C_1(x)$

$$\therefore Rate\ of\ change\ of\ momentum = \frac{change\ in\ momentum}{time\ taken}$$

$$Rate\ of\ change\ of\ momentum = \frac{-2mC_1(x)}{\frac{2L}{V_1(x)}} = \frac{-mC_1^2(x)}{L}$$

$$\text{Force exerted on the molecule} = \frac{-mC_1^2(x)}{L}$$

According to Newton's third law of motion, the force exerted by the molecule =

$$= -\frac{-mC_1^2(x)}{L} = \frac{mC_1^2(x)}{L}$$

Force exerted by all the *n* molecules is

$$F_x = \frac{mC_1^2(x)}{L} + \frac{mC_2^2(x)}{L} + \frac{mC_3^2(x)}{L} + \cdots + \frac{mC_n^2(x)}{L}$$

Pressure exerted by the molecules

$$P_x = \frac{F_x}{A}$$

$$P_x = \frac{1}{L^2}\left(\frac{mC_1^2(x)}{L} + \frac{mC_2^2(x)}{L} + \frac{mC_3^2(x)}{L} + \cdots + \frac{mC_n^2(x)}{L}\right)$$

$$P_x = \frac{m}{L^3}\left(C_1^2(x) + C_2^2(x) + C_3^2(x) + \cdots + C_n^2(x)\right)$$

Similarly, pressure exerted by the molecules along Y and Z axes are

$$P_y = \frac{m}{L^3}\left(C_1^2(y) + C_2^2(y) + C_3^2(y) + \cdots + C_n^2(y)\right)$$

$$P_z = \frac{m}{L^3}\left(C_1^2(z) + C_2^2(z) + C_3^2(z) + \cdots + C_n^2(z)\right)$$

Since the gas exerts the same pressure on all the walls of the container

$$P_x = P_y = P_z$$

$$P = \frac{P_x + P_y + P_z}{3}$$

$$P = \frac{1}{3}\frac{m}{L^3}\left[\left(C_1^2(x) + C_2^2(x) + C_3^2(x) + \cdots + C_n^2(x)\right) + \left(C_1^2(y) + C_2^2(y) + C_3^2(y) + \cdots + C_n^2(y)\right) + \left(C_1^2(z) + C_2^2(z) + C_3^2(z) + \cdots + C_n^2(z)\right)\right]$$

$$P = \frac{1}{3}\frac{m}{L^3}\left[\left(C_1^2(x) + C_1^2(y) + C_1^2(z)\right) + \left(C_2^2(x) + C_2^2(y) + C_2^2(z)\right) + \cdots + \left(C_n^2(x) + C_n^2(y) + C_n^2(z)\right)\right]$$

$$P = \frac{m}{3V}\left[C_1^2 + C_2^2 + .. + C_n^2\right]$$

$$P = \frac{mn}{3V}\left[\frac{C_1^2 + C_2^2 + .. + C_n^2}{n}\right]$$

$$P = \frac{mn}{3V} < C^2 >$$

Here V is volume of gas

Where < C^2 > is called the root mean square (RMS) velocity, which is defined as the square root of the mean value of the squares of velocities of individual molecules.

Since mn = mass of gas and density ρ = mass/volume

$$P = \frac{\rho}{3} < C^2 >$$

Relation between the pressure exerted by a gas and the mean kinetic energy of translation per unit volume of the gas

Mean kinetic energy of translation per unit volume of the gas

$$E = \frac{1}{2}\rho < C^2 >$$

Thus

$$\frac{P}{E} = \frac{\frac{\rho}{3} < C^2 >}{\frac{1}{2}\rho < C^2 >} = \frac{2}{3}$$

Or P = (2/3)E

Average kinetic energy per molecule of the gas

Let us consider one mole of gas of mass M and volume V.

$$P = \frac{\rho}{3} < C^2 >$$

$$P = \frac{M}{3V} < C^2 >$$

$$PV = \frac{M}{3} < C^2 >$$

From ideal gas equation for one mole of gas

PV = RT

$$\frac{M}{3} < C^2 > = RT$$

$$M < C^2 > = 3RT$$

$$\frac{1}{2}M < C^2 > = \frac{3}{2}RT$$

Average kinetic energy of one mole of the gas is equal to = (3/2) RT

Since one mole of the gas contains N_A number of atoms where N_A is the Avogadro number we have $M = N_A m$

$$\frac{1}{2}N_A m < C^2 > = \frac{3}{2}RT$$

$$\frac{1}{2}m < C^2 > = \frac{3}{2}\frac{R}{N_A}T$$

$$\frac{1}{2}m < C^2 > = \frac{3}{2}k_B T$$

k_B is Boltzmann constant

Average kinetic energy per molecule of the gas is equal to (3/2) k_BT

Hence, it is clear that the temperature of a gas is the measure of the mean translational kinetic energy per molecule of the gas

Degrees of freedom

The number of degrees of freedom of a dynamical system is defined as the total number of co-ordinates or independent variables required to describe the position and configuration of the system.

(i) A particle moving in a straight line along any one of the axes has one degree of freedom (e.g) Bob of an oscillating simple pendulum.

(ii) A particle moving in a plane (X and Y axes) has two degrees of freedom. (eg) An ant that moves on a floor.

(iii) A particle moving in space (X, Y and Z axes) has three degrees of freedom. (eg) a bird that flies.

A point mass cannot undergo rotation, but only translatory motion. Three degree of freedom

A rigid body with finite mass has both rotatory and translatory motion.

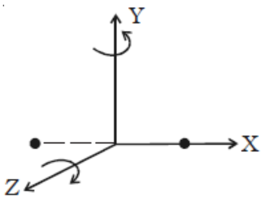
The rotatory motion also can have three co-ordinates in space, like translatory motion ; Therefore a rigid body will have six degrees of freedom ; three due to translatory motion and three due to rotator motion.

Monoatomic molecule

Since a monoatomic molecule consists of only a single atom of point mass it has three degrees of freedom of translatory motion along the three co-ordinate axes

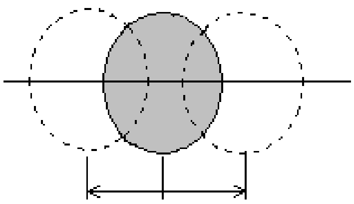
Examples : molecules of rare gases like helium, argon, etc.

Diatomic molecule rigid rotator



The diatomic molecule can rotate about any axis at right angles to its own axis. Hence it has two degrees of freedom of rotational motion in addition to three degrees of freedom of translational motion along the three axes. So, a diatomic molecule has five degrees of freedom (Fig.). Examples : molecules of O_2, N_2, Cl_2, etc

Diatomic molecule like CO : Have five freedom as stated in rigid rotator apart from that they have two more freedoms due to vibration (oscillation) about mean position



Plyatomic molecules posses rotational kinetic energy energy of vibration in addition to their translational energy. Therefore when heat energy is given to such gases, it is utilized in increasing the translational kinetic energy, rotational kinetic energy and vibrational kinetic energy of the gas molecules and hence more heat is required. This way polyatomic molecules posses more specific heat

Law of equipartition of energy

Law of equipartition of energy states that for a dynamical system in thermal equilibrium the total energy of the system is shared equally by all the degrees of freedom. The energy associated with each degree of freedom per moelcule is (1/2)kT
where *k* is the Boltzmann's constant.
Let us consider one mole of a monoatomic gas in thermal equilibrium at temperature T.
Each molecule has 3 degrees of freedom due to translatory motion.
According to kinetic theory of gases, the mean kinetic energy of a molecule is (3/2)kT
Let *Cx* , *Cy* and *Cz* be the components of RMS velocity of a molecule along the three axes.
Then the average energy of a gas molecule is given by

$$\frac{1}{2}mC^2 = \frac{1}{2}mC_x^2 + \frac{1}{2}mC_y^2 + \frac{1}{2}mC_z^2$$

$$\frac{1}{2}mC_x^2 + \frac{1}{2}mC_y^2 + \frac{1}{2}mC_z^2 = \frac{3}{2}kT$$

Since molecules move at random, the average kinetic energy corresponding to each degree of freedom is the same.

$$\frac{1}{2}mC_x^2 = \frac{1}{2}mC_y^2 = \frac{1}{2}mC_z^2 = \frac{1}{2}kT$$

∴ Mean kinetic energy per molecule per degree of freedom is (1/2) kT

Mean free path

"The linear distance travelled by a molecule of gas with constant speed between two consecutive collisions (between molecules) is called free path. The average of such free paths travelled by a molecule is called mean free path"
Suppose molecules of gas are spheres of diameter d. If the centre between the two molecules is less or equal to d then they will collide when they come close.
Consider a molecule of diameter d moving with average speed v, and the other molecule is stationary. The molecule under consideration will sweep a cylinder of $\pi d^2 vt$. In time t.
If the number of molecules per unit volume is n, then the number of molecules in this cylinder is $n\pi d^2 vt$. Hence the molecule will under go $n\pi d^2 vt$ collisions in time t
The mean free path l is the average distance between two successive collision

$$Mean\ free\ path = \frac{distance\ travelled\ in\ time\ t}{number\ of\ collisions\ in\ time\ t}$$

$$l = \frac{vt}{2\pi d^2 vt}$$

$$l = \frac{1}{2\pi n d^2}$$

In this derivation other molecules are considered stationary. In actual practice all gas molecules are moving and there collision rate is determined by the average relative velocity < V > Hence mean free path formula is $l = \frac{1}{\sqrt{2}\pi n d^2}$

Solved Numerical

Q) Find the mean translational kinetic energy of a molecules of He at 27^{O}

Solution: Since He is mono atomic degree of freedom is 3

Kinetic energy =(3/2)k_BT

Here k_B = Boltzmann's constant = 1.38×10^{-23} J K^{-1} Temperature T = 27+273 = 300 K

$$K = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21} J$$

Q) At what temperature rms velocity of O_2 is equal to rms velocity of H_2 ay 27^{O}C?

Solution

Kinetic energy

$$\frac{1}{2} m < C^2 > = \frac{3}{2} k_B T$$

$$< C^2 > = \frac{3k_B T}{m}$$

But rms velocity of O_2 rms velocity of He

$$\frac{3k_B T}{32} = \frac{3k_B \times 300}{4}$$

$$T = 2400\ K$$

Q) Find rms velocity of hydrogen at 0^{O}C temperature and 1 atm pressure. Density of hydrogen gas is 8.9×10^{-2} kg m^{-3}

Solution:

From formula

$$P = \frac{\rho}{3} < C^2 >$$

$$< C > = \sqrt{\frac{3P}{\rho}}$$

$$< C > = \sqrt{\frac{3 \times 1.01 \times 10^5}{8.9 \times 10^{-2}}} = 1845\ ms^{-1}$$

Q) If the molecular radius of hydrogen molecule is 0.5Å, find the mean free path of hydrogen molecule at 0^{O}C temperature and 1 atm pressure

Solution

d = 2×r = 1 Å

From formula P = nK_BT

$$n = \frac{P}{k_B T}$$

$$n = \frac{1.01 \times 10^5}{1.38 \times 10^{-23} \times 273} = 2.68 \times 10^{25}$$

From formula for mean path $l = \frac{1}{\sqrt{2} \times 3.14 \times 2.68 \times 10^{25} \times (1 \times 10^{-10})^2} = 8.4 \times 10^{-7}\ m$

SECTION II
Thermodynamics

Some important terms

Thermodynamic system : It is apart of the universe under thermodynamic study. A system can be one, two or three dimensional. May consists of single or many objects or radiation

Environment : remaining part of universe around the thermodynamic system is Environment. Environment have direct impact on the behavior of the system

Wall: The boundary separating the Stem from the universe is wall

Thermodynamic co-ordinates: The macroscopic quantities having direct effect on the internal state of the system are called thermodynamic coordinates. For example
Take the simple example of a sample of gas with a fixed number of molecules. It need not be ideal. Its temperature, *T*, can be expressed as a function of just two variables, volume, *V*, and pressure, *p*. We can, it turns out, express all gas properties as functions of just two variables (such as *p* and *V* or *p* and *T*). These properties include refractive index, viscosity, internal energy, entropy, enthalpy, the Helmholtz function, the Gibbs function. We call these properties 'functions of state'. The state is determined by the values of just two variables

Thermodynamic system: The system represented by the thermodynamic co-ordinate is called a thermodynamic system

Thermodynamic process: The interaction between a system and its environment is called a thermodynamic process

Isolated system: If a system does not interact with its surrounding then it is called an isolated system. Thermal and mechanical properties of such system is said to be ina definite thermodynamic equilibrium state
Heat (Q) and Work(W): The amount of heat energy exchanged during the interaction of system with environment is called heat (Q) and the mechanical energy exchanged is called work (W).

Thermodynamic variables: Thermodynamic variables describe the momentary condition of a thermodynamic system. Regardless of the path by which a system goes from one state to another — i.e., the sequence of intermediate states — the total changes in any state variable will be the same. This means that the incremental changes in such variables are exact differentials. Examples of state variables include: Density (ρ), Energy (E), Gibbs free energy (G), Enthalpy (H) , Internal energy (U), Mass (m) , Pressure (p) ,Entropy (S) Temperature (T), Volume (V)

Extensive thermodynamic state variable: The variables depending on the dimensions of the system are called extensive variables. For examples mass, volume, internal energy

Intensive thermodynamic state variable: The variables independent on the dimensions of the system are called intensive variables. For examples pressure, temperature, density

Thermal equilibrium: When two system having different temperatures are brought in thermal contact with each other, the heat flows from the system at higher temperature to that at lower temperature. When both the system attains equal temperatures, the net heat exchanged between them becomes zero. In this state they are said to be in thermal equilibrium state with each other.

Zeroth Law of thermodynamics: "If the system A and B are in the thermal equilibrium with a third system C, then A and B are also in thermal equilibrium with each other"

Temperature may be defined as the particular property which determines whether a system is in thermal equilibrium or not with its neighbouring system when they are brought into contact
adiabatic wall – an insulating wall (can be movable) that does not allow flow of energy (heat) from one to another**.**
diathermic wall – a conducting wall that allows energy flow (heat) from one to another

Specific heat capacity

Specific heat capacity of a substance is defined as the quantity of heat required to raise the temperature of 1 kg of the substance through 1K. Its unit is J $kg^{-1}K^{-1}$.

Molar specific heat capacity of a gas
Molar specific heat capacity of a gas is defined as the quantity of heat required to raise the temperature of 1 mole of the gas through 1K. Its unit is J mol^{-1} K^{-1}.
Let *m* be the mass of a gas and *C* its specific heat capacity. Then $\Delta Q = m \times C \times \Delta T$ where ΔQ is the amount of heat absorbed and ΔT is the corresponding rise in temperature.
Case (i)

If the gas is insulated from its surroundings and is suddenly compressed, it will be heated up and there is rise in temperature, even though no heat is supplied from outside

(i.e) $\Delta Q = 0 \therefore C = 0$

Case (ii)

If the gas is allowed to expand slowly, in order to keep the temperature constant, an amount of heat ΔQ is supplied from outside, then

$$C = \frac{\Delta Q}{m\Delta T} = \frac{\Delta Q}{0} = +\infty$$

($\because$ ΔQ is +ve as heat is supplied from outside)

Case (iii)

If the gas is compressed gradually and the heat generated ΔQ is conducted away so that temperature remains constant, then

$$C = \frac{-\Delta Q}{m\Delta T} = \frac{-\Delta Q}{0} = +\infty$$

($\because$ ΔQ is -ve as heat is supplied by the system)

Thus we find that if the external conditions are not controlled, the value of the specific heat capacity of a gas may vary from $+\infty$ to $-\infty$

Hence, *in order to find the value of specific heat capacity of a gas, either the pressure or the volume of the gas should be kept constant.* Consequently a gas has two specific heat capacities

(i) Specific heat capacity at constant volume
(ii) Specific heat capacity at constant pressure.

Molar specific heat capacity of a gas at constant volume

Molar specific heat capacity of a gas at constant volume C_V is defined as the quantity of heat required to raise the temperature of one mole of a gas through 1 K, keeping its volume constant

Molar specific heat capacity of a gas at constant pressure

Molar specific heat capacity of a gas at constant pressure C_p is defined as the quantity of heat to raise the temperature of one mole of a gas through 1 K keeping its pressure constant

Specific heat of gas from the law of equipartition of energy

The energy associated with each degree of freedom is $(1/2)K_BT$. It means that, if the degree of freedom of a gas molecule is f then the average heat energy of each molecule of gas is

$$E_{ave} = f \times \frac{1}{2} k_B T$$

If number of moles of an ideal gas is μ , then the number of moles in the gas is μN_A.

Therefore the **internal energy** of μ mole of ideal gas is

$U = \mu N_A E_{average}$

$$U = \mu N_A f \times \frac{1}{2} k_B T$$

$$U = \frac{f}{2} \mu RT \quad (\because R = N_A k_B)$$

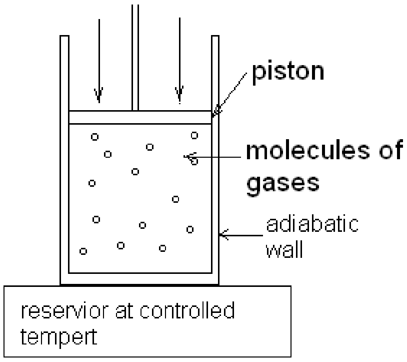
Work in thermodynamics

The amount of mechanical energy exchanged between two bodies during mechanical interaction is called work. Thus work is a quantity related to mechanical interaction. A system can possess mechanical energy, but cannot posses work

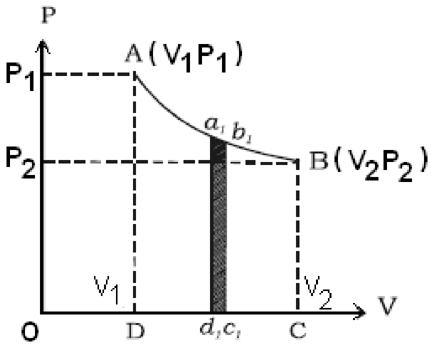
In thermodynamics the work done by the system is considered positive and the work done on the system is considered negative.

The reason behind such sign convention is due to the mode of working of heat engine in which the engine absorbs heat from the environment and converts it into work W means the energy of the system is reduced by W

Formula for the work done during the compression of gas at constant temperature



As shown in figure μ molecules of gas are enclosed in a cylindrical container at low pressure, and an air tight piston capable of moving without friction with area A is provide.. the conducting bottom of the cylinder is placed on an arrangement whose temperature can be contolled.



At constant temperature , measuring the volume of the gas for different values of pressure, the graph of P-V can be plotted as shown in figure. These types of process are called isothermal process and curved of P-V is called isotherm.

Suppose initial pressure and volume of the pas is represented by P_1 and V_1 respectively. Keeping the temperature T of the gas to be constant , volume of gas decreases slowly by pushing piston down. Let final pressure and volume of the gas be is P_2 and V_2

During the process, at one moment when pressure of the gas is P and volume V, at that time, let the piston moves inward by Δx. then the volume of the gas decreases by ΔV. this displacement is to small that there is no apparent change in pressure.

Hence work done on the gas

$\Delta W = F\Delta x$

$\Delta W = PA\Delta x$

$\Delta W = P\Delta V$

If the volume of the gas is decreasing from V_1 to V_2 through such small changes, then the total work done on the gas

$$W = \sum_{V_1}^{V_2} P \vartriangle V$$

If this summation is taking the limit as $\Delta V \rightarrow 0$ the summation results in integration

$$W = \int_{V_1}^{V_2} PdV$$

But the ideal gas equation for μ moles of gas is PV=μRT thus

$$W = \int_{V_1}^{V_2} \frac{\mu RT}{V} dV$$

$$W = \mu RT \int_{V_1}^{V_2} \frac{dV}{V}$$

$$W = \mu RT[\ln V]_{V_1}^{V_2}$$

$$W = \mu RT[V_2 - V_1]$$

$$W = \mu RT \ln\left(\frac{V_2}{V_1}\right)$$

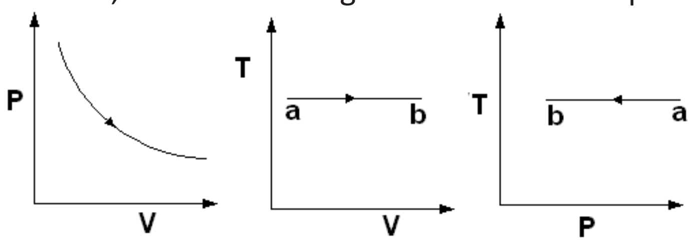
$$W = 2.303\mu RT \log_{10}\left(\frac{V_2}{V_1}\right)$$

Equation does not give the work W by an ideal gas during every thermodynamic process, but it gives the work done only for a process in which the temperature is held constant.

Since $V_2 < V_1$ hence $\log(V_2/V_1)$ is negative . Thus we get negative value of work which represents that **during the compression of gas at constant temperature, the work is done on the gas**

If the gas is expanded then Since $V_2 > V_1$ hence $\log(V_2/V_1)$ is positive . Thus we get negative value of work which represents that **during the compression of gas at constant temperature, the work is done by the gas**

The P-V, T-V and T-P diagram for isothermal process will be like the curves given below



ISOTHERMAL PROCESS

Work done at constant volume and constant pressure

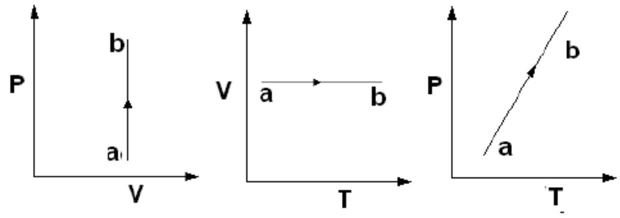
Constant volume : Also called as isochoric process

If the volume is constant then dV = from equation

$$W = \int_{V_1}^{V_2} PdV$$

Work done is zero

The P-V, V-T and P-T diagrams for isochoric process will be like curves given below



ISOCHORIC PROCES

Constant pressure :Also called as Isobaric process

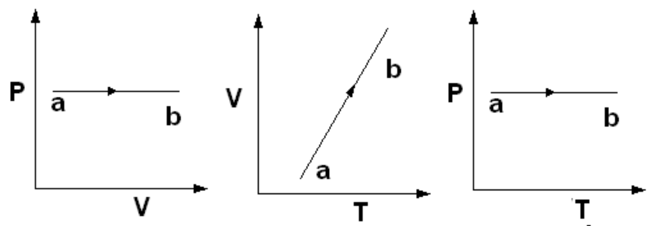
If the volume is changing while pressure is constant then from equation

$$W = \int_{V_1}^{V_2} PdV = P\int_{V_1}^{V_2} PdV$$

$$W = P[V_2 - V_1]$$

W = PΔV (for constant pressure)

The P-V, V-T and P-T diagrams for isobaric process will be like curves given below



ISOBARIC PROCESS

Work done during adiabatic process

No excahge of heat takes palce between system and it environment in this process. This is possible when (1) walls of a system are thermal insulator or (2) process is very rapid.

The relation between pressure and volume for ideal gas is

$$PV^{\gamma} = constant$$

Where $\gamma = \frac{C_P}{C_V}$

For an adiabatic process

$$W = \int_{V_1}^{V_2} PdV$$

Let

$$PV^{\gamma} = A$$

$$W = \int_{V_1}^{V_2} \frac{A}{V^{\gamma}} dV$$

$$W = A \int_{V_1}^{V_2} \frac{dV}{V^{\gamma}}$$

$$W = A \left[\frac{V^{-\gamma+1}}{-\gamma+1} \right]_{V_1}^{V_2}$$

$$W = \frac{1}{1-\gamma} \left[AV_2^{-\gamma+1} - AV_1^{-\gamma+1} \right]$$

$But\ A = P_2 V_2^{\gamma} = P_1 V_1^{\gamma}$

$$W = \frac{1}{1-\gamma} \left[P_2 V_2^{\gamma} V_2^{-\gamma+1} - P_1 V_1^{\gamma} V_1^{-\gamma+1} \right]$$

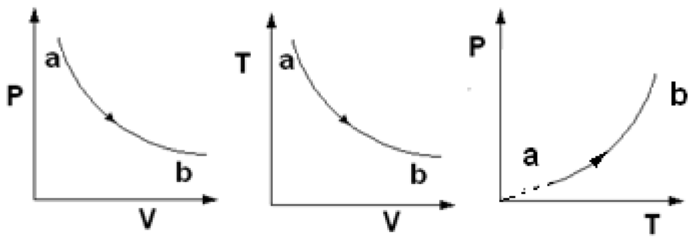
$$W = \frac{1}{1-\gamma} [P_2 V_2 - P_1 V_1]$$

$$W = \frac{1}{\gamma - 1} [P_1 V_1 - P_2 V_2]$$

From ideal gas equation PV = μRT

$$W = \frac{\mu R}{\gamma - 1} [T_1 - T_2]$$

The P-V, T-V and P-T diagrams for adiabatic process will be lie the curves given below



ADIABATIC PROCESS

Solved Numerical

Q) Calculate work done if one mole of ideal gas is compressed isothermally at a temperature 27^{O} C from volume of 5 litres to 1 litre

Solution:

Formula for work done during iso-thermal process is

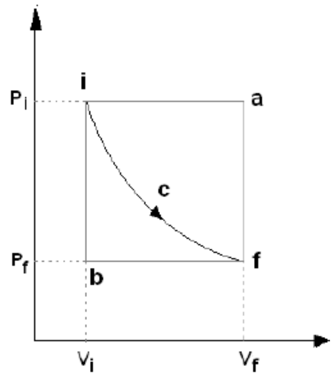
$$W = 2.303 \mu RT \log_{10} \left(\frac{V_2}{V_1} \right)$$

$$W = 2.303 \times 1 \times 8.31 \times 300 \times log \left(\frac{1}{5} \right)$$

$$W = 2.303 \times 1 \times 8.31 \times 300 \times [log1 - log5]$$
$$W = 2.303 \times 1 \times 8.31 \times 300 \times [0 - 0.6990]$$

W = -4012.5J

First law of thermodynamics



Suppose a sytem absorbs heat and as a result work is done by it (by the system). We can think of different paths (process) through which the system can be taken from initial stage (i) to final state (f)

For the process iaf, ibf, icf. Suppose the heat absorbed by the system are Q_a, Q_b , Q_c respectively and the values of the work done are respectively W_a, W_b, W_c. Here

$Q_a \neq Q_b \neq Q_c$ and $W_a \neq W_b \neq W_c$, but difference of heat and work done turns out to be same

$Q_a - W_a = Q_b - W_b = Q_c - W_c$

Thus value of Q – W depends only on initial and final state of the system. A thermodynamic state function can be defined such that the difference between any two states is equal to Q – W. Such a function is called internal energy U of system

The system gains energy Q in the form of heat energy and spends energy W to do work. Hence the internal energy of the system changes by Q-W.

If the internal energies of system in initial state is U_i and final state is U_f then

$U_i - U_f = \Delta U = Q - W$ Which is the first law of thermodynamics

The first law is obeyed in all the changes occurring in nature

Isochoric process

Since in this process volume remains constant, the work done in this process is equal to zero. Applying first law of thermodynamics to this process, we get

$\Delta Q = \Delta U + \Delta W$

$\Delta Q = \Delta W$

So heat exchange in this process takes place at the expense of the internal energy of the system.

dQ = dU

$$\left(\frac{dQ}{dT}\right)_V = \left(\frac{dU}{dT}\right)_V$$
$$since\ U = \frac{f}{2}RT$$

$$\left(\frac{dQ}{dT}\right)_V = \frac{f}{2}R$$

Thus above equation is for the energy required to increase temperature by one unit of one of ideal gas it is molar specific heat at constant volume C_V

$$\left(\frac{dQ}{dT}\right)_V = \frac{f}{2}R = C_V$$

Isobaric process

Applying first law of thermodynamics to isobaric process we get

$$\Delta Q = \Delta U + P(V_2 - V_1)$$

$$\Delta Q = \Delta U + P\Delta V$$

But PV = RT for one mole of gas

$$\therefore P\Delta V = R\Delta T \text{ thus}$$

$$\therefore \Delta Q = \Delta U + R\Delta T$$

$$since\ U = \frac{f}{2}RT$$

$$\therefore \left(\frac{dQ}{dT}\right)_P = \frac{f}{2}R\frac{dT}{dT} + \mathrm{R}\frac{dT}{dT}$$

$$\therefore \left(\frac{dQ}{dT}\right)_P = \frac{f}{2}R + \mathrm{R}$$

Since dQ/dT is specific heat at constant pressure = C_P

$$\therefore C_P = C_V + R$$

$$\text{OR } C_P - C_V = R$$

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{f}{2}R + \mathrm{R}}{\frac{f}{2}R}$$

$$\gamma = 1 + \frac{2}{f}$$

f is degree of freedom

For monoatomic molecule f = 3

$$C_V = \frac{3R}{2},\ \ C_P = \frac{5R}{2}\ ,\gamma = \frac{5}{3}$$

For the diatomic molecules (rigid rotator) f = 5

$$C_V = \frac{5R}{2},\ \ C_P = \frac{7R}{2}\ ,\gamma = \frac{7}{5}$$

For the diatomic molecules (with vibration , molecule like CO) f = 7

$$C_V = \frac{7R}{2},\ \ C_P = \frac{9R}{2}\ ,\gamma = \frac{9}{7}$$

According to the equipartion theorem the change in internal energy is related to the temperature of the system by

$$\Delta U = mC_V\Delta T$$

Isothermal process

For isothermal process ΔU = 0.

Applying first law of thermodynamics we get

ΔQ = W

$$\Delta Q = W = 2.303\mu RT \log_{10}\left(\frac{V_2}{V_1}\right)$$

Adiabatic process

Applying first law of thermodynamics we get

ΔQ = ΔU + ΔW

For adiabatic process ΔQ = 0

$$-\Delta U = \Delta W$$

The reduction in internal energy of the gas (due to which temperature fails) is equal to the work done during an adiabatic expansion. Again during an adiabatic compression the work done on the gas causes its temperature rise. Adiabatic processes are generally very fast.

Example when we use air pump to fill air in bicycle tyre, pump get heated on pumping rapidly

Solved Numerical

Q) At 27^{O}C, two moles of an ideal momoatomic gas occupy a volume V. The gas expands adiabatically to a volume 2V. Calculate (a) final temperature of the gas (b) Change in its internal energy (c) Work done by the gas during the process

Take R = 8.31 J/mole/K

Solution:

For monoatomic gas γ = 5/3.

T = 27+273 =300

(a) Gas expanded adiabatically

$$P_2V_2^{\gamma} = P_1V_1^{\gamma}$$

Since PV∝T

P∝ T/V

Thus

$$T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1}$$

$$\therefore T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\therefore T_2 = 300\left(\frac{1}{2}\right)^{5/3-1} = 189K$$

(b) For adiabatic process ΔQ = 0

$$-\Delta U = \Delta W$$

$$W = -\Delta U = \frac{\mu R}{\gamma - 1}[T_1 - T_2]$$

$$-\Delta U = \frac{2\times 8.31}{\frac{5}{3}-1}[300-189] = 2767.23\ J$$

ΔU = -2767.23 J

(c) ΔW = -ΔU

ΔW = 2767.23 J

Isothermal and adiabatic curves

The relation between the pressure and volume of gas can be represented graphically. The curve for an isothermal process is called isothermal curve or an isotherm and there are different isotherms for different temperatures for a given gas. A similar curve for an adiabatic process is called an adiabatic curve or adiabatic

Since

$$\left(\frac{dP}{dV}\right)_{isothermal} = -\frac{P}{V}$$

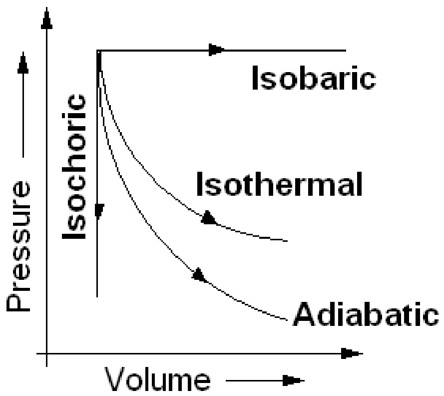
And

$$\left(\frac{dP}{dV}\right)_{adiabatic} = -\gamma\frac{P}{V}$$

So

$$\left(\frac{dP}{dV}\right)_{adiabatic} = \gamma\left(\frac{dP}{dV}\right)_{isothermal}$$

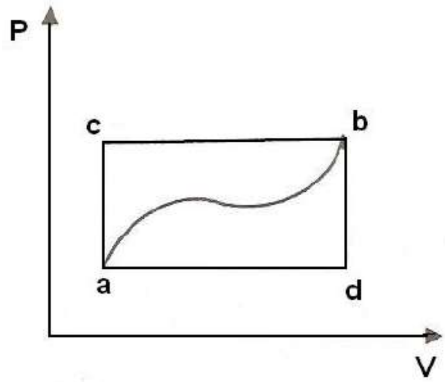
Since γ > 1, so adiabatic curve is steeper-than the isothermal curve



To permit comparison between isothermal ,adiabatic process, Isochoric and isobaric process an isothermal curve ,an adiabatic curve isochoric and Isobaric curves of gas are drawn on the same pressure-volume diagram starting from the same point.

Solved Numerical

Q)When a system is taken from state a to state b along the path acb it is found that a quantity of heat Q = 200J is absorbed by the system and a work W = 80J is done by it. Along the path adb, Q = 144J



(i)What is the work done along the path adb

(ii)IF the work done on the system along the curvered path ba is 52J, does the system absorb or linerate heat and how much

(iii)If U_a = 40J, what is U_b

(iv)If U_d = 88J, what is Q for the path db and ad?

Solution

From the first law of thermodynamics, we have

$Q = \Delta U + \Delta W$

$Q = (U_b - U_a) + W$

Where U_b is the internal energy in the state b and U is the internal energy in the stste $_a$

For the path acb, it is given that

Q = 200J (absorption) and

Q = 80J (work done by the system)

$\therefore U_b - U_a = = Q - W = 200-80 = 120J$

Which is the increase in the internal energy of the system for path acb. Whatever be the path between a and b the change in the internal energy will be 120 J only

(i) To determine the work done along the path adb

Given Q = 144J

$\Delta U = U_b - U_a = 120J$

$Q = (U_b - U_a) + W$

144 = 120 + W

W = 24J

Since W is positive, work is done by the system

(ii) For the curved return path ba, it is given that

Given W=-52J (work done on the system)

$\Delta U = -120$ J (negative sign since $\Delta U = U_a - U_b$)

$Q = (U_a - U_b) + W$

Q = (-120 – 52) J = -172 J

Negative sign indicates heat is extracted out of the system

(iii) Since $U_b - U_a = 120$ J and $U_a = 40$ j

$U_b = U_a + 120 = 40 + 120 = 160J$

(iv) For path db, the process is isochoric since it is at constant volume

Work done is zero

$Q = \Delta U + W$

$Q = \Delta U$

$Q = U_b - U_d = 160 - 88 = 72$ J

For the path ad,

$Q = Q_{adb} - Q_{db} = 144J - 72\ J = 72J$

Q) A mass of 8 g of oxygen at the pressure of one atmosphere and at temperature 27^{O}C is enclosed in a cylinder fitted with a frictionless piston. The following operations are performed in the order given

(a) The gas is heated at constant pressure to 127^{O}C

(b) then it is compressed isothermally to its initial volume and

(c) finally it is cooled to its initial temperature at constant volume

(i) What is the heat absorbed by the gas during process (A)?

(ii) How much work is done by the gas in process A

(iii)What is the work done on the gas in process B

(iv)How much heat is extracted from the gas in process (c)

[Specific heat capacity of oxygen C_V = 670 J/KgK;]

Solution:

Volume of gas at temperature 27+273 = 300K = T_2

Molecular weight of Oxygen = 32 thus 8g = 0.2 mole

At STP volume of 1 mole is 22.4 litre Thus volume of 0.25 mole is V_1 = 22.4/4

Thus for formula volume at 27^{O}C is

$$\frac{V_2}{T_2} = \frac{V_1}{T_1}$$

$$V_2 = \frac{T_2}{T_1}V_1$$

$$V_2 = \frac{300}{273} \times \frac{22.4}{4} = \frac{560}{91} \times 10^{-3}\, m^3$$

Similarly

Volume at 127^{O}C is

$$V_3 = V_2 \times \frac{400}{300} = \frac{4}{3}V_2$$

$$\frac{V_3}{V_2} = \frac{4}{3}$$

(i)

For Isothermal compression

dQ = dU + dW = $mC_v\Delta T$ + P(V_3 – V_2)

$$dQ = \frac{8}{1000} \times 670 \times 100 + 1.013 \times 10^5 \times \left[\frac{560 \times 10^{-3}}{3 \times 91}\right]$$

dQ = 536 + 207.8 = 743.8 J

(ii) dW = P(V_3 – V_2) = 207.8 J

(iii) Work done in compressing the gas isothermally =

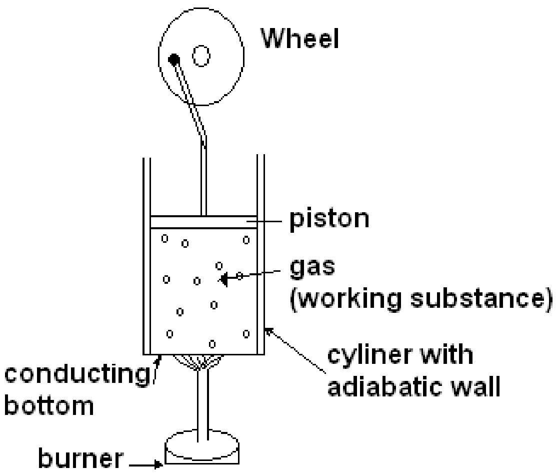
$$W = 2.303\mu\mathrm{RT}\log_{10}\left(\frac{\mathrm{V_3}}{\mathrm{V_2}}\right)$$

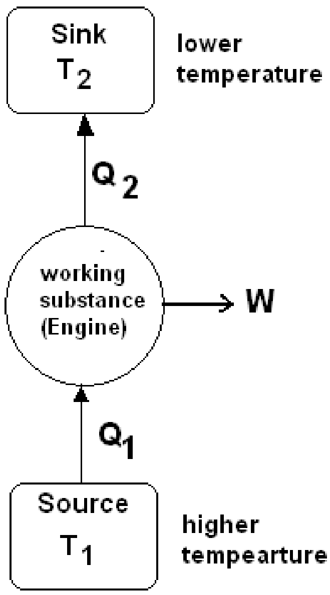
$$W = 2.303 \times \frac{\mathrm{m}}{\mathrm{M}}\mathrm{RT}\log_{10}\left(\frac{\mathrm{V}_3}{\mathrm{V}_2}\right)$$
$$W = 2.303 \times \frac{8}{32} \times 8.31 \times 400 \times \log_{10}\left(\frac{4}{3}\right)$$
$$W = 831 \times 0.2877 = 239.1\,J$$

(d) Heat given out by the gas in stage (C) = $mC_V\Delta T$

$$\frac{8}{1000} \times 670 \times 100 = 536\,J$$

Heat Engine





A device converting heat energy into mechanical work is called heat engines.

A simple heat engine is shown in figure. The gas enclosed in a cylinder with a piston receives heat from the flame of a burner. On absorbing heat energy the gas expands and pushes the piston upwards. So the wheel starts rotating. To continue the rotations of the wheel an arrangement is done in the heat engine so that the piston can move up and down periodically. For this, when piston moves more in upward direction, then hot gas is released from the hole provided on upper side

Here gas is called working substance. The flame of the burner is called heat source and the arrangement in which gas is released is called heat sink.

Following figure shows working of the heat engines by line diagram

In the heat engine, the working substance undergoes a cyclic process. For this the working substance absorbs heat Q_1, from the heat source at higher temperature T, out of which a

part of energy is converted to mechanical energy (work W) and remaining heat Q_2 is released into the heat sink.
Hence , the net amount of heat absorbed by the working substance is
$Q = Q_1 - Q_2$
But for a cyclic process, the net heat absorbed by the system is equal to the net work done
$\therefore Q = W$
$Q_1 - Q_2 = W$
In the cyclic process, the ratio of the network (W) obtained during one cycle is called the efficiency (η) of the heat engine. That is

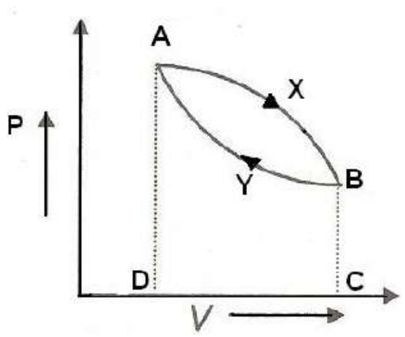
$$\eta = \frac{Net\ work\ obtaine\ per\ cycle}{Heat\ absorbed\ per\ cycle}$$

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$\eta = 1 - \frac{Q_2}{Q_1} \quad ---eq(1)$$

From equation(1) it can be said that if $Q_2 = 0$, then the efficiency of the heat engine is $\eta = 1$. This means that the efficiency of heat engine becomes 100% and total heat supplied to the working substance gets completely converted into work.
In practice, for any engine $Q_2 \neq 0$ means that some heat Q_2 is always wasted hence $\eta<1$

Cyclic process and efficiency calculation



When a system after passing through various intermediate steps returns to its original state, then it is called a cyclic process.
Suppose a gas enclosed in cylinder is expanded from initial stage A to final stage B along path AXB as shown in figure
If W_1 be the work done by the system during expansion, then
W_1 = + Area AXBCDA
Now late the gas be compressed from state B to state along the path BYA, so as to return the system to the initial state. If W_2 be the work done on the system during compression, then W_2 = -Area BYADCB
According to sign convention, work done on the system during compression is negative and the net work done in the cyclic process AXBYA is
W = Area AXBCDA – Area BRADCB = Area AXBYA
Which is a positive quantity and hence net work will be done by the system
So the net amount of work done during a cyclic process is equal to the area enclosed by the cyclic path. It is evident from the figure that if the cyclic path is being traced in anticlockwise direction, the expansion curve will be below the compression curve and net

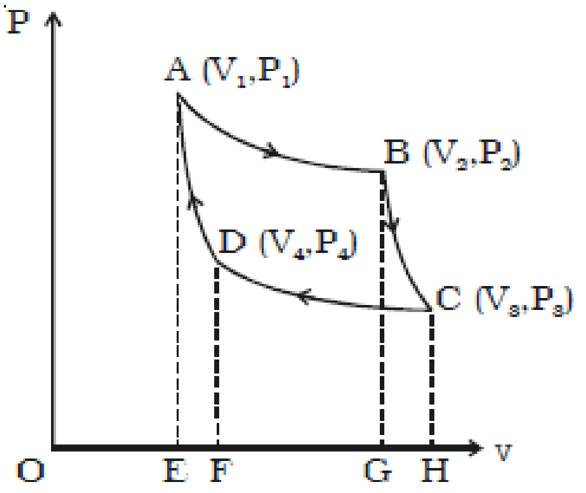
work done during the process will be negative. This implies that the net work will now be done on the system. Applying first law of thermodynamics to cyclic process, we get
$\Delta Q = \Delta U + \Delta W$
But ΔU = for cyclic process
So $\Delta Q = \Delta W$

Solved Numerical

Q) An ideal monoatomic gas is taken round the cycle ABCD where co-ordinates of ABCDP_V diagram are A(p, V), B(2p, V), C(2p, 2V) and D(p, 2V). Calculate work done during the cycle
Solution Area enclosed = pV

Carnot Cycle and Carnot Engine

Carnot engine consists of a cylinder whose sides are perfect insulators of heat except the bottom and a piston sliding without friction. The working substance in the engine is μ mole of a gas at low enough pressure (behaving as an ideal gas). During each cycle of the engine, the working substance absorbs energy as heat from a heat source at constant temperature T_1 and releases energy as heat to a heat sink at a constant lower temperature $T_2 < T_1$.
The cyclic process, shown by P-V graph in figure a, is completed in four stages.



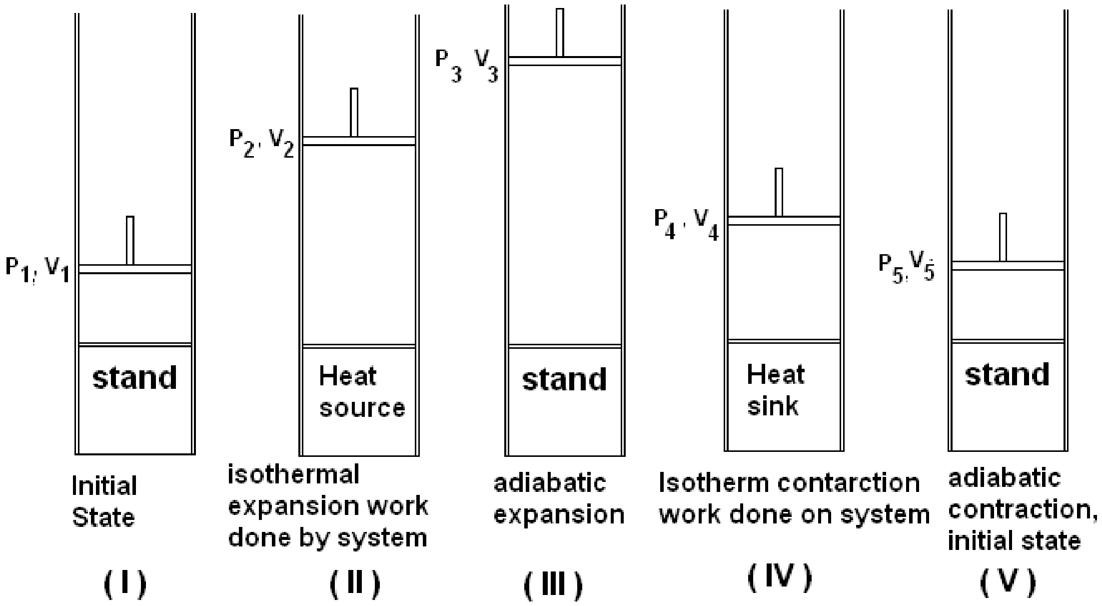
(a)

The Carnot engine and its different stages are shown in figure b

(I) First stage Isothermal expansion of gas from (a → b)
Initial equilibrium state (P_1, V_1, T_1)final equilibrium state (P_2, V_2, T_1) Suppose gas absorbs heat Q_1 during the process. Hence work done is

$$W_1 = Q_1 = \mu RT_1 \ln\left(\frac{V_2}{V_1}\right) \quad --- \text{eq}(1)$$

Further $P_1V_1 = P_2V_2$ ---- eq(2)



(b)

(II) Second stage Adiabatic expansion of gas (b → c)

Now , the cylinder is placed on a thermally insulated stand and the gas is adiabatically expanded to attain the state c (P_3, V_3, T_2).

During this (adiabatic process the gas does not absorb any heat but does work while expanding, so its temperature decreases. For this process

$$P_2V_2^{\gamma} = P_3V_3^{\gamma} \quad ---eq(3)$$

(III) Third Stage: Isothermal compression of gas (c→d)

Now, the cylinder is brought in contact with heat sink at temperature T_2 and isothermally compressed slowly to attain an equilibrium state d (P_4, V_4, T_2). Work done on the gas during this process of isothermal compression is negative as work is done on the gas from state c → d is

$$W_2 = Q_2 = -\mu RT_2 \ln\left(\frac{V_4}{V_1}\right)$$

$$W_2 = Q_2 = \mu RT_2 \ln\left(\frac{V_3}{V_4}\right) \quad ---\mathrm{eq}(4)$$

Here Q_2 is released by the gas into heat sink

Further for isothermal process

$$P_3V_3 = P_4V_4 \quad ---- \text{eq}(5)$$

(IV) Fourth Stage: Adiabatic compression of gas (d →a)

Now, the cylinder is placed on a thermally insulated stand and compressed adiabatically to its original state a (P_1V_1 T_1). This process is adiabatic, therefore, there're is exchange of heat with surrounding, but the work is done on the gas and hence temperature increases from T_2 to T_1

For this adiabatic process

$$P_4V_4^{\gamma} = P_1V_1^{\gamma} \quad ---eq(6)$$

Note that over the whole cycle, the heat absorbed by the gas is Q_1 and the heat given out by the gas is Q_2. Hence the efficiency η of the Carnot engine is

$$\eta = 1 - \frac{Q_2}{Q_1}$$

From equation (1) and (4)

$$\eta = 1 - \frac{\mathrm{T_2}\ \ln\left(\frac{\mathrm{V_3}}{\mathrm{V_4}}\right)}{\mathrm{T_1}\ \ln\left(\frac{\mathrm{V_1}}{\mathrm{V_4}}\right)} \quad ---eq(7)$$

Multiplying equation (2), (3), (5) and 6 we get

$$P_1V_1P_2V_2^{\gamma}P_3V_3P_4V_4^{\gamma} = P_2V_2P_3V_3^{\gamma}P_4V_4P_1V_1^{\gamma}$$

$$\therefore (V_2V_4)^{\gamma-1} = (V_3V_1)^{\gamma-1}$$

$$\therefore V_2V_4 = V_3V_1$$

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

$$\therefore ln\left(\frac{V_2}{V_1}\right) = ln\left(\frac{V_3}{V_4}\right)$$

Using this result in equation (7) We get efficiency of Carnot engine as

$$\eta = 1 - \frac{T_2}{T_1} \quad ---eq(8)$$

Or

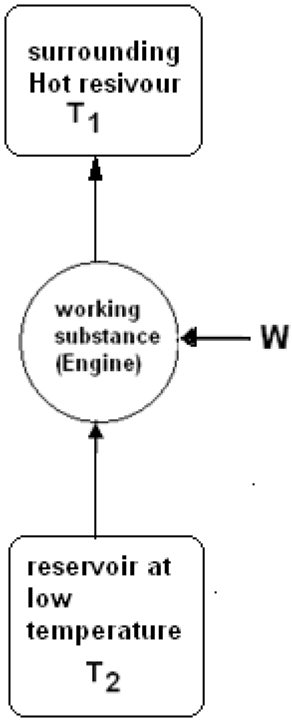
$$\eta = 1 - \frac{low\ temp\ of\ sink}{high\ temperature\ of\ source}$$

Equation (8) shows that the efficiency of the Carnot engine depends only on the temperature of the source and the sink. Its efficiency does not depends on the working substance (if it is ideal gas).

If the temperature of the source (T_1) is infinite or the temperature of the sink (T_2) is absolute zero 9 which is not possible) then only, the efficiency of Carnot engine will be 100%, which is impossible.

Refrigerator / Heat pump and Coefficient of Performance

If the cyclic process performed on the working substance in heat engine is reversed, then the system woks as a refrigerator or heat pump. Figure below shows the block diagram of refrigerator/ heat pump



In the refrigerator, the working substance absorbs heat Q_2 from the cold reservoir at lower temperature T_2, external work W, is performed on the working substance and the working substance releases heat Q_1 into the hot reservoir at higher temperature T_1
The ratio of the heat Q_1 absorbed by the working substance to the work W performed on it, is called the coefficient of performance (α) of the refrigerator. That is

$$\alpha = \frac{Q_2}{W}$$

Here heat is released in surrounding

$Q_1 = W + Q_2$

$Q_1 = W + Q_2$

$W = Q_1 - Q_2$

$$\alpha = \frac{Q_2}{Q_1 - Q_2}$$

Here the value of α can be more than 1 ($\because Q_2 > Q_1 - Q_2$), but it can not be infinite

LESSON 1

ELECTRIC CHARGES AND FIELDS

SECTION I

ELECTRIC CHARGE

Electric charge and its characteristic

(i) ***Electric charges is the fundamental intrinsic property due to which electric force acts***

(ii) Electric charges are two types. Traditionally charge on proton is positive and charge on electron negative. Magnitude of charge on both is same $1.6X10^{-19}C$

(iii) Fundamental charge is the charge on the electron or proton denoted by e.

(iv) Quantization of charges: ***Magnitude of all charges are found to be integral multiple of fundamental charge thus If Q is total charge then Q = ne here n = 1, 2, 3 ,…***

(v) Law of conservation of electric charge state that "The ***algebraic sum of electric charge in an electrically isolated system always remains constant irrespective of any process taking place.*** Or in other words "***In an electrically isolated system only those processes are possible in which charges of equal magnitude and opposite types are either produced or destroyed.*** Example Before rubbing glass rod on silk algebraic sum of charge is zero. On rubbing and separating the rod from silk, we find equal and opposite amount of charge is developed on silk and glass rod.

(vi) Two charges exerts equal and opposite force on each other. Like charges repeals while unlike attracts

(vii) S.I Unit of charge is "coulomb' denoted by C. CGS unit of charge esu

1 coulomb = $3X10^9$esu

(viii) Charge cannot exists without mass though mass can exist without charge

(ix) Charge is invariant: This means that ***charge is independent of frame of reference. i.e. change of the body does not charge with whatever be its speed***

Ways of charging body

(A) Charging by friction:

When two bodies are rubbed together, a transfer of electrons take place from one body to another. The body from which electrons have been transferred is left with an excesses of positive charge, so get positively charged. The body which receives the electrons becomes negatively charged.

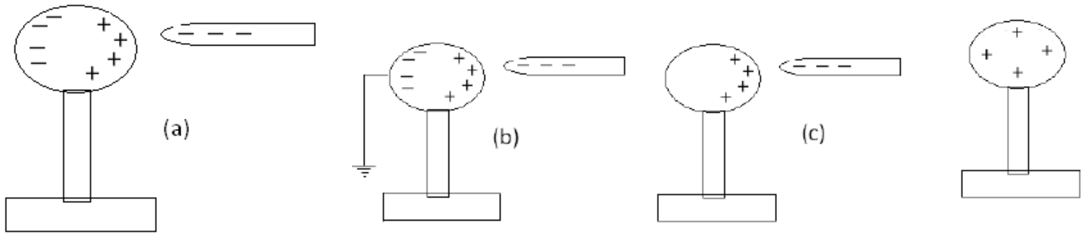
"The positive charge and negative charges produced by rubbing are always equal in magnitude"

When glass rod is rubbed on silk, glass rod loses its electrons and gets positive charges, while silk acquires equal negative charges.

An ebonite or plastic rod acquires a negative charge, if it is rubbed with wool. The piece of wool acquires an equal positive charge

(B) Charging by electrostatic induction

If a negatively charge rod is brought near the conductor mounted on insulated base as free electrons of conducting spheres close to rod experiences a force of repulsion and go to the other part of the sphere as shown in fig a .



Consequently the part of sphere close to rod becomes positively charge due to deficiency of electrons in that region.
As shown in figure b when the sphere is connected to the earth through a conducting wire, the some of the electrons of the spheres will flow to the ground.
As shown in figure c, even if the connection with the earth is removed, the sphere retains the positive charge. When the negatively charged rod is moved away from the sphere, the electrons get redistributed on the sphere such that the same positive charge is spread all over the surface of the sphere as shown in figure d
Important points regarding electrostatic induction
(a) Inducing body neither gains nor loses charges
(b) The nature of induced charge is always opposite to that of inducing charge
(c) Induced charge can be lesser or equal to inducing charge but it is never greater than the inducing charge
(d) Induction takes place only in bodies (either conducting or non conducting) and not particles

(C) Charging by conduction

Let us consider two conductor, one charged and other uncharged. We bring the conductors in contact with each other. The charge under its own repulsion will spread over both the conductors. Thus the conductors will be charged with the same sign. If charged and uncharged conductors are of same size charge will be equally divided if separated after contact.

Solved numerical

Q) A copper penny has a mass of 3.1g. Being electrically neutral, it contains equal amount of positive and negative charges. What is the magnitude of these? A copper atom has a positive nuclear charge of $4.6\text{X}10^{-18}$ C. Atomic weight of copper is 64 g/mole and Avogadro's number is $6\text{X}10^{23}$ atoms/mole
Solution : 1 mole of copper i.e 64 g of copper has $6\text{X}10^{23}$ atoms. Therefore, the number of atoms in copper penny of 3.1 g is

$$\frac{6\times10^{23}}{64}\times 3.1 = 2.9\text{X}10^{22}$$

One atom of copper has each positive and negative charge of $4.6\text{X}10^{-18}$C. So each charge on the penny is $(4.6\text{X}10^{18})\ \text{X}\ (2.9\text{X}10^{22}) = 1.3\text{X}10^{5}$ C

QUESTIONS (A)

Q) How many number of protons of the charge is equivalent to a 1μC?
Q) To identical metal spheres of equal radius are taken one is charged with 100 electrons and other is with 60 protons. When the two spheres are brought in contact and the separated , What will be the charges on each spheres
Q) What will happen when a charged body is placed near an uncharged body.
Q) Can we charge a body having charge $10X10^{-19}$C
Q) How many electrons must be removed from a piece of metal so as to leave it with a positive charge of 10^{-7} coulomb?
Q) A metal sphere is suspended through a nylon thread. When another charged sphere (identical to A) is brought near A and kept at a distance d, a force of repulsion F acts between them. Now A is brought in contact with identical uncharged sphere D and then they are separated from each other. What will be the force between the sphere A and B when they are at a distance d/2 [Ans F]
Q) Write the conservation law of charges
Q) The electric charge of a macroscopic body is either a surplus or deficit of electrons. Why not protons.
Q) How does the force between two point charges, if the dielectric constant of the medium in which they are kept decreases?
Q) Can two bodies, both carrying same type of charge be attracted to each other? [hint yes, if second body have very large charge than to first, explain on the basis of induction]
Q) Write down the value of constant of proportionality $\frac{1}{4\pi\varepsilon_0}$ involved in expression for Coulomb's force between the two charges
Q) Write down the value of absolute permittivity of free space (vacuum), give its dimensional formula
Q) Name the experiment which established quantum nature of electric charge?
[Millikan's oil drop experiment for measurement of electric charge]
Q) Can a body have a charge of $0.8X10^{-19}$C
Q) Why does an ebonite rod get negative charge on rubbing with fur
Q) What does $q_1 + q_2 = 0$ signify in electrostatics?
Q) What is the basic cause of quantization of charge
Q) What is the least possible value of charge
Q) Does the motion of body affect its charge?
Q) What is the cause of charging?
Q) How mass of a body is affected on charging
Q) Making use of conservation of charges, identify the element x in the following nuclear reactions
(i) $H^1 + Be^9 \rightarrow X + n$
(ii) $C^{12} + H^1 \rightarrow X$
(iii) $N^{15} + H^1 \rightarrow He^4 + X$
Q) Ordinary rubber is an insulating. But the special rubber tyres of aircrafts are mage slightly conducting. Why this is necessary

Q) Vehicle carrying inflammable material usually have metallic ropes touching the ground during motion why?

Q) Give two point differences between charge and mass

Q) How you can charge a metal sphere positively without touching it

Coulomb's law :

" Two point charges repel or attract each other with force which is directly proportional to the product of the magnitude of their charges and inversely proportional to the square of the distance between them"

Let 'r' be the distance between two point charges q_1 and q_2 the according to Coulomb's law $F \propto \frac{|q_1||q_2|}{r^2}$ or $F = \frac{K|q_1||q_2|}{r^2}$,

K is proportionality constant . The value of K depends on the medium in which two point charges are placed

In SI system K = $\frac{1}{4\pi\varepsilon_0}$ for vacuum (or air)

The constant ε_0 (=8.85X10^{-12} Nm^{2}C^{-2}) is called "permittivity" of the free space

Value of K = 9X10^9

Permittivity of medium

If medium between the charges is not a vacuum (or air) then

$$\boxed{F = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{|q_1||q_2|}{r^2}}$$

Where ε_r is called ' relative permittivity" of medium , it is a dimension less quantity, it is also called as dielectric constant or specific inductive capacity

Term $\varepsilon = \varepsilon_0 \varepsilon_r$ are called as " absolute permittivity" or "permittivity" of the medium

Solved numerical

Q) The repulsive force between two particles of same mass and charge , separated by a certain distance equal to the weight of one of them. Find the distance between them .

Mass of particle = 1.6X10^{-27} kg, charge on particle = 1.6X10^{-19}C, g = 10 ms^{-2}

Solution: Here repulsive force between two particles = weight of one of particle

$$\frac{Kq_1q_2}{r^2} = mg$$

$$r = \left(\frac{Kq_1q_2}{mg}\right)^{1/2}$$

On substituting values we get r = 1.44X10^{-2}

Q) Two point charges placed at certain distance r in air exerts a force of F on each other. Then if dielectric of constant K is placed between the charges then what should be the thickness of the dielectric slab to get same force F between the charges

Solution

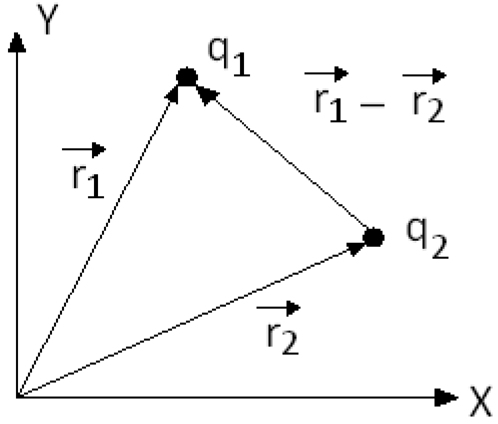
Let r’ be the thickness of dielectrics having dielectric constant k . since force remain same

$$\frac{Kq_1q_2}{r^2} = \frac{Kq_1q_2}{kr'^2}$$

Thus r’ = r/√k

Coulomb’s law in vector form

Force is a vector quantity, so Coulomb’s law can be represented in vector form as follow
Let two charges q_1 and q_2 are like charges (both positive or both negative charges)
Let $\mathbf{r_1}$ and $\mathbf{r_2}$ be the position vector s of the charge q_1 and q_2 .
Let $\mathbf{r_{12}}$ be the vector pointing q_2 to q_1, then displacement vector $\mathbf{r_{12}} = \mathbf{r_1} - \mathbf{r_2}$



According to coulomb’s law, force acting on charge q1 due to charge q2 is

$$\overrightarrow{F_{12}} = \frac{kq_1q_2}{(r_{12})^2}\hat{r}_{12}$$

Where $r_{12} = |\mathbf{r_1} - \mathbf{r_2}|$ is the distance between two charges and $\hat{r}_{12}$ us the unit vector of $\mathbf{r}_{12}$ in direction from q_2 to q_1

$$\hat{r}_{12} = \frac{\overrightarrow{r_1} - \overrightarrow{r_2}}{|\overrightarrow{r_1} - \overrightarrow{r_2}|}$$

∴

$$\overrightarrow{F_{12}} = \frac{kq_1q_2}{(|\overrightarrow{r_1} - \overrightarrow{r_2}|)^2}\frac{\overrightarrow{r_1} - \overrightarrow{r_2}}{|\overrightarrow{r_1} - \overrightarrow{r_2}|}$$

$$\overrightarrow{F_{12}} = \frac{kq_1q_2}{(|\overrightarrow{r_1} - \overrightarrow{r_2}|)^3}\overrightarrow{r_1} - \overrightarrow{r_2} \text{ .....(1)}$$

Above equation is valid for any sign of charge whether positive or negative
If charges are unlike then force will be negative indicating attractive force , if F is positive force is repulsive .
Force on q_2 due to q_1 in vector form can be represented as

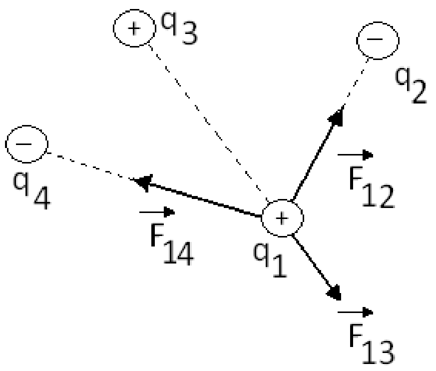
$$\boxed{\overrightarrow{F_{12}} = \frac{kq_1q_2}{(|\overrightarrow{r_2} - \overrightarrow{r_1}|)^3}(\overrightarrow{r_2} - \overrightarrow{r_1})}$$

If we take out negative sign common from $\overrightarrow{r_1} - \overrightarrow{r_2}$ from equation (1), then we get $-(\overrightarrow{r_2} - \overrightarrow{r_1})$. Now $(\overrightarrow{r_2} - \overrightarrow{r_1})$ is a vector pointing from charge q_1 to q_2. Also $(|\overrightarrow{r_2} - \overrightarrow{r_1}|)^3$= $(|\overrightarrow{r_1} - \overrightarrow{r_2}|)^3$ thus from equation (1) and equation (2) we get $\overrightarrow{F_{21}}$ = - $\overrightarrow{F_{12}}$

Thus force of interaction between two bodies is equal and opposite Or Coulomb's law agree with Newton's Third Law

Principle of super position

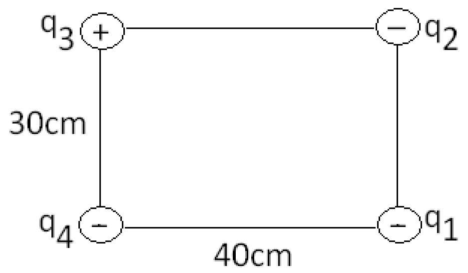
According to the principle of super position, the force acting on one charge due to another is independent of the presence of other charges. So we can calculate the force separately for each pair of charges and then their vector sum or find the net force on any charge.



The figure shows a charge q_1 interacting with other charges. Thus, to find the force on q_1, we first calculate the forces exerted by each of the other charges, one at a time. The net force $\overrightarrow{F_1}$ on q_1 is simply the vector sum $\overrightarrow{F_1}$ = $\overrightarrow{F_{12}}$ + $\overrightarrow{F_{13}}$+$\overrightarrow{F_{14}}$+……….

Where $\overrightarrow{F_{12}}$ is the force on the charge q_1 due to the q_2 and so on

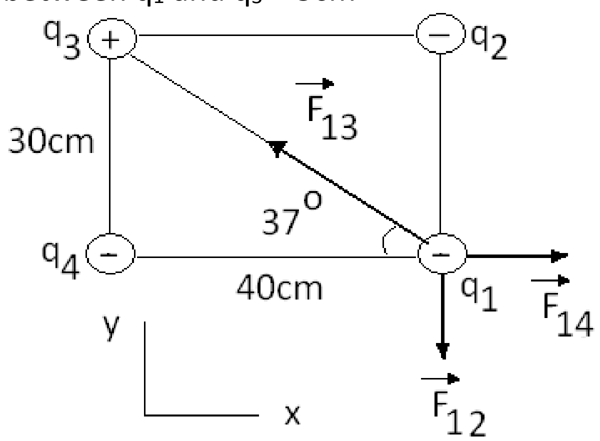
Solved numerical



Q) Find the net force on charge q_1 due to the three other charges as shown in figure. Take q1 = -5μC; q2 = -8 5μC; q3 =15 5μC; and q4 = -16

Solution : Angle between lines joining q_1, q_2 and q_1 ,q_3 is 37^{O}

The direction of the forces on q 1 and coordinate axes are as shown in figure and distance between q_1 and q_3 = 5cm



Now, F_{12}= $9 \times 10^9 \frac{q_1 q_2}{r^2}$

$F_{12} = \frac{9\times10^9\times(5\times10^{-6})(8\times10^{-6})}{(3\times10^{-1})^2} = 4N$

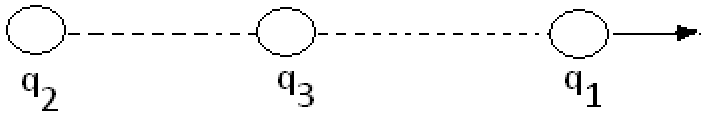
Similarly F_{13} = 2.7 N and F_{14} = 4.5 N

Forces in vector forms are

$\vec{F}_{12} = -4\hat{j}$, $\vec{F}_{13} = 2.7(-cos37\ \hat{\imath} + sin37\hat{j})$, $\vec{F}_{14} = 4.5\hat{\imath}$

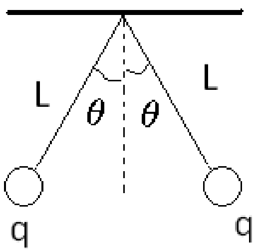
Resultant force on q_1 is vector addition of all the forces
$\vec{F}_1 = -4\hat{j} + 2.7(-cos37\ \hat{\imath} + sin37\hat{j}) + 4.5\hat{\imath}$
$\vec{F}_1 = 2.3\hat{\imath} - 2.38\hat{j}$ N

Q) Three charges lie along the x axis as shown in figure. The positive charge q_1 = 15.0 μC is at c = 2.0 m and the positive charge q2 = 6.0 μC is at the origin. Where a negative charge q_3 must be placed on the x-axis such that the resultant force on it is zero?

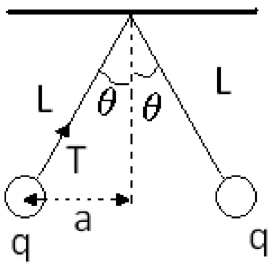


Solution
Since q3 is negative and both q_1 and q_2 are positive, the force $\vec{F}_{31}$ and $\vec{F}_{32}$ are both attractive. Let x be the coordinate of q_3 we have
F31 = $\frac{kq_1q_3}{(2-x)^2}$ and F32 = $\frac{kq_3q_2}{(x)^2}$
Since net force on the charge q_3 is a zero we have
$$\frac{kq_1q_3}{(2-x)^2} = \frac{kq_3q_2}{(x)^2}$$
Or (4 – 4x +x) ($6X10^{-6}$) = x^2 ($15X10^{-6}$)
On solving quadric equation we get x = 0.775 m

Q) Two identical small charged spheres, each having a mass of 3.0X10 -2 kg, hang in equilibrium as shown below, if the length of each string is 0.15 m and the angle θ = 5°, find the magnitude of the charge on each sphere.

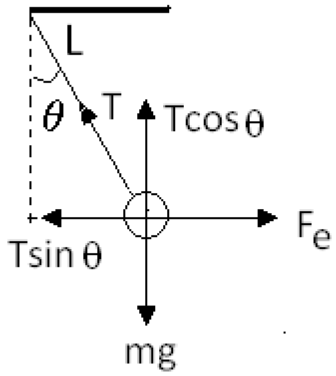


Solution :



From figure a = L sinθ = 0.15sin5 = 00.013m . Hence spheres are at separation 2a = 0.26m

From FBD of one sphere



Since the sphere is in equilibrium, the resultant forces in the horizontal and vertical direction must separately added up to zero, thus

Tsinθ = F_e = 0 –(i)

Tcosθ = mg --(ii)

Dividing equation (i) by (ii) we get

tanθ = F_e/mg or Fe = mg tanθ

F_e =(3X10-2) X(9.8)(tan5)

F_e = 2.6X10 $^{-2}$ N

Let q be charge on each sphere, According to coulomb's law

$F_e = \frac{9\times10^9 q^2}{r^2}$

2.6X10^{-2} = $\frac{9\times10^9 q^2}{(0.026)^2}$

Or q = 4.4X10^{-8}C

Force due to continuous charge distributions

To find the force exerted by a continuous charge distribution on a point charge, we divide the charge into infinitesimal charge element. Each infinitesimal charge element is then considered as a point charge. The magnitude of the force dF exerted by the charge dq on the charge q_0 is given by

$$dF = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{|dq||q_0|}{r^2}$$

Where r is the distance between dq and q_0 . The total force is then found by adding all the infinitesimal force element, which involves integral

Each type of the charge distribution is described in table below by an appropriate Greek-letter parameter λ, σ, ρ

Charge distribution	Relative parameter	SI unit	Charge on element
Along a line	λ, charge per unit length Q/L Q is charge L is length	C/m	dq = λ dx
On surface	σ , charge per unit area Q/A Q is charge , A is area	C/m^2	dq = σ dA

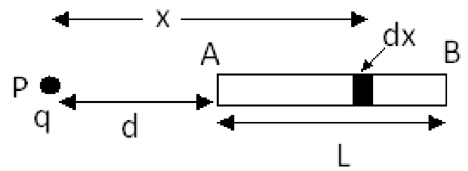
Throughout volume	ρ, charge per unit volume Q/V Q is charge V is volume	C/m^3	dq = ρ dV

Note : charge distribution is continuous but may not be uniform thus charge distribution is function of position

Solved numerical

Q) A point charge is situated at a distance d from one end of a thin non-conducting rod of length L having a charge Q (uniformly distributed along the length) as shown. Find the magnitude of the electric force between the two

Solution



Consider an element of rod of length 'dx' at a distance x from the point charge q. treating the element as point charge, the force between q and the charge element

$$dF = \frac{1}{4\pi\varepsilon_0}\frac{q\,dQ}{x^2}$$

But dQ = $\frac{Q}{L}dx$ so

∴
$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L}\int_d^{(d+L)}\frac{dx}{x^2}$$

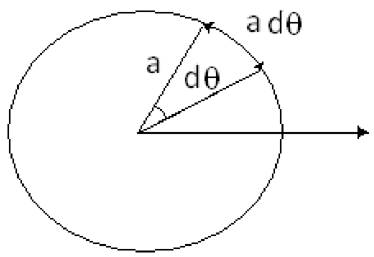
$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L}\left[\frac{-1}{x}\right]_d^{(d+L)}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L}\left[\frac{1}{d} - \frac{1}{d+L}\right]$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{L(d+L)}$$

Q) A circle, having radius 'a' has liner charge distribution over its circumference having linear charge density $\lambda = \lambda_0 \cos^2\theta$. Calculate the total charge on the circumference of the circle

Solution . The length of an infinitesimally small line element have charge dq = λdl .



But dl = a (dθ)

dq = $a\lambda_0\cos^2\theta.\ d\theta$

In order to calculate the total charge Q residing on the surface, we have to integrate dq over the entire surface Q = $\oint dq$ here symbol $\oint$ indicates the integer over the entire close path (circumference of the circle)

Q = $\oint a\lambda_0\cos^2\theta.\ d\theta = a\lambda_0\int_0^{2\pi}\cos^2\theta.\ d\theta = a\lambda_0\pi$

QUESTIONS (B)

Q) Three equal charges each of magnitude of 2.0X10^{-6}C are placed at the three corners of right angled triangle of sides 3cm, 4am 5cm. Find the force on the charge at the right angle corner and direction of force [ANs (-22.5,40)N and tanθ = -1.777]

Q) Two electric charges having magnitude 8.0 μC and -2.0μC are separated by 230cm. Where should a third charge be placed so that the resultant force acting on it is zero
[20 cm from -2.0μC]

Q) Two spheres having same radius and mass re suspended by two strings of equal length from the same point, in such a way that their surfaces touches each other. On depositing 4X10^{-7}C charge on them, they repel each other in such a way that in equilibrium the angle between their string becomes 60^{O}. If the distance from the point of suspension to the centre of the sphere is 20cm, find the mass of each sphere g = 10 m/s^2 [Ans : 1.56X10^{-3}kg]

Q) Three identical charges q are placed on the vertices of n equilateral triangle . Find resultant force acting on the charge 2q kept at its centroid. (distance of centroid from vertices is 1m)

Q) Two identically charged spheres are suspended by stings of equal length. When they are suspended in kerosene, the angle between their strings remains the same as it was in the air. Find the density of the spheres. The dielectric constant of kerosene is 2 and its density is 800 kg m^{-3} [Ans 1600 kg m^{-3}]

Q) If $q_1q_2 > 0$, which type of the force acting between the charges

Q) Two large conducting spheres carrying charges q_1 and q_2 are brought close to each other. Is the magnitude of electrostatic force between them exactly given by

$$F = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{|q_1||q_2|}{r^2}$$

{Hint: No, explain on the basis of induction value of q_1 and q_2 will alter]

Q) State Coulomb's law in electrostatics.

Q) Force between two point electric charges kept at distance d apart in air is F. If these charges are kept at the same distance in water, how does the force between them charge? Given dielectric constant of water = 80

Q) Dielectric constant of water is 80. What is its permittivity

Q) Define unit of electric charge in terms of electric force

Q) State the principle of superposition of forces in electrostatics

Q) Given two point charges q_1 and q_2 such that q_1 q_2 = <0. What is the nature of the force between them?

Q) Consider three charged bodies P, Q and R. If P and Q repel each other and P attracts R, what is the nature of force between Q and R

Q) In Coulombs law, on what factor the value of electrostatic force constant K depends?

Q) Does Coulombs law of electric force obeys Newton's third law of motion?

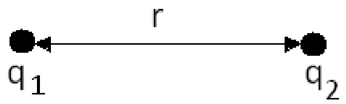
Q) Is the Coulomb force that one charge exerts on other changes if other charges are brought nearby?
Q) Two small balls having equal positive charge q coulomb are suspended by two insulating strings of equal length 1 meter from a hook fixed to a stand. The whole set up is taken in a satellite into space where there is no gravity. What is the angle between the two strings and the tension in each string?
Q) An ebonite rod held in hand can be charged by rubbing with flannel but a copper rod cannot be charged like this why?
Q) What is the importance of Coulomb law in vector form?
Q) Electrostatic forces are much stronger than gravitational forces. Give one example

SECTION II
ELECTRIC FIELD

An electric field is defined as a region in which there should be a force on a charge brought into that region. Whenever a charge is being placed in an electric field, it experiences a force

Electric fields are usually produced by different types of charged bodies, point charges, charged plates, charged spheres etc

If two point charges are placed as shown in figure, we can describe the force on them in two ways



(i) The charge q_2 is in the electric field of charge q_1. Thus the electric field of charge q_1 exerts force on q_2.

(ii) The charge q_1 is in the electric field of charge q_2. Hence the electric field of charge q_2 exerts a force on q_1

Electric field intensity or Electric field Strength ($\vec{E}$)

The electric field intensity at a point in an electric field is the force experienced by a unit positive charge placed at that point, it is being assumed that the unit charge does not affect the field.

Thus, if a positive test charge q_0 experiences a force $\vec{F}$ at a point in an electric field, then the electric field intensity $\vec{E}$ at a point is given by

$$\boxed{\vec{E} = \frac{\vec{F}}{q_0}}$$

<u>Important points regarding electric field intensity</u>

(i) It is vector quantity. The direction of the electric field intensity at a point inside the electric field is the direction in which the electric field exerts force on the unit positive charge.

(ii) Direction of electric field due to positive charge is outward while direction of electric field due to negative charge is inward

(iii) Dimensions of electric field intensity E = $[MLT^{-3}A^{-1}]$

S.I. unit of Electric field is C/C or V/m as

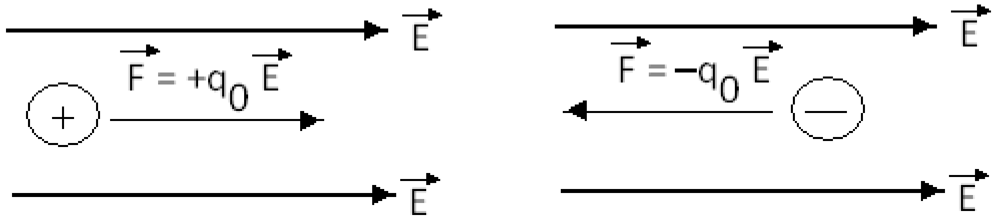
$$\frac{N}{C} = \frac{N \times m}{C \times m} = \frac{J}{C \times m} = \frac{V}{m}$$

Force exerted by a field on a charge inside it

By definition $\vec{E} = \frac{\vec{F}}{q_0}$ or $\vec{F} = q_0\vec{E}$

If q_0 is positive charge force $\vec{F}$ on it is in the direction of $\vec{E}$

If q_0 is negative charge force $\vec{F}$ on it is opposite to the direction of $\vec{E}$



Solved numerical

Q) An electron (q =-e) is placed near a charged body experiences a force in the positive y direction of magnitude 3.6X10^{-8} N

() What is the electric field at that location

(b) What would be the force exerted by the same charged body on an alpha particle (q = +2e) placed at the location initially occupied by the electron

Solution (a)

From equation E = F/q = 3.6X10^{-8}/ 1.6X10^{19} = 2.25X10^{11} N/c

We know electron is negatively charged particle hence will move opposite to the direction of electric field . Given electron experiences for is positive direction, electric field must be in negative direction

(b) The force on alpha particle will be F = Eq = 2.25X10^{11} (2X1.6X10^{19}) = 7.20X10^{-8} N

Since alpha particle is positively charge it will experience force in the direction of electric field

Electric field due to point charge

Let a positive test charge q_0 be placed at a distance r from a point charge q. The magnitude of force acting on q_0 is given by Coulomb's law.

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r^2}$$

The magnitude of the electric field at the site of the charge is

$$\boxed{E = \frac{F}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$

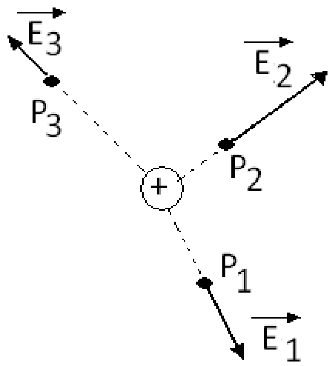
Electric field in vector form is

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}}$$

Here $\vec{r}$ is the position vector of point

The direction of $\vec{E}$ is the same s the direction of $\vec{F}$, along a radial line q, point outward if q is positive and inward is q is negative.

The figure given shows the direction of the electric field $\vec{E}$ at various points near a positive point charge.
Note length of arrow is more where electric field is more and point near to charge have more electric field



Electric field intensity due to a group of point charges

Since the principle of linear superposition is valid for Coulomb's law, it is also valid for the electric field. To calculate the electric field at a point due to a group of N point charges. We find the individual field strength $\vec{E}_1$ due to Q_1, $\vec{E}_2$ due to Q_2 and so on.. The resultant field strength is the vector sum of individual field strengths

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

Electric field at a point depends on the charge and position of point
Consider a system of charges q_1, q_2, q_3 , ……q_n with position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, ---$\vec{r}_n$ relative to some origin O.
Electric field at point P having position vector $\vec{r}$. For this purpose place a very small test charge q_0 at that point and use the superposition principle.
Electric field at point P due to q_1 is given by

$$\vec{E}_1 = \frac{\vec{F}_1}{q_0} = k\frac{q_1}{|\vec{r} - \vec{r}_1|^3}(\vec{r} - \vec{r}_1)$$

Electric field at point P due to q_2 is given by

$$\vec{E}_2 = \frac{\vec{F}_2}{q_0} = k\frac{q_2}{|\vec{r} - \vec{r}_2|^3}(\vec{r} - \vec{r}_2)$$

Electric field at point P due to q_3 is given by

$$\vec{E}_3 = \frac{\vec{F}_3}{q_0} = k\frac{q_3}{|\vec{r} - \vec{r}_3|^3}(\vec{r} - \vec{r}_3)$$

Same way, electric field at point P due to charge q_n is

$$\vec{E}_n = \frac{\vec{F}_n}{q_0} = k\frac{q_n}{|\vec{r} - \vec{r}_n|^3}(\vec{r} - \vec{r}_n)$$

According to superposition principle, net electric field at a point P is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots + \vec{E}_n$$

$$\vec{E} = k\frac{q_1}{|\vec{r} - \vec{r}_1|^3}(\vec{r} - \vec{r}_1) + k\frac{q_2}{|\vec{r} - \vec{r}_2|^3}(\vec{r} - \vec{r}_2) + \cdots + k\frac{q_n}{|\vec{r} - \vec{r}_n|^3}(\vec{r} - \vec{r}_n)$$

$$\vec{E} = k\sum_{j=1}^{n}\frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)$$

Here q_1, q_2, q_3 ,…. Are the sources of electric field

Physical significance of electric field

a) Equation for electric field is given by

$$\vec{E} = k\sum_{j=1}^{n}\frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)$$

Equation of force acting on a unit positive charges at point $\vec{r}_{(x,y.z)}$ once $\vec{E}_{\vec{r}}$ is kown, we do not have to worry about the source of electric field. In this since, the electric field itself is a special representation of the system of charges producing electric field, as far as the effect on other charges are concerned. Once the representation is done, the force acting on charge q kept at that point in the electric field can be determined using following equation

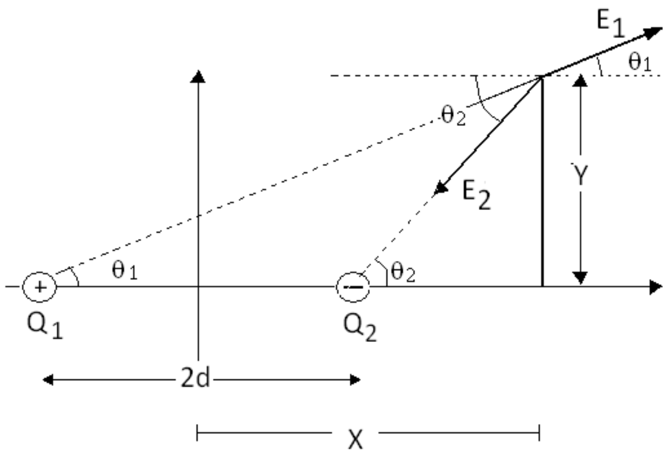
$$\vec{F}_{\vec{r}} = q\vec{E}_{\vec{r}}$$

b) True significance of electric field is when there is an accelerated motion of charge q_1 and q_2. For effect of q_1 and q_2 there will be a time delay between the force on q_2 and the cause (motion of q_1) .

The field picture is this: the accelerated motion of charge q_1 produces electromagnetic waves, which then propagate with the speed of light and reaches q_2 and cause a force on q_2. The notation of field elegantly accounts for the time delay. Thus even though electric and magnetic fields can be detected only by their effect (force) on charge, they are regarded as physical entity. Electric and magnetic field transport energy. Thus, a source of time-dependent electromagnetic fields, turned on briefly and switch off, leaves behind propagating electromagnetic field transporting energy

Solved numerical

Q) A point charge Q_1 = 20μC is at (-d,0) while Q_2 = 10 μC is at (+d, 0). Find the resultant field strength at a point with coordinates (x, y). Take d = 1.0m and x=y=2m



Solution

From figure we have

$r_1 = \sqrt{(x^2+y^2)^2+y^2}$ = √13 = 3.6 m

$r_2 = \sqrt{(x^2-y^2)^2+y^2}$ = √5 = 2.2 m

The magnitudes of the fields are

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{|Q_1|}{r_1^2} = \frac{(9.0\times10^9)(2\times10^{-5})}{13} = 1.385\text{X}10^4 \text{ N/C}$$

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{|Q_2|}{r_2^2} = \frac{(9.0\times10^9)(10^{-5})}{5} = 1.8\text{X}10^4 \text{ N/C}$$

The components of the resultant field strength are

$E_x = E_{1x} + E_{2x} = E_1\cos\theta_1 - E_2\cos\theta_2$

$E_y = E_{1y} + E_{2y} = E_1\sin\theta_1 - E_2\sin\theta_2$

From figure $\sin\theta_1 = y/r_1$; $\sin\theta_2 = y/r_2$; $\cos\theta_1 = (x+d)/r_1$; $\cos\theta_2 = (x-d)/r_2$

Therefore

$$E_x = (1.385\times10^4)\frac{3}{3.6} - (1.8\times10^4)\frac{1.0}{2.2} = 3.32\times10^3 \text{N/C}$$

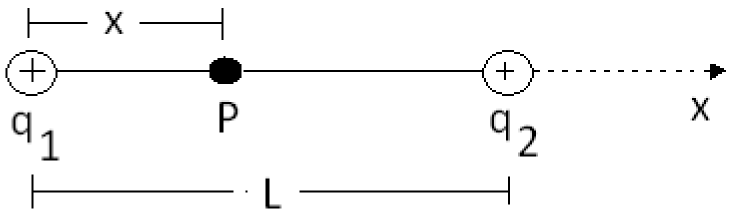
$$E_y = (1.385\times10^4)\frac{2}{3.6} - (1.8\times10^4)\frac{2.0}{2.2} = -8.66\times10^3 \text{N/C}$$

Hence

$$\vec{E} = (3.32\times10^3)\hat{\imath} - (8.66\times10^3)\hat{\jmath} \text{ N/C}$$

Q) A point charge q_1 of +1.5 µC is placed at a origin of a coordinate system, and the second charge q_2 of +2.3 µC is at position x = L, where L = 13 cm. At what point P along the x-axis is the electric field zero

Solution



The point must lie between the charges because only in this region the force exerted by q_1 and q_2 on a test charge oppose each other. If $\vec{E}_1$ is the electric field due to q_1 and $\vec{E}_2$ is that due to q_2, the magnitudes of these vectors must be equal to or $E_1 = E_2$

We have

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{(L-x)^2}$$

Where x is the coordinate of point P

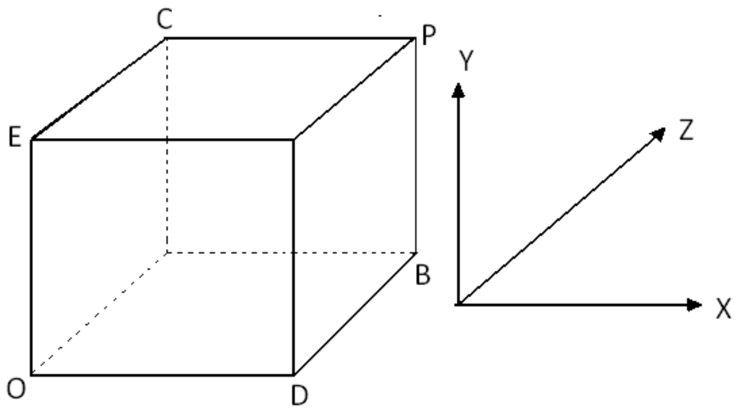
On solving we get $x = \frac{L}{1\mp\sqrt{q_1/q_2}}$ substituting the values we get

X = 5.8 cm and -54.8 cm

But the negative value of x is unacceptable

Hence x = 5.8cm

Q) A cube of edge 'a' carries a point charge q at each corner. Calculate the resultant force on any one of the charges



Solution :

Let us take one corner of the cube as origin O(0, 0, 0) and the opposite corner P as (a,a,a). We will calculate the electric field at P due to the other seven charges at corners.
Expressing the field of a point charge in vector form

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

Coordinates of A = (a,a,0)
Coordinates of B = (a,0,a)
Coordinates of C = (0,a,a)
Coordinates of D = (a,0,0)
Coordinates of E = (0,a,0)
Coordinates of F = (0,0,a)
Electric field strength at P due to charges A, B, C

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}\left[\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP}\right]$$

$\overrightarrow{AP}$ = P - A = (a,a,a) – (a,a,0) = (0,0,a) and $|\overrightarrow{AP}|$ = a
$\overrightarrow{BP}$ = P-B = (a,a,a) – (a,0,a) = (0,a,0) and $|\overrightarrow{BP}|$ = a
$\overrightarrow{CP}$ = P-C= (a,a,a) – (0,a,a) = (a,0,0) and $|\overrightarrow{CP}|$ = a

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}\left[\hat{\imath} + \hat{\jmath} + \hat{k}\right]$$

Electric field strength at P due to charges at D, E , F

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}\left[\overrightarrow{DP} + \overrightarrow{EP} + \overrightarrow{FP}\right]$$

$\overrightarrow{DP}$ = (a,a,a) – (a,0,0) = (0,a,a) and $|\overrightarrow{DP}|$ = a√2
$\overrightarrow{EP}$ = (a,a,a) – (0,a,0) = (a,0,a) and $|\overrightarrow{EP}|$ = a√2
$\overrightarrow{FP}$ = (a,a,a) – (0,0,a) = (a,a,0) and $|\overrightarrow{EP}|$ = a√2

$$\left[\overrightarrow{DP} + \overrightarrow{EP} + \overrightarrow{FP}\right] = (0,\mathrm{a},\mathrm{a}) + (\mathrm{a},0,\mathrm{a}) + (\mathrm{a},\mathrm{a},0)$$

$$\left[\overrightarrow{DP} + \overrightarrow{EP} + \overrightarrow{FP}\right] = (2\mathrm{a},2\mathrm{a},2\mathrm{a})$$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(a\sqrt{2}\right)^3}\left[2a\hat{\imath} + 2a\hat{\jmath} + 2a\hat{k}\right]$$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\sqrt{2}a^2\right)}\left[\hat{\imath} + \hat{\jmath} + \hat{k}\right]$$

Electric field strength at point P due to O

$\vec{E}_3 = \frac{1}{4\pi\varepsilon_0}\frac{q}{(a\sqrt{3})^3}\left[\overrightarrow{OP}\right]$ (We have OP = a√3]

$\overrightarrow{OP}$ = = (a,a,a) – (0,0,0) =(a,a,a)

Hence resultant electric field at P

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$\vec{E} = \frac{q[\hat{\imath}+\hat{\jmath}+\hat{k}]}{4\pi\varepsilon_0 a^2}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{3\sqrt{3}}\right]$ outward along OP

Force on the charge at P is F = qE

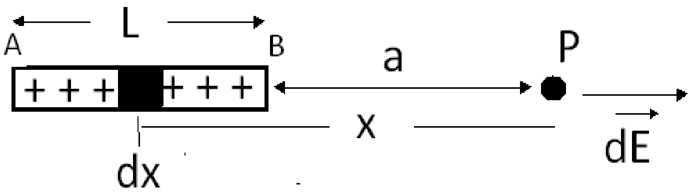
$\vec{F} = \frac{q^2[\hat{\imath}+\hat{\jmath}+\hat{k}]}{4\pi\varepsilon_0 a^2}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{3\sqrt{3}}\right]$ outward along OP

Magnitude

$F\ \frac{q\sqrt{3}}{4\pi\varepsilon_0 a^2}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{3\sqrt{3}}\right]$ outward along OP

Q) A thin rod of length L carries a uniformly distributed charge Q. Find the electric field strength at a point along its axis at a distance ‘a’ from one end

Solution:



Let us consider an infinitesimal element of length d at a distance x from the point P. The charge on this element is dq = λdx. Where λ = Q/L is the linear charge density.

The magnitude of the electric field at P due to this element is

$\mathrm{dE} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{x^2}$ and its direction is to the right since λ is positive. The total electric field strength E is given by

$$\mathrm{E} = \frac{\lambda}{4\pi\varepsilon_0}\int_a^{a+L}\frac{dx}{x^2}$$

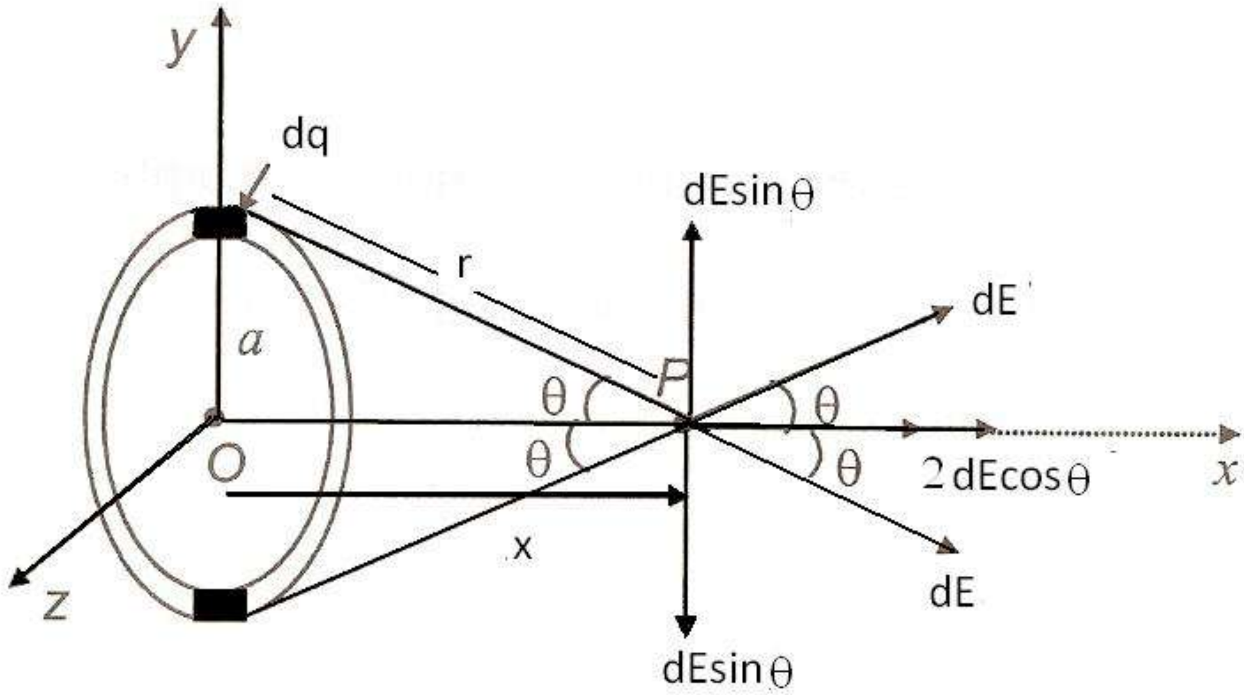
$$\mathrm{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{-1}{x}\right]_a^{a+L}$$

$$\mathrm{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+L}\right]$$

But Q = λL

$$\mathrm{E} = \frac{Q}{(4\pi\varepsilon_0)a(a+L)}$$

Electric field due to a uniformly charged ring at a point on the axis of the ring



Let us consider a charge Q distributed uniformly on a thin, circular, non-conducting ring of radius a. We have to find electric field E at the point P on the axis of the ring, at a distance x from the centre.
From symmetry we observe that every element dq be paired with a similar element on the opposite side of the ring. Every component dEsinθ perpendicular to the x-axis is thus cancelled by a component dEsinθ in the opposite direction. In a summation process, all the perpendicular components add to zero. Thus we only add dE_X components

Now $r^2 = a^2 + x^2$ and $\cos\theta = \frac{x}{\sqrt{a^2+x^2}}$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(d^2+x^2)}$$

Hence, the resultant electric field at P is given by

$$E = \int dE_X = \int dE cos\theta$$

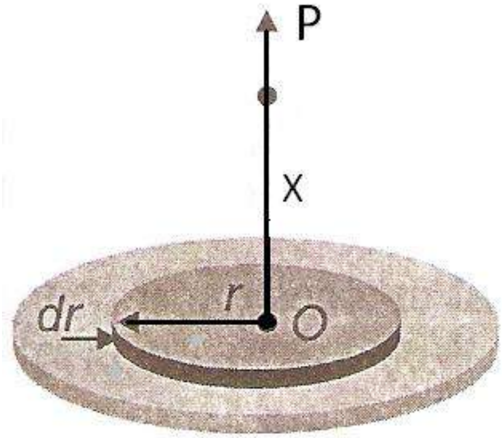
$$E = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(d^2+x^2)}\left(\frac{x}{\sqrt{a^2+x^2}}\right)$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{x}{(d^2+x^2)^{3/2}}\int dq$$

As we integrate around the ring, all the terms remain constant and $\int dq = q$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{xQ}{(d^2+x^2)^{3/2}}}$$

Electric field due to a uniformly charged disc at a point on the axis of the disc



Let us consider a flat, circular, non-conducting thin disc of radius *R* having a uniform surface charge density σ C/m^2. We have to find the electric field intensity at an axial point at a distance *x* from the disc.

Let O be the centre of a uniformly charged disc of radius R and surface charge density σ. Let P be an axial point, distance x from O, at which electric field intensity is required.

From the circular symmetry of the disc, we imagine the disc to be made up of large number of concentric rings and consider one such ring of radius 'r' and infinitesimally small width dr.

The area of the elemental ring = Circumference X width = (2πr dr)

The charge dq on the elemental ring = (2πr dr) σ

Therefore, the electric field intensity at P due to the elementary ring is given by

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{x(2\pi \mathrm{r\ dr})\ \sigma}{(r^2+x^2)^{3/2}}$$

And is directed along x –axis. Hence, the electric intensity E due to the whole disc is give by

$$E = \frac{x\sigma}{2\varepsilon_0}\int_0^R \frac{\mathrm{r\ dr}}{(r^2+x^2)^{3/2}}$$

$$E = \frac{x\sigma}{2\varepsilon_0}\left[\frac{-1}{(r^2+x^2)^{1/2}}\right]_0^R$$

$$E = \frac{x\sigma}{2\varepsilon_0}\left[\frac{-1}{(R^2+x^2)^{1/2}}+\frac{1}{x}\right]$$

$$E = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{1}{(R^2+x^2)^{1/2}}\right]$$

Electric lines of Forces

The properties of electric lens of forces are the following

(i) The electric lines of force are continuous curves in an electric field starting from a positively charged body and ending on a negatively charged body.

(ii) The tangent to the curve at any point gives the direction of the electric field intensity at that point

(iii) Electric field lines of forces do not pass but leave or end on a charged conductor normally.

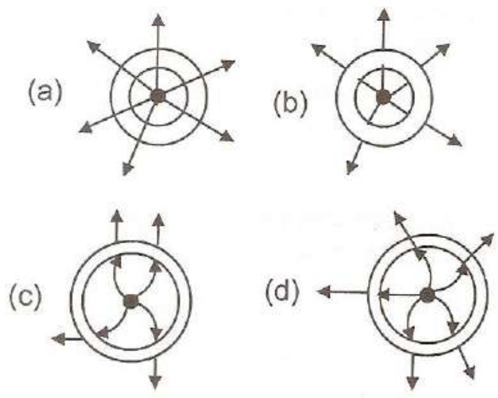
Suppose the line of forces is not perpendicular to the conductor surface. In this situation, the component of electric field parallel to the surface cause the electrons to move and hence conductor will not remain equipotential which is absurd as electrostatics conductor is an equipotential surface.

(iv) Electric field line of forces never intersects since if they cross at a point, electric field intensity at that point will have two directions, which is not possible.

(v) The number of electric lines of force that originate from or terminate on a charge is proportional to the magnitude of the charge.

(vi) As number of lines of force per unit area normal to the area at point represents magnitude of intensity, crowded lines represent strong field while distant lines weak field. Further, if the lines of force are equidistant straight lines, the field is uniform.

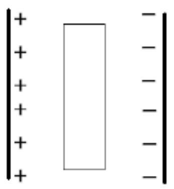
Solved numerical



Q) A metallic shell has a point charge 'q' kept inside its cavity. Which one of the following diagrams correctly represents the electric field lines of force

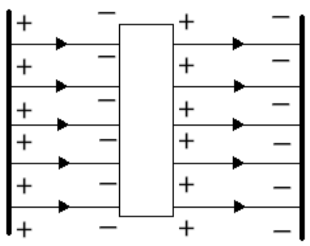
Solution:
Option B is correct because it satisfy following property (i) Electric field lines due to point charge is radial (ii) Electric field lines of forces do not pass but leave or end on a charged conductor normally.

Q) A metallic slab is introduced between the two charged parallel plates as shown in figure. Sketch the electric field lines of force between the plates



Solution: Keeping in mind that

(i) Electric lines of force starts from positive charge and end on negative charge



(ii) Electric lines of force start and end normally on the surface of a conductor

(iii) Electric lines of force do not exist inside a conductor.

QUESTIONS (C)

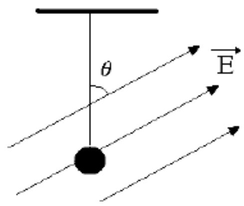
Q) A thin glass rod is bent into a semicircle of radius r. A charge +q is distributed over it. Find the electric field E at a centre of the semicircle

Q) A charged $+10^{-9}$C is located at the origin of a Cartesian co-ordinate system and another charge Q at (2, 0, 0) m. IF X-component of electric field at (3,1,1)m is zero calculate the value of Q [Ans -0.43×10^{-9}C]

Q) Four electric charges +q, +q, -q and –q are respectively placed on the vertices A, B, C and D of a square. The length of the square is 'a', calculate the intensity of the resultant electric field at the centre[Ans E = $\frac{4\sqrt{2}kq}{a^2}$
Q) Four particles. each of charge q is placed on the four vertices of a regular pentagon. The distance of each corner from the centre is 'a'. Find the electric field at the centre of the pentagon [E = kq/a^2]

Q) An electron falls through a distance of 1.5cm in a space, devoid of gravity, having uniform electric field of intensity $2.0X10^4$ N/C. The direction of electric field intensity is then reversed keeping its magnitude same, in which a proton falls through the same distance. Calculate the time taken by both of them $m_e = 9.1X10^{-31}$kg and $m_P = 1.7X10^{-27}$kg and $e = 1.6X10^{-19}$C

Q) An arc of radius r subtends an angle θ at the centre with the x-axis in a Cartesian coordinate system. A charge is distributed over the arc such that the linear charge density is λ. Calculate the electric field at the region. Ans $E = \frac{K\lambda}{r}[(-sin\theta)\hat{\imath} + (cos\theta - 1)\hat{\jmath}]$
Q) A simple pendulum is suspended in a uniform electric field $\vec{E}$ as shown in figure. What will be its period if its length is l? Charge on the bob of pendulum is q and mass m



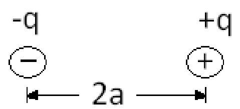
Ans. $T = 2\pi\sqrt{\frac{l}{\left((g)^2+\frac{q^2E^2}{m^2}-\frac{2gqE}{m}cos\theta\right)^{1/2}}}$

Q) What is test charge? What should be its magnitude
Q) A charged particle is fired with velocity $\vec{v}$ making a certain angle with an electric line of force. Will the charged particle move along the line of force?
Q) A small test charge is relased at rest at a apoint in an electrostatic field configuration. Will it travel along the line of force passing through that point?
Q) What is the relation between electric field strength and force?
Q) Draw line of force to represent a uniform electric field
Q) Define electric field at a point
Q) What electric field lines represent?
Q) Define dielectric constant of a medium in terms of force between two electric harges
Q) What is the dielectric constant of metal ? [Ans infinity]
Q) Name the physical quanity whose SI unit is N/C
Q) Four charges of same magnitude and same sign are placed at the corners of a square, of each side 0.1m. What is electric field intensity at the centre of the square?
Q) Name any for vector field? [Ans. Electric , Magnetic , gravitational, flow fild of flude]
Q) Force experenced by an electron in an electric field $\vec{E}$ is F newton. What will be the force experienced by a proton in the same field? Take mass of proton to be 1836 times the mass of electron

Q) Define electric field intensity at a point
Q) Two point charges of +3μC each are 100cm apart. At what point on the line joining the charges will the electric intensity be zero?
Q) How does a free electron at rest move in an electric field?
Q) A plastic comb run through one's hair attracts small bits of paper. Why? What happens if hair is wet or if it is raining?
Q A charge dparticle is free to move in an electric field. Will it always move along an electric line of force?
Q) Give two propertties of electric field lines of force. Sketch them for an isolated positive charge.
Q) Sketch the electric field lines of force due to point charge +q and –q
Q) Two point charges of unkown magnitued and sign are palced some distance apart. The electric field intensity is zero at a point not between the charges but on the line joining them. Write two essential condition for this to happen
Q) What is an electric line of force? What is its importance?
Q) A block of mass m carrying charge q is placed on a frictionless horizontal surface. The block is connected to a rigid wall through an unstressed spring of spring constant k. A horizontal uniform electric field E parallel to the spring is swithched on,. Find the amplitude of the sresulting simple harmonic motion of the block [ANs a = qE/k]
Q) A metal sphere is held fixed on a smooth horizontal insulated plate and another metal sphere is palced some distance away. If the fixed sphere is given a charge , how will other sphere react?

Electric Dipole

A system of equal and opposite charges, separated by a finite distance is called as an electric dipole.



As shown in figure, the two electric charges of electric dipole are +q and –q and distance between them is 2a. Electric dipole moment ($\vec{P}$) of the system can be defined as follows

$$\vec{P} = q(2\vec{a})$$

Importatnt points regarding electric dipole
(i) The SI unit of electric dipole is coulomb metre (C m)
(ii) Electric dipole is a vector quantity and its direction is from negative charge (-q) to positive charge (+q)
(iii)The net electric charge on an electric dipole is zero but its electric field is not zero, since the position of the two charges is different .
If lim q→ ∞ and a→ 0 in $\vec{P} = q(2\vec{a})$, then electric dipole is called point dipole.

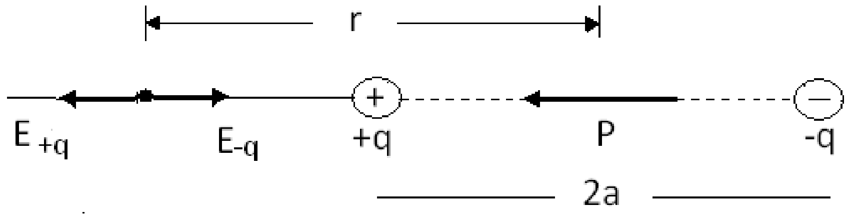
Electric field of a dipole

The electric field of the pair of charges (-q and +q) at any point in space can be found out from Coulombs law and the superposition principle.

The resultants are simple for the following two cases:
(i) When the point is on the dipole axis
(ii) When it is in the equatorial plane

(i) Electric field due to dipole for point on the axis

Let the point P be at distance r from the centre of the dipole on the side of the charge q as shown in figure, then



Elctric field due to negative charge at point P is given by

$$\vec{E}_{-q} = -\frac{q}{4\pi\varepsilon_0 (r+a)^2}\hat{p}$$

where $\hat{p}$ is the unit vector along the dipole axis (from –q to +q)
Electric field at point P due to positive charge is given by

$$\vec{E}_{+q} = \frac{q}{4\pi\varepsilon_0 (r-a)^2}\hat{p}$$

Total electric field at P $\vec{E} = \vec{E}_{+q} + \vec{E}_{-q}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right]\hat{p} = \frac{q}{4\pi\varepsilon_0}\left[\frac{4ar}{(r^2-a^2)^2}\right]\hat{p}$$

But 2qa = P

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2pr}{(r^2-a^2)^2}\right]\hat{p}}$$

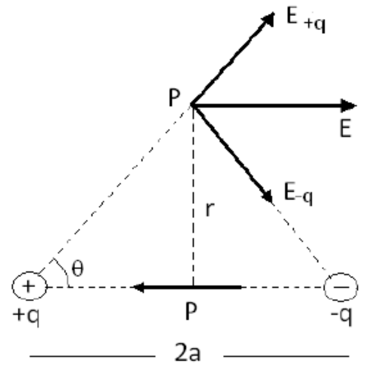
For r >> a

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2p}{r^3}\right]\hat{p}}$$

Direction of electric field is in the direction of electric dipole moment

(ii) Electric field due to dipole for point on the equatorial plane

Let the point P be at distance r from the centre of the dipole on the equatorial plane as shown in figure. Then



Magnitude of electric field at point P due to positive charge is given by

$$E_{+q} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2+a^2}$$

Magnitude of electric field at point P due to negative charge is given by

$$E_{-q} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2+a^2}$$

Both magnitude are equal . Directions of $\vec{E}_{-q}$ and $\vec{E}_{+q}$ are as shown in figure .

Clearly componants perpendicular to axis cancel away. The componant along the dipole axis add up. The total electric field is opposite to $\hat{p}$. We have

$\vec{E} = -(E_{+q} + E_{-q})cos\theta\hat{p}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{2}{r^2+a^2}cos\theta\hat{p}$$

From figure

$$cos\theta = \frac{a}{\sqrt{r^2+a^2}}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{2}{r^2+a^2}\frac{a}{\sqrt{r^2+a^2}}\hat{p}$$

$$\boxed{\vec{E} = \frac{2qa}{4\pi\varepsilon_0(r^2+a^2)^{3/2}}\hat{p}}$$

At large distance (r>>a), this reduces to

$$\vec{E} = \frac{2qa}{4\pi\varepsilon_0(r)^3}\hat{p}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{p}{(r)^3}\right]\hat{p}}$$

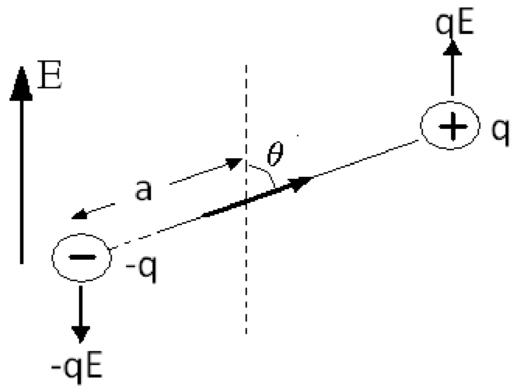
Important points

(i) The electric field at large distances falls of as $1/r^3$

(ii) The magnitude and direction of the dipole field depends not only on the distance r but also on the angle between the position vector $\vec{r}$ and dipole moment $\vec{p}$

(iii) Electric field of dipole is cylindrically symmetrical.

Dipole in uniform external electric field



As shown in figure an electric dipole of magnitude p = q(2a) is kept in a uniform electric field . Let θ be the angle between dipole momentum $\vec{p}$ and electric field $\vec{E}$.

The force q$\vec{E}$ and -q$\vec{E}$ are acting on the charges q and –q respectively. These forces are equal but opposite in direction.

The resultant force being zero keeps dipole in translational equilibrium. But, the two forces have different line of action, hence the dipole will experience a torque.

When the net force is zero, the torque (couple) is independent of the origin. Its magnitude is equal to the magnitude of the force multiplied by the arm of the couple (perpendicular distance between the two antiparallel forces)

Magnitude of torque = qEX2asinθ = 2qaEsinθ but $q(2a) = p$

Thus $\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$

Direction of torque is perpendicular to paper, coming out.
The torque rotates the dipole in such a way that the angle θ reduces,
(i) when the dipole aligns itself along the direction of electric field, the torque becomes zero. This is the normal position of dipole in electric field.
(ii)If the dipole is to be rotated by an angle θ from this position, work is required to be done against torque. This work is equal to the change in potential energy of dipole.

Behaviour of electric dipole in non-uniform electric field

If the electric field is not uniform, the intensity of electric field will be different at different points as a result electric force acting on the positive charge and negative charge of the dipole will also be different. In this situation the net force and torque are acting on the dipole. As a result dipole experiences a linear displacement in addition to roatation. This rotation will continue only till the dipole aligns in the direction of the electric field. But linear motion will continue.
When charged comb is brough near the piece of parer, the non-uniform electric field is produced by the comb. Electric dipole is induced along the direction of non-uniform eletric field in small piece of paper. Now non uniform electric field exters a net force on piece of paer and paper moves in the direction of comb.

QUESTIONS (D)

Q) Define the elctric dipole moment and give SI unit
Q) What will be the torque acting on the dipole, if ut is palced parallel to the electric field.
Q) Why bits of paper gets attarcted to charged comb
Q) Why does ythe two electric field lines do not intersecting each other?
Q) Draw the electric field lines of electric dipole
Q) What is the direction of elecric dipole moment vector of an electric field
Q) What is an ideal dipole
Q) Define electricdipole
Q) Two equal and opposite point charges are separeated by a certain distance. What are the points at which the resultant electric field is parallel to the line joining two charges
Q) How does the torque affact the dipole in an electric field
Q) What is the net force on an elelectric dipole palced in a uniform electric field?
Q) Is torque on the an electric dipole a vector or scalar?
Q) Give SI unit of electric moment of dipole
Q) When is the elelctric dipole in ustable equilibrium
Q) What is meant by the statement that the electric field of a point charge has spherical symmetry whereas that of an electric dipole is cylindrically symmetrical?
Q) What is the orientation of an electric dipole ina uniform electric field corresponding to stable equilibrium?

Flux of an electric field or Electric flux

Let us consider a plane surface of area S placed in an electric fied $\vec{E}$.

Electric flux through an elementary area $\overrightarrow{dS}$ is defind as the scalar product of $\overrightarrow{dS}$ and $\vec{E}$ .i.e

d ϕ_E = $\vec{E} \cdot \overrightarrow{dS}$, where $\overrightarrow{dS}$ is the area vector, whose magnitude is the area dS of the element and whose direction is along the outward normal to the elementary area.

Hence, the electric flux through the entire surface is given by

$\phi_E = \int \vec{E} \cdot \overrightarrow{dS}$ or $\phi_E = \int E\, dS \cos\theta$

θ is the angle between area vector and electic field

Area vector is always outward for closed surface. While for other surface it can be considered as inward outward.

If the electric field is uniform, then $\phi_E = E \cos\theta \int dS$

When the electric flux through a closed surface is required, we use a small circular sign on the integration symbol

$\phi_E = \oint \vec{E} \cdot \overrightarrow{dS}$

Thus general defination of electic flux can be given as " The flux linked with any surface is the surface integration of the electric field over the given surface"

Unit of flux N-m/C

Importatnt points regarding electric flux

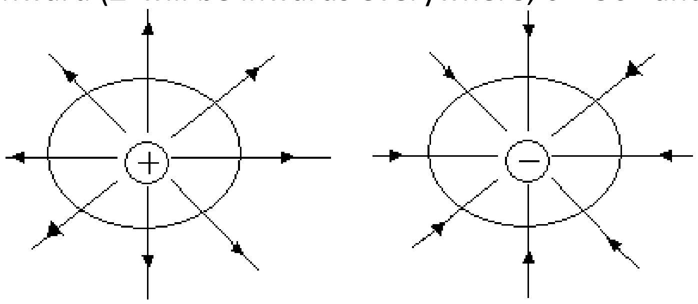
(i) The number of lines of force passing normally to the given area gives the measure of flux of electric field over the given area.

(ii) It is a real scalar physical quantity with units (volt xm)

(iii) It will be maximum when cosθ = max = 1. i.e θ = 0^O, i.e. electric field is normal to the surface with $(d\phi_E)_{max} = E(dS)$

(iv) It will be minimum when |cosθ| = min = 0 , θ = 90^O i.e. field parallel the area with $(d\phi_E)_{min} = 0$

(v) For closed surface, ϕ_E is positive if the lines of force point outward everywhere ($\vec{E}$ will be outward everywhere, θ < 90^O and $\vec{E} \cdot \overrightarrow{dS}$ will be positive) and negative if they point inward ($\vec{E}$ will be inwards everywhere, θ > 90^O and $\vec{E} \cdot \overrightarrow{dS}$ will be negative)



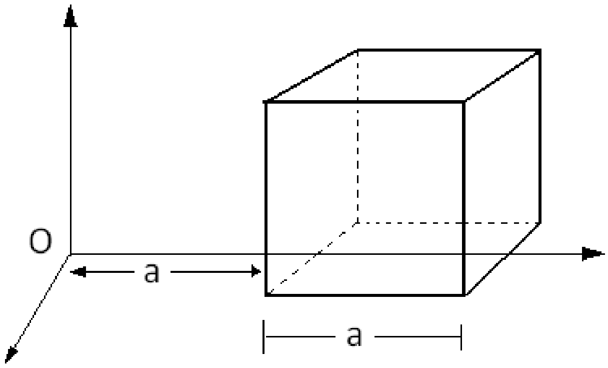
Positive flux $\phi > 0$ Negative flux $\phi < 0$

Solved numerical

Q) In a region of space the electric field is given by $\vec{E} = 8\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$. Calculate the eletrci flux through a csurface of area 100 units in x-y plane

Solution: Since surface is in x-y plane area vector is along z direction thus $\vec{S}$=100 $\hat{k}$.
Electric flux ϕ_E = $\vec{E}.\vec{S}$ = $(8\hat{\imath} + 4\hat{\jmath} + 3\hat{k})(100\hat{k}.)$ = 300 units

Q) Calculate the electric flux through a cube of side 'a' as shown, where $E_x = bx^{1/2}$, $E_Y = E_z$, =0 , a = 10cm and b = 800 N/C-m$^{1/2}$

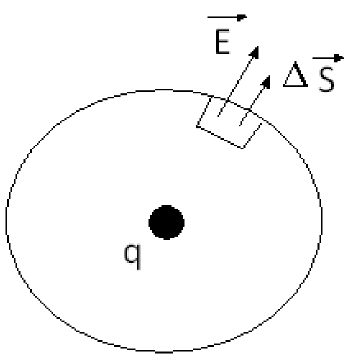


Solution:
The electric field throughout the region is no-uniform and its x-component is given by
$E_x = bx^{1/2}$.
Now for the left face peroppendicular to x-axis, we have x = a = 10cm,
Electric field for left face $E_X = 800(10)^{1/2}$
Hence flux for left face = $-E_X(a^2)$ Here sign is negative as flux is negative

while for the right face x = 2a = 20 cm
Electric field for left face $E'_X = 800(20)^{1/2}$
Hence flux for left face = $E'_X(a)^2$ Here sign is positive as flux is positive
Flux through remaining faces is zero as electric field is zero
thus total flux passing through a cube = $E'_X(a)^2 - E_X(a^2)$ = $a^2 [E'_X - E_X]$= $[(20)^{1/2}-(10)^{1/2}]$ $(10X10^{-2})^2$ = 1.05 N-m/C

Q) A charge q is placed at the centre of a sphere. Find the flux of the electric field through the surface of the sphere due to the enclosed charge

Solution



Let us take a small element ΔS on the surface of the sphere. The electric field here is radially outward and has the magnitude Kq/r^2 , where r is the radius of the sphere
The electric flux through this element is
$\Delta \phi_E = \vec{E}.\Delta \vec{S} = \frac{Kq}{r^2}\Delta S$ (as $\theta = 0^O$)
Hence electric flux through entire sphere is given by
$\phi_E = \sum\Delta \phi_E = \frac{q}{4\pi\varepsilon_0}\sum \Delta S$

$\phi_E = \frac{q}{4\pi\varepsilon_0}(4\pi r^2) = \frac{q}{\varepsilon_0}$

Gauss's Law

This law gives a relation between the electric flux through any closed hypothetical surface (called gaussian surface) and the charge enclosed by the surface. It states ***"The electric flux (ϕ_E) through any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the /net' charge enclosed by the surface"*** That is

$\phi_E = \oint \vec{E} \,.\, \overrightarrow{dS} = \frac{\Sigma q}{\varepsilon_0}$ --(i)

Where $\sum q$ denotes the algebric sum of all the charges enclosed by the surface.

If there are several charges $+q_1$, $+q_2$, $+q_3$, $-q_4$, $-q_5$ …etc inside the Gaussian surface then

$$\sum q = q_1 + q_2 + q_3 - q_4 - q_5 \dots..$$

It is clear form above equation that flux inked with a closed body is independent of the sphape and size of the body and position of charge inside it

The law implies that the total elecric flux through a closed surface is zero, if no net charge is enclosed by the surface

Important points regarding law

(i) Gaus's law is true for any closed surface, no matter what its shape or size

(ii) The term q on the right side of the eqation , include the sum of all charges enclosed by the surface. The charges may be located any where inside the surface

(iii) The electric field appearing on the left hand side of the equation(1) is the electric field produced due to a system of charges, whether enclosed by the surface or outside it

(iv) The surface that we choose for application of Gauss's law is called Gaussian surface

(v) Gauss law is useful towards a much easier calculation of electric field when system has some symmetry.

Applications of Gauss's Law

Gauss's law is useful when there is symmetry in the charge distribution, as in the case of uniformly charged sphere, long cylinders and flat surface over which the surface integral gives by equation (1) can be easily evaluated

These are steps to apply the Gauss's law

(i) Use symmetry of the charge distribution to determine the pattern of the lines

(ii) Choose a Gaussian surface for which $\vec{E}$ is either parallel to $\overrightarrow{dS}$ or perpendicular to $\overrightarrow{dS}$

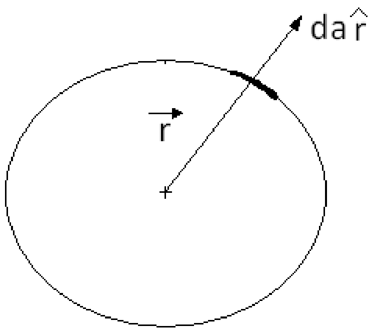
(iii) If $\vec{E}$ is parallel to $\overrightarrow{dS}$, then the magnitude of $\vec{E}$ should be constant over this part of the surface.

The integral then reduces to sum over area elements.

Solved numerical

Q) An electric field prevailing in a region depends on x and y co-ordinates according to an equation $\vec{E} = b\,\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ where b is a constant. Find the flux passing through a sphere of radius r whose centre is on the origin of the co-ordinate system.

Solution



As shown in figure, $\hat{r}$ is the unit vector in the direction of $\vec{r}$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}$$

Now $\vec{E} = b\,\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$

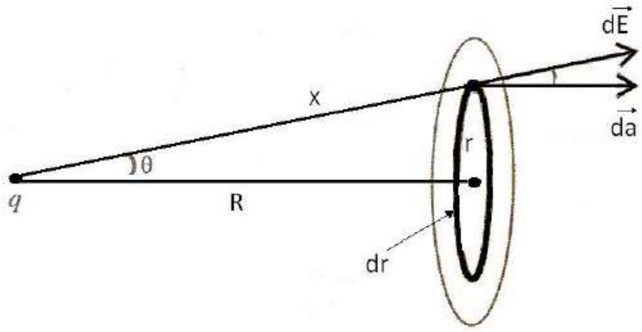
$$\therefore \vec{E}.\overrightarrow{da} = b\left(\frac{x\hat{i}+y\hat{j}}{x^2+y^2}\right).\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}da$$

$$\vec{E}.\overrightarrow{da} = \frac{b\ da}{r}\frac{x^2+y^2}{x^2+y^2} = \frac{b}{r}da$$

$$\int \vec{E}.\overrightarrow{da} = \frac{b}{r}\int da = \frac{b}{r}4\pi r^2 = 4\pi r$$

Q) Calculate the total electric flux linked with a circular disc of radius a. situated at a distance R from a point charge q

Solution



Consider a thin circular ring of radius r and dr as shown in figure. The electric field intensity at some point P on ring is given by.

$$\left|\overrightarrow{dE}\right| = \frac{Kq}{x^2}$$

The area of the ring is $\left|\overrightarrow{da}\right| = 2\pi r dr$

$\overrightarrow{da}$ is perpendicular to the plane of the ring and makes an angle θ with $\overrightarrow{dE}$. The flux passing through the small area element of the disc is given by

d ϕ $=\left|\overrightarrow{dE}\right|\left|\overrightarrow{da}\right|cos\theta$

d ϕ $=\frac{Kq}{x^2}\times 2\pi r dr \times \frac{R}{x}$

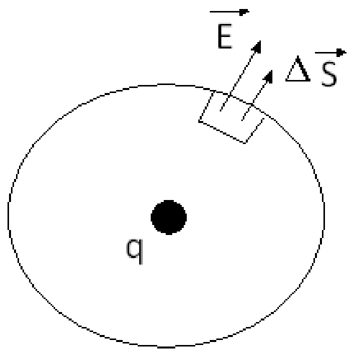
d ϕ $= 2\pi KqR \times \frac{rdr}{x^3}$

d ϕ $= 2\pi KqR \times \frac{rdr}{(R^2+r^2)^{3/2}}$ (as $x^2 = R^2 + r^2$)

Total flux

$$\emptyset = 2\pi kqR\int_0^a \frac{rdr}{(R^2+r^2)^{3/2}} = 2\pi kqR\left[\frac{-1}{\sqrt{R^2-r^2}}\right]_0^a = 2\pi kqR\left[\frac{1}{R} - \frac{1}{\sqrt{R^2-r^2}}\right]$$

Q) Starting form Gauss's law, calculate the electric field due to an isolated point charge q and show that Coulomb's law follows from this result



Solution

For this situation, we choose a spherical Gaussian surface of radius r and centered on the point charge, as shown. The electric field of a positive point charge is radially outward normal to the surface at every point. That is $\vec{E}$ is parallel to $\overrightarrow{dS}$ at each point, and so $\vec{E}.\overrightarrow{dS}$ =EdS.

According to Gauss's law $\phi_E = \oint \vec{E}\,.\,\overrightarrow{dS} = \oint$ E .dS= $\frac{q}{\varepsilon_0}$ --(i)

By smmetry, E is constat verywhere on the surface. Therefore

$\oint$ E .dS = E$\oint$dS = E(4πr^2) = $\frac{q}{\varepsilon_0}$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

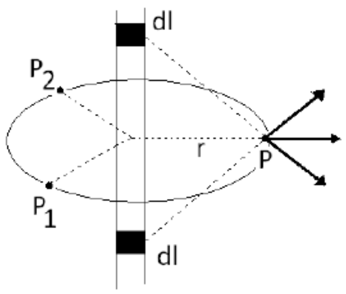
is nothing but Coulomb's law. Hence they are equivalent.

Field due to an infinite line of charge

Consider an infinite line of charge has a linear charge density λ. Using Gauss's law, let us find the electric field at a distance 'r' from the line.

The cylindrical symmetry tells us that the field strength will be the same at all points at a fixed distance r from the line.

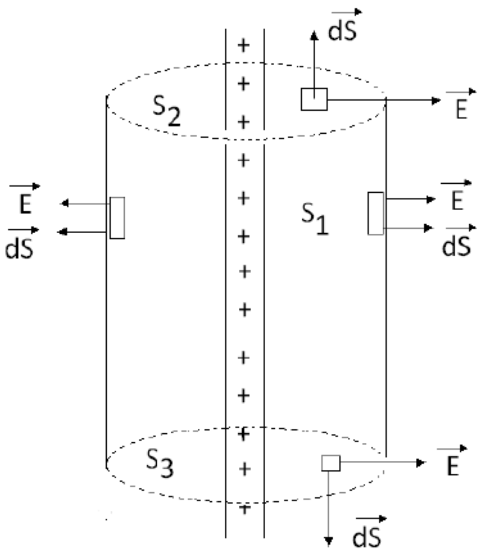
Since the line is infinite and uniform, for every charge element on the other side the component along the line of the fields due to all such element cancels in pairs. Thus the field lines are directed radially outward, perpendicular to the line charge. Also perpendicular distance from line is same magnitude at all points P_1,P_2 will be same



The appropriate choice of Gaussian surface is a cylinder of radius r and length L. On the flat faces S_2 and S_3 , $\vec{E}$ is perpendicular to $\overrightarrow{dS}$, which means no flux cross them.

On curved surface S_1, $\vec{E}$ is parallel to $\overrightarrow{dS}$ so that $\vec{E}.\overrightarrow{dS} = EdS$.

The charge enclosed by the cylinder is Q = λL



Applying Gauss's law to the curved surface, we have

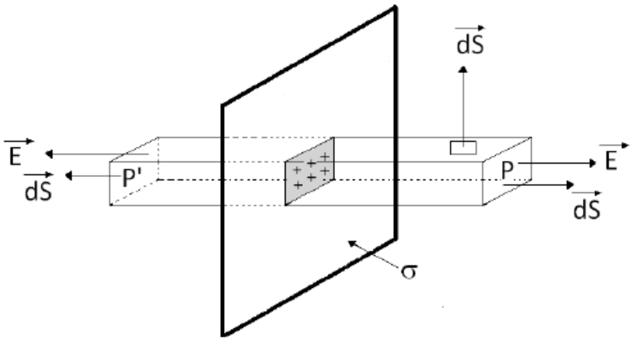
E$\oint$dS = E(2πrL) = $\frac{\lambda L}{\varepsilon_0}$

$$\boxed{E = \frac{\lambda}{2\pi r\varepsilon_0}}$$

Vectorially , $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{r}$ here $\hat{r}$ is the radial unit vector in the plane normal to the wire passing through the point.

Note that although only the charge enclosed by the surface was included above, the electric field E is due to the charges on the entire wire.

Field due to an infinite plane sheet of charge



Let us consider a thin non-conducting plane sheet of charge, infinite in extent and having a surface charge density σ C/m^2 . Let point P be a point at distance r from the sheet, at which the electric intensity is required.
Let us choose a point P' symmetrical with P, on the other side of sheet. Let us now draw a Gaussian surface to be rectangular parallelepiped of cross sectional area A, as shown in figure
By symmetry, the electric field at all points on either side near the sheet will be perpendicular to the sheet, directed outward (if sheet is positively charged). Thus $\vec{E}$ is perpendicular to the plane ends contain point P and P' .Also magnitude of $\vec{E}$ will be same at P and P'. Therefore, the flux passing through the points containing P and P'

$$\emptyset_E = \int \vec{E}.\overrightarrow{dS} + \int \vec{E}.\overrightarrow{dS}$$

$$\emptyset_E = \int EdS + \int EdS$$

$$\emptyset_E = 2EA$$

The flux through remaining surface is zero because $\vec{E}$ is perpendicular to $\overrightarrow{dS}$ and do not contribute to the total flux
The charge enclosed by the Gaussian surface shown by shaded area. q = σA
Applying Gauss's law, we have

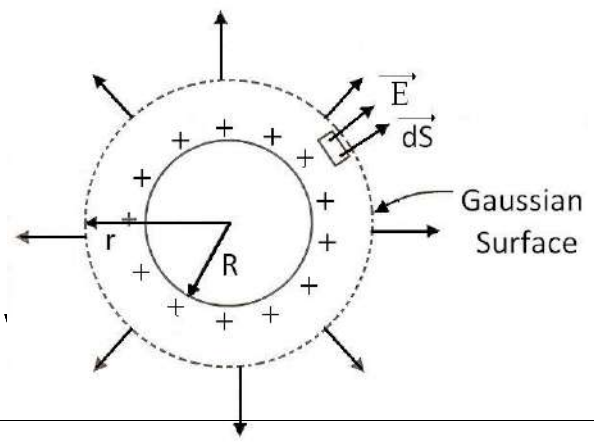
$$2EA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma A}{2\varepsilon_0}}$$

Electric field intensity is independent of the distance from an infinite sheet of charges

Electric field due to a uniformly charged spherical shell

Using Gauss's law, we can find the intensity of the electric field due to a uniformly charged spherical shell or a solid conducting sphere at



Case(I): At an external point

In an isolated charged spherical conductor an excess charge on it is distributed uniformly over its surface. Since charge lines is radially

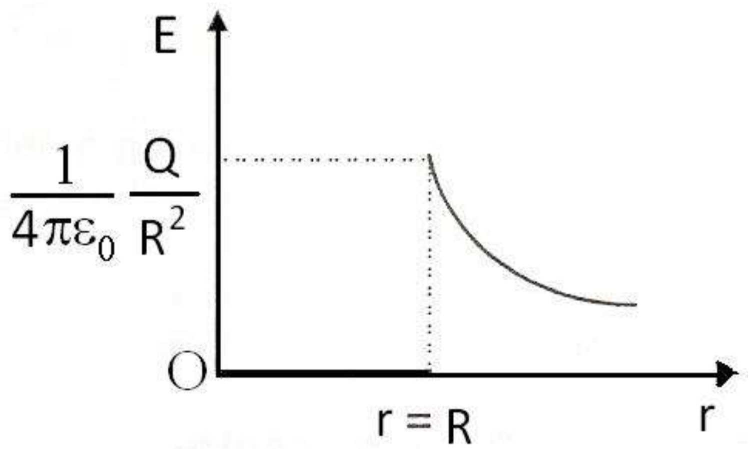
outward. Also, field strength will have the same value at all points on any imaginary spherical surface concentric with the charged conducting sphere or shell. This symmetry leads us to choose the Gaussian surface to be a sphere of radius r > R.
Any arbitrary element of area $\overrightarrow{dS}$ is parallel to the local $\vec{E}$, so $\vec{E}.\overrightarrow{dS}$ = EdS at all points on the surface
According to Gauss's law

$$\oint \vec{E}.\overrightarrow{dS} = \oint EdS = E\oint dS = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

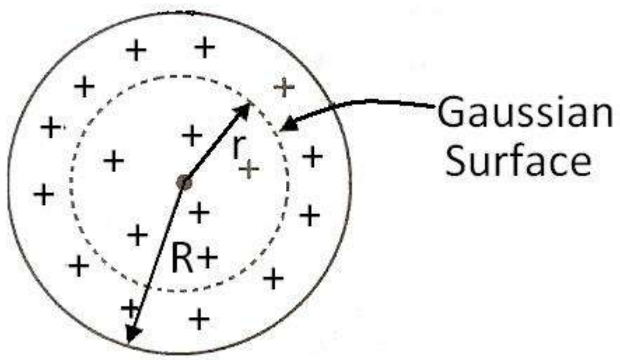
For all points outside the charged conducting sphere or the charge spherical shell, the field is same as that appoint charge at the centre

Case(ii) At an internal point (r <R)
The field still has the same symmetry and so we again pick a spherical Gaussian surface, but now with radius r less than E. Since the enclosed charge is zero, from Gauss' law, we have E = 0
Thus , we conclude that E=0 at all points inside a uniformly charged conducting sphere or the charged spherical shell



Variation of E with the distance from the centre (r)

Electric field due to uniformly charged sphere or sphere of charge



A non conducting uniformly sphere of radius R has a total charge Q uniformly distributed throughout its volume. Using Gauss's law Positive charge Q is uniformly distributed throughout the volume of sphere of radius R. Density of charge

Case(i): At an internal point (r <R)
For finding the electric field at a distance (r <R) from the centre, we choose a spherical Gaussian surface of radius r, concentric with the charge distribution. Let charge enclosed is q. From symmetry the magnitude E of the

electric field has the same value at every point on the Gaussian surface, and the direction of $\vec{E}$ is radial at every point on the surface
So, applying Gauss's law

$$\oint \vec{E}.\vec{dS} = \oint EdS = E\oint dS = E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

Here

$$q = \left(\frac{4}{3}\pi r^3\right)\rho$$

$$q = \left(\frac{4}{3}\pi r^3\right)\frac{3Q}{4\pi R^3} = Q\frac{r^3}{R^3}$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}}$$

Or

$$E = \frac{\rho r}{3\varepsilon_0}$$

Case (ii) At an external point (r >R)
To find the electric field outside the charged sphere, we use a spherical Gaussian surface of radius r (r>R). This surface encloses the entire charged sphere, So, from Gauss's law, we have

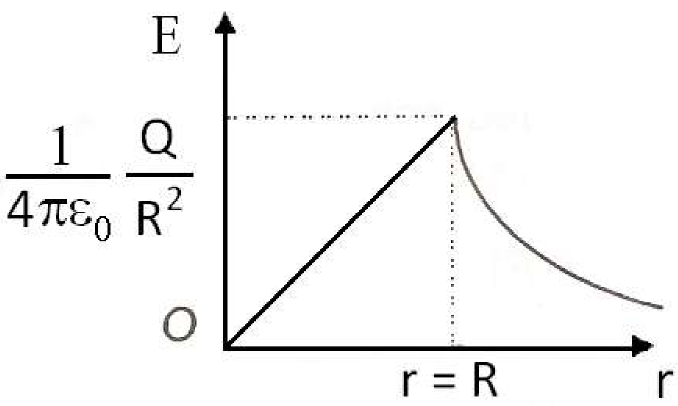
$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Or

$$E = \frac{\rho R^3}{3r^2\varepsilon_0}$$

Variation of E with the distance from centre (r)



QUESTIONS (E)

Q) Give the statement of Gauss' Law

Q) What is the electric field intensity at a point inside a uniformly charged rubber ballon?

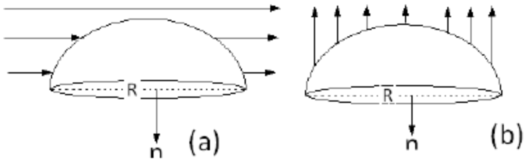
Q) Using Gauss's law, find the intensity of the electric field produced due to uniformly charged infinite plane

Or Prove that electric field intensity is independent of the distance from an infinite sheet of charges

Q) Obtain the expression of electric field due to an infinitely long linear charged wire along the perpendicular distance from the wire.

Q) Using Gauss's Law, find the intensity of the electric field inside and outside the charged sphere having uniform charge density.

Q) A hemispherical body placed in uniform electric field E. What is the flux linked with the curved surface, (i) if field parallel to base fig (a) and (ii)perpendicular to base fig(b)



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LESSON 2

ELECTROSTATIC POTENTIAL AND CAPACITANCE

SECTION I

ELECTROSTATIC POTENTIAL

ELECTRIC FIELD IS CONSERVATIVE

In an electric field work done by the electric field in moving a unit positive charge from one point to the other, depends only on the position of those two points and does not depend on the path joining them.

ELECTROSTATIC POTENTIAL

Electrostatic potential is defined as **“Work required to be done against the force by electric field in bringing a unit positive charge from infinite distance to the given point in the electric field us called the electrostatic potential (V) at that point”**

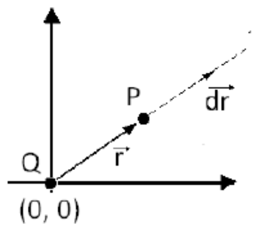
According to above definition the electric potential at point P is given by the formula

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot \vec{dr}$$

Electric potential is scalar quantity. SI units (J/C) called as volt (V)

POTENTIAL AT APOINT DUE TO A POINT CHARGE

Consider a point charge positive Q at the origin. To deter let P be the point at a distance ‘r’ from origin of coordinate axis . Since work done in electric field is independent of path, we will consider radial path as shown in figure.



According to definition of electric potential we can use the equation

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot \vec{dr}$$ And electric field E is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\vec{r}$ given by

$$V_P = -\int_{\infty}^{P} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\vec{r} \cdot \vec{dr}$$

$$V_P = -\int_{\infty}^{P} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} dr$$

$$V_P = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]$$

$$V_P = \frac{Q}{4\pi\varepsilon_0 r}$$

ELECTRIC POTENTIAL DUE TO GROUP OF POINT CHARGE

The potential at any point due to group of point charges is the algebraic sum of the potentials contributed at the same point by all the individual point charges

$V = V_1 + V_2 + V_3 + \ldots.$

ELECTRIC POTENTIAL DIFFERENCE

Electric potential difference is defined as "Work required to be done to take a unit positive charge from one point (say P) to another point (say Q) against the electric field

According to formula for potential at point P

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot \vec{dr}$$

Thus potential at point Q is given by

$$V_Q = -\int_{\infty}^{Q} \vec{E} \cdot \vec{dr}$$

From above formula potential difference between points Q and P is given by

$$V_Q - V_P = \left(-\int_{\infty}^{Q} \vec{E} \cdot \vec{dr}\right) - \left(-\int_{\infty}^{P} \vec{E} \cdot \vec{dr}\right)$$

$$V_Q - V_P = \int_{Q}^{\infty} \vec{E} \cdot \vec{dr} + \int_{\infty}^{P} \vec{E} \cdot \vec{dr}$$

$$V_Q - V_P = \int_{Q}^{P} \vec{E} \cdot \vec{dr}$$

$$V_Q - V_P = -\int_{P}^{Q} \vec{E} \cdot \vec{dr}$$

Or work done in moving charge from point P to point Q

SI unit of potential is V and dimensional formula $[M^1L^2T^{-3}A^{-1}]$

ELECTROSTATIC POTENTIAL ENERGY

The electric potential energy is defined as "The work required to be done against the electric field in bringing a given charge *(q),* from infinite distance to the given point in the electric field motion without acceleration is called the electric potential energy of that charge at that point."

From definition of electric potential energy and the electric potential we can write electric potential energy of charge q at point P, as

$$U_p = -\int_{\infty}^{p} q\vec{E}.\overrightarrow{dr} = q\int_{\infty}^{p} \vec{E}.\overrightarrow{dr} = qV_p$$

The absolute value of the electric potential energy is not at all important, only the difference in its value is important. Here, in moving a charge *q,* from point P to Q, without acceleration, the work required to be done by the external force, shows the difference In the electric potential energies (U_Q - Up) of this charge *q,* at those two points.

$$U_Q - U_P = -q\int_P^Q \vec{E}.\overrightarrow{dr}$$

Electric potential energy is of the entire system of the sources producing the field and the charge, for some configuration, and when the configuration changes the electric potential energy of the system also changes.

POTENTIAL ENERGY OF A SYSTEM OF TWO POINT CHARGES

The potential energy possessed by a system of two – point charges q_1 and q_2 separated by a distance 'r' is the work done required to bring them to this arrangements from infinity. This electrostatic potential energy is given by

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

ELECTRIC POTENTIAL ENERGY OF A SYSTEM OF POINT CHARGES

The electric potential energy of such a system is the work done in assembling this system starting from infinite separation between any two-point charges.
For a system of point charges q_1, q_2, q_3 …. q_n the potential energy is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}\left(i \neq j\right)$$

It simply means that we have to consider all the pairs that are possible

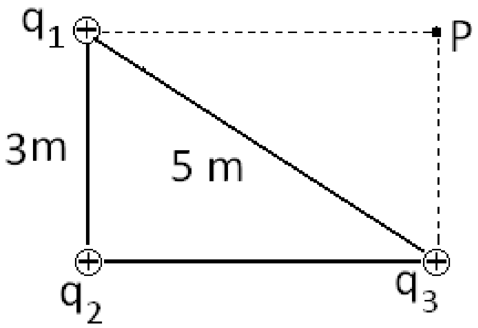
Important points regarding electrostatic potential energy
(i) Work done required by an external agency to move a charge *q* from *A* to B in an electric field with constant speed
(ii) When a charge *q* is let free in an electric field, it loses potential energy and gains kinetic energy, if it goes from *A* to B, then loss in potential energy = gain in kinetic energy
Or

$$q\left(V_B - V_A\right) = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$$

Solved numerical

Q) Find work done by some external force in moving a charge q = 2μC from infinity to a point where electric potential is 10^4V
<u>Solution</u>
Work W = Vq = (10^4V)(2 $\times 10^{-6}$C) = 2×10^{-2}J

Q) Three point charges $q_1 = 1\ \mu C$, $q_2 = -2\mu C$ and $q_3 = 3\mu C$ are fixed at position shown in figure (a) What is the potential at point P at the corner of the rectangle? (b) How much work would be needed to bring a charge $q_4 = 2.5\ \mu C$ from infinity to place it at P



Solution

(a) The total potential at point P is the scalar sum

$V_P = V_1 + V_2 + V_3$

$$V_P = \frac{q_1}{4\pi\varepsilon_0 r_1} + \frac{q_2}{4\pi\varepsilon_0 r_2} + \frac{q_3}{4\pi\varepsilon_0 r_3}$$

$$V_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{1\times10^{-6}}{4} + \frac{-2\times10^{-6}}{5} + \frac{3\times10^{-6}}{3}\right)$$

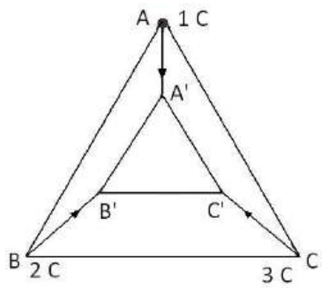
$$V_P = 9\times10^9\left(1.7\times10^{-6}\right)$$

$$V_P = 7.65\times10^3 V$$

(b) External work is $W_{ext} = q[V_f - V_i]$, In this case $V_i = 0$

So $W_{ext} = q_4 V_P = (2.5X10^{-6})(7.65X10^3) = 0.019J$

Q) Three point charges 1C, 2C, 3C are placed at the corner of an equilateral triangle of side 1m. Calculate the work required to move these charges to the corners of a smaller equilateral triangle of side 0.5m as shown



Solution

As the potential energy of two point charges separated by a distance 'r' is given by

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

The initial potential energy of the system will be

$$U_i = \frac{1}{4\pi\varepsilon_0}\left[\frac{1\times2}{1} + \frac{2\times3}{1} + \frac{3\times1}{1}\right]$$

$$U_i = 9\times10^9\times11$$

$$U_i = 9.9\times10^{10} J$$

The final potential energy of system

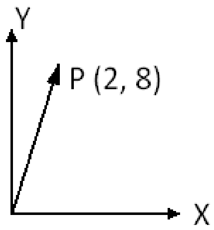
$$U_f = \frac{1}{4\pi\varepsilon_0}\left[\frac{1\times 2}{0.5}+\frac{2\times 3}{0.5}+\frac{3\times 1}{0.5}\right]$$

$$U_f = 9\times 10^9 \times 22$$

$$U_f = 19.8\times 10^{10}\,J$$

So, the work done in changing the configuration of the system
W = U_f - U_i = (19.8 – 9.8)×10^{10} = 9.9×10^{10} J

Q) Suppose an electric field due to a stationary charge distribution is given by
$\vec{E} = ky\,\hat{\imath} + kx\,\hat{\jmath}$, where k is a constant. Obtain the formula for electric potential at any point on the line OP, with respect to (0, 0)



Solution
Equation of line is y = 4x
Let potential at origin is zero
In order to obtain potential at any point Q(x, y) on the line OP with respect to (0, 0) we can use

$$V_Q - V_P = -\int_{(0,0)}^{Q} \vec{E}\cdot\vec{dr}$$

$$V_Q - 0 = -\int_{(0,0)}^{Q} \left(ky\,\hat{i} + kx\,\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$V_Q = -\int_{(0,0)}^{Q} \left(kydx + kxdy\right)$$

$$as\quad y = 4x$$

$$dy = 4dx$$

$$V_Q = -\int_{(0,0)}^{Q} 4kxdx + 4kxdx$$

$$V_Q = -\int_{0}^{x} 8kxdx$$

$$V_Q = -8k\left[\frac{x^2}{2}\right]_0^x$$

$$V_Q = 4kx^2$$

Q) The electric field at distance r perpendicularly from the length of an infinitely long wire is $E_{(r)} = \frac{\lambda}{2\pi\varepsilon_0 r}$, where λ is the linear charge density of the wire. Find the potential at a point having distance b from the wire with respect to a point having distance a from the wire (a >b)

Solution: Let V_a be reference point thus $V_a = 0$

$$V_b - V_a = -\int_a^b \vec{E}\cdot\vec{dr}$$

$$V_b - V_a = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0 r}dr \therefore \left(\vec{E} \parallel \vec{dr}\right)$$

$$V_b - V_a = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r}$$

$$V_b - V_a = -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln r\right]_a^b$$

$$V_b - V_a = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{b}\right)$$

$$\because V_a = 0 \ \ reference$$

$$V_{ba} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{b}\right)$$

Q) An electric field is represented by $\vec{E} = Ax\hat{\imath}$, where A = 10 V/m^2. Find the potential of the origin with respect to the point (10, 20) m

Solution : $\vec{E} = Ax\hat{\imath} = 10x\,\hat{\imath}$

$$V_{(0,0)} - V_{(10,20)} = -\int_{(10,20)}^{(0,0)} \vec{E}\cdot\vec{dr}$$

$$V_{(0,0)} - V_{(10,20)} = -\int_{(10,20)}^{(0,0)} \left(10x\,\hat{i}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$V_{(0,0)} - V_{(10,20)} = -\int_{10}^{0} \left(10x\,dx\right)$$

$$V_{(0,0)} - V_{(10,20)} = -10\left[\frac{x^2}{2}\right]_{10}^{0}$$

$$V_{(0,0)} - V_{(10,20)} = \left[0 - (-500)\right]$$

$$V_{(0,0)} - V_{(10,20)} = 500\ V$$

Since V(10,20) is to be taken zero V(0, 0) = 500 volts

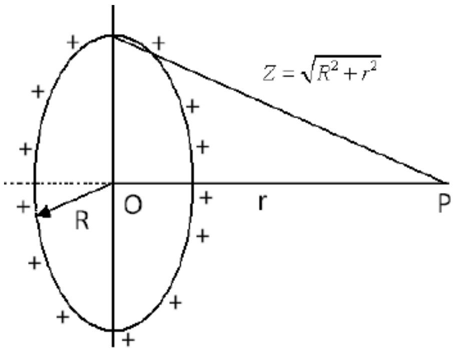
ELECTRIC POTENTIAL DUE TO CONTINUOUS CHARGE DISTRIBUTION

The electric potential due to continuous charge distribution is the sum of potential of all the infinitesimal charge elements in which the distribution may be divided

$V = \int dV = \int \frac{dq}{4\pi\varepsilon_0 r}$

ELECTRIC POTENTIAL DUE TO A CHARGED RING

A charge Q is uniformly distributed over the circumference of a ring. Let us calculate the electric potential at an axial point at a distance r from the centre of the ring. The electric potential at P due to the charge element dq of the ring is given by



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{R^2 + r^2}}$$

Hence electric potential at P due to the uniformly charged ring is given by

$$V = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{R^2 + r^2}} = \frac{1}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2 + r^2}}\int dq$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2 + r^2}}$$

ELECTRIC POTENTIAL DUE TO A CHARGED DISC AT A POINT ON THE AXIS

A non-conducting disc of radius 'R' has a uniform surface charge density σ $C/m^2$to calculate the potential at a point on the axis of the disc at a distance from its centre. Consider a circular element of disc of radius x' and thickness dx. All points on this ring are at the same distance $Z = \sqrt{x^2 + r^2}$, from the point P. The charge on the ring is dq = σA dq= σ(2πxdx) and so the potential due to the ring is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{Z} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi x dx)}{\sqrt{x^2 + r^2}}$$

Since potential is scalar, there are no components. The potential due to the whole disc is given by

$$V=\frac{\sigma}{2\varepsilon_0}\int_0^R\frac{x}{\sqrt{x^2+r^2}}dx=\frac{\sigma}{2\varepsilon_0}\left[\left(x^2+r^2\right)^{1/2}\right]_0^R$$

$$V=\frac{\sigma}{2\varepsilon_0}\left[\left(R^2+r^2\right)^{1/2}-r\right]$$

$$V=\frac{\sigma}{2\varepsilon_0}\left[r\left(\frac{R^2}{r^2}+1\right)^{1/2}-r\right]$$

For large distance R/r << 1 thus

$$r\left(\frac{R^2}{r^2}+1\right)^{1/2}\approx r\left(1+\frac{R^2}{2r^2}\right)$$

Substituting above value in equation for potential

$$V=\frac{\sigma}{2\varepsilon_0}\left[r\left(1+\frac{R^2}{2r^2}\right)-r\right]$$

$$V=\frac{\sigma}{2\varepsilon_0}\frac{R^2r}{2r^2}$$

$$V=\frac{\sigma}{2\varepsilon_0}\frac{R^2}{2r}\left(\frac{\pi}{\pi}\right)$$

$$Since\ \ Q=\sigma\pi R^2$$

$$V=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Thus, at large distance, the potential due to disc is the same as that of point charge

ELECTRIC POTENTIAL DUE TO A SHELL

A shell of radius R has a charge Q uniformly distributed over its surface.

(a) At an external point

At point outside a uniform spherical distribution, the electric field is

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

Since $\vec{E}$ is radial, $\vec{E}.\overrightarrow{dr}$ = E dr

Since V(∞)=0, we have

$$V(r) = -\int_0^r \vec{E}\cdot\vec{dr} = -\int_0^r \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]_\infty^r$$

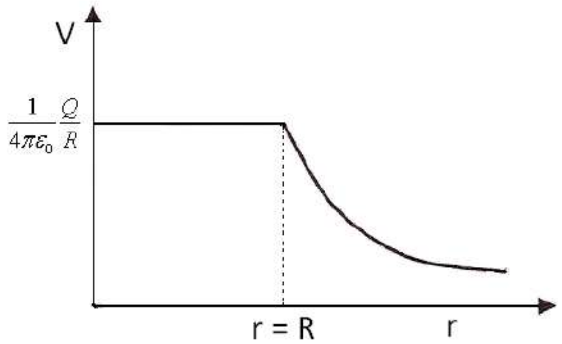
$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \quad (r > R)$$

Thu potential due to uniformly charged shell is the same as that due to a point charge Q at the centre of the shell.

(b) At an internal point

At point inside the shell, E = 0. So work done in bringing a unit positive charge from a point on the surface to any point inside the shell is zero. Thus, the potential has a fixed value at all points within the spherical shell and is equal to the potential at the surface.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$



Above results hold for a conducting sphere also whose charge lies entirely on the outer surface.

ELECTRIC POTENTIAL DUE TO A NON-CONDUCTING CHARGED SPHERE

A charge Q is uniformly distributed through a non-conducting volume of radius R.

(a) Electric potential at external point is given by equation. ‘r’ is the distance of point from the center of the sphere

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

(b) Electric potential at an internal point is given by equation

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R^3}\left[3R^2 - r^2\right]$$

Here R is the radius of the sphere and r is the distance of point from the centre

RELATION BETWEEN THE ELECTRIC FIELD AND ELECTRIC POTENTIAL

We know that electric potential from electric field is given by

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot \vec{dr}$$

And potential difference between two points is given by

$$V_Q - V_P = -\int_{P}^{Q} \vec{E} \cdot \vec{dr}$$

If points P and Q are very close to each other, then for such a small displacement $\vec{dr}$, integration is not required and only term $\vec{E}.\vec{dr}$ can be kept thus

dV = - $\vec{E}.\vec{dr}$

I) If $\vec{dr}$, is the direction of electric field $\vec{E}$, $\vec{E}.\vec{dr} = Edr\cos\theta = Edr$

$$dV = -Edr$$

$$E = -\frac{dV}{dr}$$

This equation gives the magnitude of electric field in the direction of displacement $\vec{dr}$.

Here $\boldsymbol{\frac{dV}{dr}}$ **= potential difference per unit distance. It is called the potential gradient.** It unit is Vm^{-1}, which is equivalent to N/C

II) If $\vec{dr}$, is not in the direction of $\vec{E}$, but in some other direction, the $-\boldsymbol{\frac{dV}{dr}}$ **would give us the component of electric field in the direction of that displacement**

If electric field is in X direction and displacement is in any direction (in three dimensions) then

$\vec{E} = E_X\hat{\imath}$ and $\vec{dr} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

∴ dV = - $(E_X\hat{\imath}).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$ = $-E_X dx$

∴ $E_x = -\frac{dV}{dx}$

Similarly, if the electric field is Y and only in Z direction respectively, we would get

$E_y = -\frac{dV}{dy}$ and $E_Z = -\frac{dV}{dz}$

Now if the electric field also have three (x,y,z) components then

$$E_x = -\frac{\partial V}{\partial x}. \quad E_y = -\frac{\partial V}{\partial y}. \quad E_Z = -\frac{\partial V}{\partial z}.$$

And $\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right).$

Here $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ shows the partial differentiation of V(x, y, z) with respect to x, y, z respectively. Moreover, the potential differentiation of V(x, y, z) with respect to x means the differentiation of V with respect to x only, by taking y and z in the formula of V as constant

Solved numerical

Q) The electric potential in a region is represented as V = 2x+3y-z. Obtain expression for the electric field strength

Solution

We know

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right).$$

Here

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}[2\mathrm{x} + 3\mathrm{y} - \mathrm{z}] = 2$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial x}[2\mathrm{x} + 3\mathrm{y} - \mathrm{z}] = 3$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial x}[2\mathrm{x} + 3\mathrm{y} - \mathrm{z}] = -1$$

$$\vec{E} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$$

Q) The electrical potential due to a point charge is given by $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ Find

a) the radial component of the electric field

b) the x-component of the electric field

Solution

a) The radial component of electric field

$E_r = -\frac{dV}{dr} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$.

(b) In terms of rectangular components, the radial distance r= $(x^2 + y^2+z^2)^{1/2}$; therefore the potential function

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2)^{\frac{1}{2}}}$$

To find the x-component of the electric field, we treat y and z constants. Thus

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2)^{\frac{3}{2}}}$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{r^3}$$

EQUIPOTENTIAL SURFACE

An equipotential surface is that surface on which the electric potentials at all points are equal

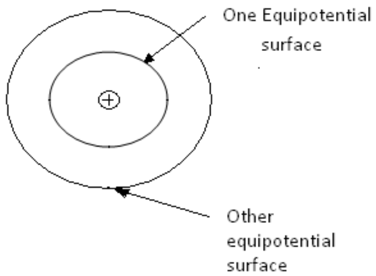
Important points regarding equipotential surfaces

(i) The line of forces are always normal to equipotential surface.

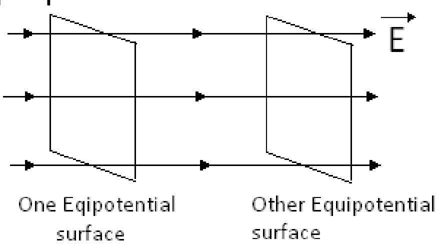
(ii) The net work done in taking a charge from A to B is zero, if A and B are on same equipotential surface.

Suppose a unit positive charge is given a small displacement dl on the equipotential surface from a given point.
In this process the work required to be done against the electric field is
dW = $-\vec{E}.\overrightarrow{dl}$ = potential difference between those two point
But the potential difference on the equipotential surface = 0
$\therefore \vec{E}.\overrightarrow{dl} = 0 \Rightarrow$ E dl cosθ = 0, where θ = angle between $\vec{E}$ and $\overrightarrow{dl}$
But E ≠ 0 and dl ≠ 0
$\therefore \cos\theta = 0 \Rightarrow \theta = \pi/2$
$\vec{E} \perp \overrightarrow{dl}$
But $\overrightarrow{dl}$ is along this surface. Hence electric field is normal to the equipotential surface at that point
(iii) Equipotential surface never intersect each other. If they intersect then electric field lines will also intersect which is not possible.

Examples
(i) In the field of a point charge, the equipotential surfaces are spheres centred on the point charge.

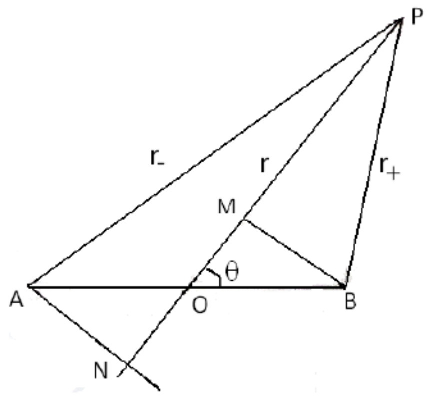


(ii) In a uniform electric field, the equipotential surfaces are planes which are perpendicular to the field lines.



(iii) In the field of an infinite line charge, the equipotential surfaces are co-axial cylinders having their axes at the line charge.
(iv)The surface of a conductor is an equipotential surface and the inside of conductor is equipotential space. Hence there is no electric field (and charge) inside the conductor's surface. The lines of forces are always normal to the surface of a conductor.

ELECTRIC POTENTIAL DUE TO DIPOLE



Let a dipole consisting of equal and opposite charge q separated by a distance 2a. Let zero of coordinate system is at centre of the dipole as shown in figure. Let p be any point in x-y plane. Let PO = r. AP = r_- BP = r_+ . Let r makes angle of θ with the axis of dipole.

Potential at point P is the sum of potential due at point P due to –q and +q charges.

BM is perpendicular on OP and AN is perpendicular on ON

$$V_{(P)} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_+} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_-}$$

$$V_{(P)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

$$V_{(P)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{r_- - r_+}{r_- r_+}\right]$$

From figure r_- = r + ON but ON = acosθ

∴ r_- = r + acosθ

Similarly

r = r_+ + OM but OM = acosθ

∴ r = r_+ + a cosθ

r_+ = r – acosθ

r_- - r_+ = 2acosθ

Substituting the values we get

$$V_{(P)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{2a\cos\theta}{(r + a\cos\theta)(r - a\cos\theta)}\right]$$

$$V_{(P)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{2a\cos\theta}{(r^2 - a^2\cos^2\theta)}\right]$$

$$V_{(P)} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2qa\cos\theta}{(r^2 - a^2\cos^2\theta)}\right]$$

$$V_{(P)} \approx \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

Since r >>2a

Case I) Potential on the axis :

For point on the axis of the dipole θ 0 or π

$$V = \pm \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}$$

From the given point, if the nearer charge is +q, then we get V as positive. And if it is –q, then we get V as negative

Case II) Potential on the equator

From a point on the equator $\theta = \pi/2 \therefore V = 0$

POTENTIAL ENERGY OF DIPOLE

When a dipole is placed in external uniform electric field $\vec{E}$, a torque $\vec{\tau} = \vec{p} \times \vec{E}$ acts on it. If we rotate the dipole through a small angle dθ, the work done by torque is

dW = τ dθ or dW = -PEsinθ dθ

The work is negative as the rotation dθ is opposite to the torque. The change in electric potential energy of the dipole is therefore

dU = -dW = -PEsinθ dθ

If dipole is rotated from angle θ_1 to θ_2, then

$$\int_{\theta_1}^{\theta_2} dU = \int_{\theta_1}^{\theta_2} pE\sin\theta\, d\theta$$

$$U(\theta_2) - U(\theta_1) = pE\left[-\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$U(\theta_2) - U(\theta_1) = -pE\left(\cos\theta_2 - \cos\theta_1\right)$$

Work done by external force = $U(\theta_2) - U(\theta_1)$

OR $W_{ext} = pE(\cos\theta_1 - \cos\theta_2)$

Work done by electric force

$W_{electric\ force} = - W_{ext} = pE(\cos\theta_2 - \cos\theta_1)$

PERIODIC TIME OF DIPOLE

When a dipole is suspended in a uniform electric field, it will align itself parallel to the field. Now if it is given a small angular displacement θ about its equilibrium, the restoring couple will be

C = -pEsinθ

Or C = -pEθ [as for small θ, sinθ = θ]

Also couple

$$C = I\frac{d^2\theta}{dx^2}$$

Thus

$$I\frac{d^2\theta}{dx^2} = -pE\theta$$

$$\frac{d^2\theta}{dx^2} = -\frac{pE}{I}\theta$$

Comparing above equation with standard equation for SHM we get

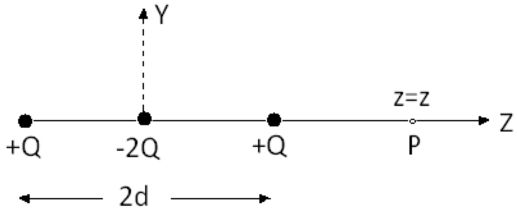
$$\omega^2 = \frac{pE}{I}$$

Thus periodic time

$$T = 2\pi\sqrt{\frac{I}{pE}}$$

Solved numerical

Q) When two dipoles are lined up in opposite direction, the arrangement is known as quadruple (as shown in figure) Calculate the electric potential at a point z = z along the axis of the quadruple



Electric potential at point p is given by

$V = V_1 + V_2 + V_3$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{z+d} + \frac{1}{z-d} - \frac{2}{z}\right]$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{2d^2}{z\left(z^2-d^2\right)}\right]$$

SECTION II
ELECTROSTATICS OF CONDUCTORS

a) EFFECT OF EXTERNAL FIELD ON CONDUCTOR

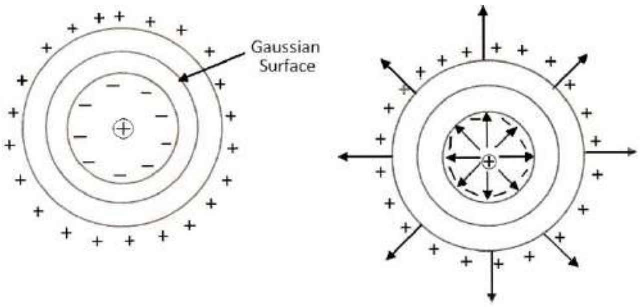
In a metallic conductor there are positive ions situated at the lattice points and the free electrons are moving randomly between these ions. They are free to move within the metal but not free to come out of the metal.

When such a conductor is placed in an external electric field, the free electrons move under the effect of the force in the direction opposite to the direction of electric field and get deposited on the surface of one end of the conductor, and equal amount of positive charge can be considered as deposited on the other end.

These induced charges produce an electric field inside the conductor, in the direction opposite to external electric field. When these two electric fields become equal in magnitude, the resultant net electric field inside the conductor becomes zero. Now the

motion of charges in the conductor stops and the charges become steady on the end surfaces
Thus in the case of metallic conductor, placed in an external electric field
(1) A steady electric charge distribution is induced on the surface of the conductor
(2) The net electric field inside the conductor is zero
(3) The net electric charge inside the conductor is zero
(4) On the outer surface of the conductor, the electric field at every point is locally normal to the surface.
If the electric field were not normal (perpendicular) a component of electric field parallel to the surface would exist and due to it the charge would move on the surface. But now the motion is stopped and the charges have become steady. Thus the component of electric field parallel to the surface would be zero, and hence the electric field would be normal to the surface.
(5) Since $\vec{E}$ =0 at every point inside the conductor, the electric potential everywhere inside the conductor and equal to the value of potential on the surface

CAVITY INSIDE A CONDUCTOR



Consider a +q_0 suspended in a cavity in a conductor. Consider. Consider a Gaussian surface just outside the cavity and inside the conductor $\vec{E}$ =0 on this Gaussian surface as it is inside the conductor form Gauss's law

$$\oint \vec{E} . \overrightarrow{ds} = \frac{\sum q}{\varepsilon_0}$$

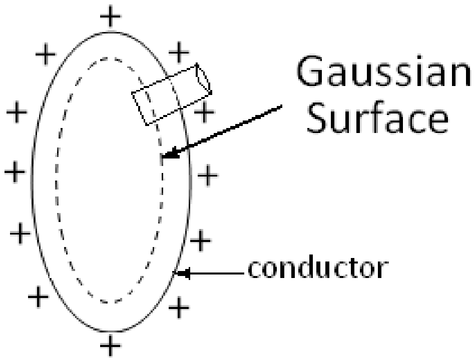
we have

$$\sum q = 0$$

This concludes that a charge of –q must reside on the metal surface on the cavity so that the sum of this induced charge –q and the original charge +q within the Gaussian surface is zero. In other words, a charge q suspended inside a cavity in a conductor is electrically neutral, a charge +q is induced on the outer surface of the conductor. As field inside the conductor, as shown in figure
ELECTROSTATIC SHIELDING
When a conductor with a cavity is placed inside the electric field, Charge induces on the surface of the conductor. These induce charges produce electric field inside the conductor such that net electric field inside the conductor and inside the cavity is zero. Thus electric field everywhere inside the cavity is zero. This fact is called electrostatic shielding

EFFECT PRODUCED BY PUTTING CHARGE ON THE CONDUCTOR

The charge placed on a conductor is always distributed only on the outer surface of the conductor. We can understand this by the fact that the electric field inside a conductor is zero. Consider a Gaussian Surface shown by the dots inside the surface and very close to it. Every point on it is inside the surface and not on the surface of conductor. Hence the electric field at every point on this surface is zero. Hence according to Gauss's theorem the charge enclosed by the surface is also zero



Consider a Gaussian surface of a pill-box of extremely small length and extremely small cross-section ds. A fraction of it is inside the surface and the remaining part is outside the surface . The total charge enclosed by this pill-box is q=σds, where σ = surface charge density of the charge on the conductor. At every point on the surface of the conductor $\vec{E}$ is perpendicular to the local surface element. Hence it is parallel to the surface vector $\overrightarrow{ds}$

But inside the surface $\vec{E}$ is zero. Hence flux coming out from the cross-section of pillbox inside the surface is zero. The flux coming out from the cross-section of pill-box outside the surface is $\vec{E}\,.\overrightarrow{ds} = E\ ds$

According to Gauss's theorem E ds = $\frac{q}{\varepsilon_0} = \frac{\sigma ds}{\varepsilon_0}$

$$E = \frac{\sigma}{\varepsilon_0}$$

In vector form

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

SECTION III

CAPACITORS AND CAPACITANCE

CAPACITY OF AN ISOLATED CONDUCTOR

When charge is given to an isolated body, its potential increases and the electric field also go on gradually increasing. In this process at some stage the electric field becomes sufficiently strong to ionize the air particles around the body as a result body is not able to store any additional charge. During the process the ratio of charge Q on the body and potential (V) on the body remains constant. This ratio is called the capacity of the body

C = Q/V

In SI system, the unit of capacity is coulomb/volt and is called Farad (F)

The capacity of a body is independent of the charge given to it and depends on the shape and size only.

CAPACITOR

Capacitor is an arrangement of two conductors carrying charges of equal magnitude and opposite sign and separated by an insulating medium. The following points may be carefully noted.

(i) The net charge on the capacitor as a whole is zero. When we say that a capacitor has a charge Q, we mean that positively charged conductor has a charge +Q and the negatively charged conductor has a charge *-Q.*

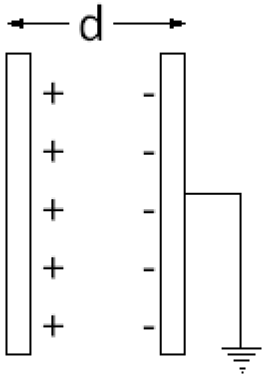
(ii) The positively charged conductor is at a higher potential than negatively charged conductor. The potential difference V between the conductors is proportional to the magnitude of charge Q and the ratio *Q/V* is known as capacitance C of the capacitor.

Q C= V

Unit of capacitance is farad (F). The capacitance is usually measured in microfarad (μF)

(iii) Circuit symbol is -||-

PARALLEL PLATE CAPACITOR



A parallel plate capacitor consists of two metal plates placed parallel to each other and separated by a distance *'d'* that is very small as compared to the dimensions of the plates. Due to this, the non-uniformity of the electric field near the ends of the plates can be neglected and in the entire region between the plates the electric field can be taken as constant. The area of each plate is *A.* Let Q be the charge on each plates. Surface charge is σ. Direction of electric field produced by both the plates is in same direction. Where outside the plates electric field is opposite in direction hence zero

Then electric field between the plates is given by

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

The potential difference (V) between plates is given by V = Ed

$$V = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{A\varepsilon_0} d$$

Hence

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$$

ISOLATED SPHERE AS A CAPACITOR

A conducting sphere of radius R carrying a charge Q can be treated as a capacitor with high potential conductor as the sphere itself and low potential conductor as sphere of infinite radius. The potential difference between these two spheres is

$$V = \frac{Q}{4\pi\varepsilon_0 R} - 0$$

Hence Capacitance C = Q/V = $4\pi\varepsilon_0 R$

ENERGY STORED IN CHARGED CAPACITOR

In order to establish a charge on the capacitor, work has to be done on the charge. This work is stored in the form of the potential energy of the charge. Such a potential energy is called the energy of capacitor.

Suppose the charge on a parallel plate capacitor is Q. In this condition each plate of the capacitor is said to be lying in the electric field of the other plate.

The magnitude of the uniform electric field produced by one plate of capacitor is = $\frac{\sigma}{2\varepsilon_0}$

Where σ $is \frac{Q}{A}$ and A is area of plate

Hence taking arbitrarily the potential on this plate as zero, that of the other plate at distance d from it will be = $\frac{\sigma}{2\varepsilon_0} d$

The potential energy of the second plate will be = (potential) (charge Q on it)

Potential energy stored in capacitor =$\frac{\sigma}{2\varepsilon_0} dQ$

$$U_E = \frac{\sigma dQ}{2\varepsilon_0} = \frac{Q}{A}\frac{d\ Q}{2\varepsilon_0} = \frac{Q^2}{2\left(\varepsilon_0 A/d\right)} = \frac{1}{2}\frac{Q^2}{C}$$

OR

$$U_E = \frac{1}{2}CV^2 = \frac{1}{2}VQ$$

ENERGY DENSITY OF A CHARGED CAPACITOR

Energy stored in capacitor is localized on the charges or the plates but is distributed in the field. Since in case of parallel plate capacitor, the electric field is only between the plates i.e. in a volume (A X d), the energy density

$$\rho_E = \frac{U_E}{volume} = \frac{\frac{1}{2}CV^2}{A \times d} = \frac{1}{2}\left[\frac{\varepsilon_0 A}{d}\right]\frac{V^2}{Ad}$$

$$\rho_E = \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2$$

as E = V/d

$$\rho_E = \frac{1}{2}\varepsilon_0 E^2$$

FORCE BETWEEN THE PLATES OF A CAPACITOR

In a capacitor as plates carry equal and opposite charges, there is a force of attraction between the plates. To calculate this force, we use the fact that the electric field is conservative and in conservative field F = -dU/dx. In case of parallel plate capacitor

$$U_E = \frac{1}{2}\frac{Q^2}{C}$$

But $C = \frac{\varepsilon_0 A}{x}$

$$U_E = \frac{1}{2}\frac{Q^2}{\varepsilon_0 A}x$$

So

$$F = -\frac{d}{dx}\left[\frac{1}{2}\frac{Q^2}{\varepsilon_0 A}x\right] = \frac{-1}{2}\frac{q^2}{\varepsilon_0 A}$$

The negative sign indicates that the force is attractive

Solved numerical

Q) The plates of a parallel plate capacitor are 5 mm apart and 2 m^2 in area. The plates are in vacuum. A potential difference of 1000 V is applied across a capacitor. Calculate

(a) the capacitance;

(b) the charge on each plate;

(c) the electric field in space between the plates;

(d) the energy stored in the capacitor.

Solution

(a) Capacitance

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.85\times 10^{12}\times 2}{5\times 10^{-3}}$$

$$C = 0.0034\mu F$$

(b) Charge

Q = CV = (0.00354 X 10^{-6}) x (1000) = 3.54μC

The plate at higher potential has a positive charge +3.54 μC and the plate at lower potential has a negative charge of -3.54 μC

(c) E = V/d = 1000 / 0.005 = 2X10^5 V/m

(d) Energy = $\frac{1}{2}CV^2 = \frac{1}{2}(0.00354)\times 10^{-6}\times 10^6 = 1.77\times 10^{-3}J$

Q) A parallel plate air capacitor is made using two square plates each of side 0.2m spaced 1 cm apart. It is connected to a 50V battery

a) What is the capacitance?

b) What is the charge on each plate?

c) What is the electric field between the plates?

e) If the battery is disconnected and then the plates are pulled apart to a separation of 2cm, what are the answers to the above parts?

Solution

(a) $C = \frac{\varepsilon_0 A}{d} = \frac{(8.85\times 10^{-12})\times 0.2\times 0.2}{0.0.1} = 3.54\times 10^{-5}\mu F$

(b) Q = CV = 3.54x10^{-5}x50 = 1.77x10^{-3}μC

(c) $U = \frac{1}{2}CV^2 = \frac{1}{2}\times(3.54\times 10^{-11})(50^2) = 4.42\times 10^{-8}J$

(d) $\mathrm{E} = \mathrm{V/d} = 50/0.01 = 5000\ \mathrm{V/m}$

(e) If the battery is disconnected the charge on the capacitor plates remains constant while the potential difference between the plates can change

$$C' = \frac{\varepsilon_0 A}{d} = \frac{C}{2} = 1.77 \times 10^{-5} \mu F$$

$$Q' = Q = 1.77 \times 10^{-3} \mu C$$

$$V' = \frac{Q'}{C'} = \frac{Q}{C/2} = 2V$$

$$U' = \frac{1}{2} C' V'^2 = \frac{1}{2} \frac{C}{2} (2V)^2$$

$$U' = CV^2 = 8.84 \times 10^{-6} J$$

$$E' = \frac{V'}{d'} = \frac{2V}{2d} = E = 5000\ V/m$$

DIELECTRICS AND PLOLARISTAION

Non-conducting material are called dielectric. Dielectric materials are of two types (i) non-polar dielectric (ii) Polar dielectric

(i) Non-polar dielectric

In a non-polar molecule, **the centre of the positive and negative charge coincides with each other.** Hence they do not possess a permanent dipole moment. Now when it is placed in a uniform electric field, these centres are displaced in mutually opposite directions. Thus an electric dipole is induced in it or molecule is said to be polarized. If extent of electric field is not very strong, it is found that this dipole moment of molecule is proportional to external electric field $\vec{E}_0$

$$\therefore \vec{p} = \alpha \vec{E}_0$$

Where α is called the polarisability of the molecule

The units of α is C^2 m N^{-1}

(ii) Polar Molecule

A polar molecule possesses a permanent dipole moment *p,* but such dipole moments of different molecules of the substance are randomly oriented in all possible directions and hence the resultant dipole moment of the substance becomes zero.

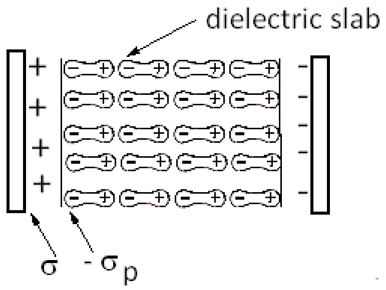
On applying an external electric field a torque acts on every molecular dipole. Therefore, it rotates and tries to become parallel to the electric field. Thus a resultant dipole moment is produced. In this way the dielectric made up of such molecules is said to be polarised.

Moreover, due to thermal oscillations the dipole moment also gets deviated from being parallel to electric field. If the temperature is T, the dipoles will be arranged in such an Equilibrium condition that the average thermal energy per molecule ($\frac{3}{2} k_B T$) balances the potential energy of dipole ($U = -\vec{p}.\vec{E}$) in the electric field. At 0 K temperature since the thermal energy is zero, the dipoles become parallel to the electric field.

EFFECT OF DIELECTRIC ON CAPACITANCE

Capacitance of a parallel plate capacitor in vacuum is given by charge density on plates is σ

$$C_0 = \frac{\varepsilon_0 A}{d}$$



Consider a dielectric inserted between the plates of capacitor, the dielectric is polarized by the electric field, the effect is equivalent to two charged sheets with surface charge densities σ_P and $-\sigma_P$. The electric field in the dielectric will be

$$E = \frac{\sigma - \sigma_P}{\varepsilon_0}$$

So the potential difference across the plates is

$$V = Ed = \frac{\sigma - \sigma_P}{\varepsilon_0}\mathrm{d}$$

For linear dielectric, we expect σ_P to be proportional to electric field due to plates E_0
Thus $(\sigma - \sigma_P)$ is proportional to σ and can be written as
$(\sigma - \sigma_P) = \frac{\sigma}{K}$
Where K is a constant characteristic of the dielectric. Then we have

$$V = \frac{\sigma}{K\varepsilon_0}\mathrm{d} = \frac{\mathrm{Qd}}{\mathrm{A}K\varepsilon_0}$$

The capacitance C, with dielectric between the plates , is then

$$C = \frac{Q}{V} = \frac{K\varepsilon_0 A}{d} = \mathrm{KC_0}$$

The product $K\varepsilon_0$ is called the **permittivity** of the medium denoted by ε , $\varepsilon = K\varepsilon_0$
Or $K = \frac{\varepsilon}{\varepsilon 0}$
For vacuum K = 1 and for other dielectric medium K >1.

INTRODUCTION OF A DIELECTRIC SLAB OF DILECTRIC CONSTANT K BETWEEN THE PLATES

(a) When battery is disconnected

Let q_0, C_0, V_0, E_0 and U_0 represents the charge, capacity, potential difference, electric field and energy associated with charged air capacitor respectively. With the introduction of a dielectric slab of dielectric constant *K* between the plates and the battery disconnected.

(i) Charge remains constant, i.e., $q = q_0$, as in an isolated system charge is conserved.

(ii) Capacity increases, i.e., $C = KC_0$, as by the presence of a dielectric capacity becomes *K* times.

(iii) Potential difference between the plates decreases,

$$V = \frac{q}{C} = \frac{q_0}{KC_0} = \frac{V_0}{K}\ [\because q = q_0\ and\ C = KC_0]$$

$$V = \frac{V_0}{K}$$

(iv) As Field between the plates decreases,

$$E = \frac{V}{d} = \frac{V_0}{Kd} = \frac{E_0}{K}\left[As\ V = \frac{V_0}{K}\right]$$

$$E = \frac{E_0}{K}$$

(v) Energy stored in the capacitor decreases

$$U = \frac{q^2}{2C} = \frac{q_0^2}{2KC_0} = \frac{U_0}{K}\quad [as\ q = q_0\ and\ C = KC_0]$$

(b) When battery remains connected (potential held constant)
(i) Potential remains constant i.e $V = V_0$
(ii) Capacity increases i.e $C = KC_0$
(iii) Charge on the capacitor increases i.e $q = Kq_0$
(iv) Electric field remains unchanged $E = E_0$
(v) Energy stored in the capacitor increases

Solved numerical

Q) A parallel plate capacitor has plates of area $4m^2$ separated by a distance of 0.5 mm. The capacitor is connected across a cell of emf 100V
(a) Find the capacitance, charge and energy stored in the capacitor
(b) A dielectric slab of thickness 0.5mm is inserted inside this capacitor after it has been disconnected from the cell. Find the answers to part (a) if K = 3

<u>Solution:</u>

Part a

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{8.85\times 10^{12}\times 4}{0.5\times 10^{-3}} = 7.08\times 10^{-2}\,\mu F$$

$$Q_0 = C_0 V_0 = 7.08\times 10^{-2}\times 100 = 7.08\mu C$$

$$U_0 = \frac{1}{2}C_0 V_0^2 = 3.54\times 10^{-4} J$$

Part b) As the cell has been disconnected Q = Q_0

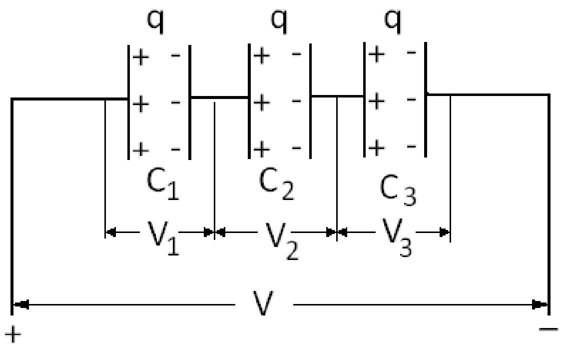
$$C = \frac{K\varepsilon_0 A}{d} = KC_0 = 0.2124\mu F$$

$$V = \frac{Q}{C} = \frac{Q}{KC_0} = \frac{V_0}{K} = \frac{100}{3}V$$

$$U_0 = \frac{Q_0^2}{2C} = \frac{Q_0^2}{2KC_0} = \frac{U_0}{K} = 118\times 10^{-6} J$$

GROUPING OF CAPACIORS

SERIES COMBINATION OF CAPACITORS



Capacitor are said to be connected in series if charge on each individual capacitor is same. In this situation

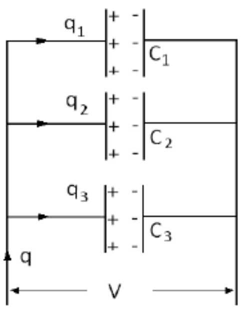
$V = V_1 + V_2 + V_3$

If C is the effective capacitance of combination then we know that V = q/C and $V_1 = q/C_1$, $V_2 = q/C_2$, $V_3 = q/C_3$

$$\frac{q}{C} = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}$$

OR

PARALLEL COMBINATION OF CAPACITORS



When capacitors are connected in parallel, the potential difference V across each is same and the charge on C_1, C_2 and C_3 is different i.e q_1, q_2 and q_3

The total charge q is given by

$q = q_1 + q_2 + q_3$

potential across each capacitor is same thus $q_1 = C_1V$

$q_2 = C_2V$ and $q_3 = C_3V$

If C is equivalent capacitance then

q = CV

Thus $CV = C_1V + C_2V + C_3V$

Or $C = C_1 + C_2 + C_3$

Charge on capacitor

Let two capacitors are connected in parallel, let Q be the total charge , Let Q_1 be the charge on capacitor of capacity C_1 and Q_2 be the charge on capacitor of capacity C_2

Since both capacitor have same potential from formula V = Q/C we get

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2}$$

$$\frac{Q_1+Q_2}{Q_2}=\frac{C_1+C_2}{C_2}$$

Since Q = Q_1+Q_2

$$\frac{Q}{Q_2}=\frac{C_1+C_2}{C_2}$$

$$Q_2=\frac{C_2}{C_1+C_2}Q$$

Similarly

$$Q_1=\frac{C_1}{C_1+C_2}Q$$

Solved numerical

Q) Two capacitors of capacitance C_1 = 6 μF and C_2 = 3 μF are connected in series across a cell of emf 18V. Calculate
(i) the equivalent capacitance
(ii) the potential difference across each capacitor
(iii) the charge on each capacitor

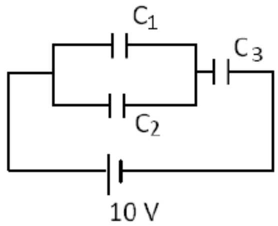
<u>Solution</u>

(i) $C=\frac{C_1C_2}{C_1+C_2}=\frac{6\times 3}{6+3}=2\mu F$

(ii) $V_1=\left(\frac{C_2}{C_1+C_2}\right)V=\left(\frac{6}{6+3}\right)\times 18=12V$

(iii) In series combination charge on each capacitor is same Q = CV = $2\times 10^{-6}\times 18=36$ μC

Q) In the circuit shown the capacitors are C_1 = 15μF, C_2 = 10μF and C_3 = 25μF. Find



(i) the equivalent capacitance of the circuit
(ii) the charge on each capacitor
(iii) the potential difference across each capacitor

<u>Solution</u>

(i) C_1 and C_2 are parallel thus C_{12} = 15+10 = 25 μF

This C_{12} is in series with C_3 thus

$$\frac{1}{C}=\frac{1}{25}+\frac{1}{25}=\frac{2}{25}$$

C = 12.5 μF

(ii) Q is the total charge supplied by the cell = CV = (12.5 × 10) C

Charge on C_1 = Q_1 = $\left(\frac{C_1}{C_1+C_2}\right)Q=\left(\frac{15}{15+10}\right)\times 125=75\mu C$

Charge on C_2 = Q_2 = $\left(\frac{C_2}{C_1+C_2}\right)Q=\left(\frac{10}{15+10}\right)\times 125=50\mu C$

Charge on C_3 = Q = 125 μC

(iii) p.d across C_1 = V_1 = Q_1/C_1 = 75/15 = 5 V

p.d across $C_2 = V_2 = V_1 = 5$ V
p.d across $C_3 = V_3 = Q_3/C_3 = 125/25 = 5$V

REDISTRIBUTION OF CHARGES

If there are two spherical conductors of radius R_1 and R_2 at potential V_1 and V_2 respectively Far apart from each other (so that charge on one does not affect the other). The charges on them will be
$Q_1 = C_1V_1$ and $Q_2 = C_2V_2$
The total charge on the system is $Q = Q_1 + Q_2$
The capacitance $C = C_1 + C_2$
Now if they are connected through a wire, charge will flow from conductor at higher potential to lower potential till both acquires same potential let charge on first becomes q_1 and charge on second sphere becomes q_2
Since potential is same

$$\frac{q_1}{C_1} = \frac{q_2}{C_2}$$

We know that capacity of sphere $C = 4\pi\varepsilon_0 R$. Thus $C \propto R$

$$\frac{q_1}{R_1} = \frac{q_2}{R_2}$$

$$\frac{q_1}{q_2} = \frac{R_1}{R_2}$$

$$\frac{q_1 + q_2}{q_2} = \frac{R_1 + R_2}{R_2}$$

But $Q = q_1 + q_2$

$$\frac{Q}{q_2} = \frac{R_1 + R_2}{R_2}$$

$$q_2 = \frac{R_2}{R_1 + R_2} Q$$

similarly

$$q_1 = \frac{R_1}{R_1 + R_2} Q$$

Solved numerical

Q) Two isolated metallic solid spheres of radius R and 2R are charged such that both of these have same charge density σ. The spheres are located far away from each other and connected by a thin conducting wire. Find the new charge density on the bigger sphere

Solution: given both spheres have same charge density thus
$Q_1 = 4\pi R^2 \sigma$. And $Q_2 = 4\pi(2R)^2 \sigma$

Total charge on both spheres Q = 20 $\pi R^2 \sigma$

Now in sharing, charge is shared in proportion to capacity i.e. radius, so charge on the bigger spheres

$$q_2 = \frac{R_2}{R_1+R_2}Q = \frac{2R}{R+2R}Q = \frac{2Q}{3} \qquad \text{eq(1)}$$

So charge density on the bigger sphere after sharing

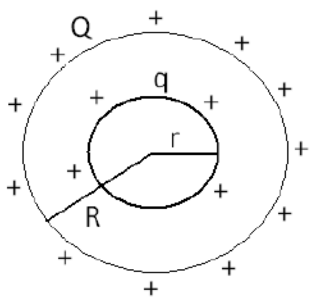
$$\sigma_2' = \frac{q_2}{4\pi(2R)^2} = \frac{2Q/3}{16\pi(R)^2} = \frac{Q}{24\pi(R)^2}$$

Putting the value of Q from equation (i) we get

$$\sigma_2' = \frac{20\pi R^2\sigma}{24\pi R^2} = \frac{5\sigma}{6}$$

VAN DE GRAFF GENERATOR

Principle:



Suppose there is a positive charge Q, on an insulated conducting spherical shell of radius R, as shown in figure. At the centre of this shell, there is a conducting sphere of radius r (r<R), having a charge q.

Here electric potential on the shell of radius R is,

$$V_R = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}+\frac{q}{R}\right)$$

And the electric potential on the spherical shell of radius r is,

$$V_r = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}+\frac{q}{r}\right)$$

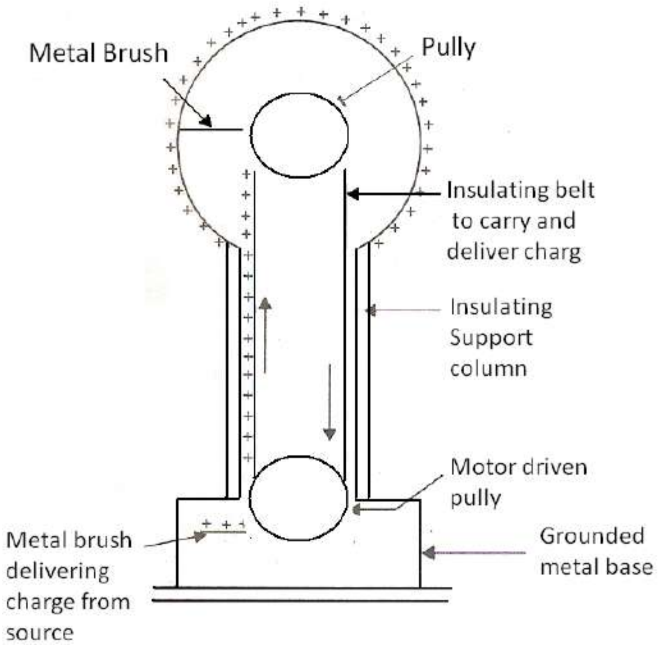
It is clear from these two equations that the potential on the smaller sphere is more and the potential difference between them is

$$V_r - V_R = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}+\frac{q}{r}\right) - \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}+\frac{q}{R}\right)$$

$$V_r - V_R = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{1}{R}\right]$$

Hence if the smaller sphere is brought in electrical contact with bigger sphere then the charge goes from smaller to bigger sphere. Thus charge can be accumulated to a very large amount on the bigger sphere and there by its potential can be largely increased

Construction



As shown in the figure a spherical shell of a few meter radius, is kept on an insulated support, at a height of a few meters from the ground.
A pulley is kept at the centre of the big sphere and another pulley is kept on the ground. An arrangement is made such that a non-conducting belt moves across two pulleys. Positive charges are obtained from a discharge tube and are continuously sprayed on the belt using a metallic brush (with sharp edges) near the lower pulley. This positive charge goes with the belt towards the upper pulley.
There it is removed from the belt with the help of another brush and is deposited on the shell (because the potential on the shell is less than that of the belt on the pulley.) Thus a large potential difference (nearly 6 to 8 million volt) is obtained on the big spherical shell.
Uses : With the help of this machine, a potential difference of a few million (1 million = 10^6 = ten lac) volt can be established. By suitably passing a charged particle through such a high potential difference it is accelerated (to very high velocity) and hence acquires a very high energy ($\frac{1}{2}mv^2$). Because of such a high energy they are able to penetrate deeper into the matter. Therefore, fine structure of the matter can be studied with the help of them.

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ELECTRIC CURRENT

Electric current:

The net amount of charge flowing through a cross section in unit time is defined as current.

Mathematical Equation: If rate of flow of charge is independent of time, then the electric current is said to be steady then I = Q/t

If rate of flow of charge varies with time, the current at any time i.e. instantaneous current is given by I = dQ/dt .

If n be the number of conduction electrons crossing cross –section in time t then I = ne/t .

SI unit of current is ampere.

One ampere is defined as If one coulomb of charge flows across any of its cross-section in one second.

Note: Electric current has direction as well as magnitude but it is not a vector quantity. This is because currents do not add like vectors.

Solved Numerical

Q) An electric device sends out 78 coulombs of charge through a conductor in 6 seconds. Find current flow

Solution

Given Q = 78C, time of flow t = 6s

The current I = Q/t = 78/6 = 13A

Q) What is the quantity of electricity required to provide a current o 10A for one hour

Solution

Given current I = 10A, time of flow t = 1 hour = 3600s

The quantity of electricity = amount of charge flowing

Q = It

Q = (10)(3600) = 36000C

Q) The current through a wore varies with time as $I = I_O + \alpha t$, where I_o = 10A and $\alpha = 4$ A/s. Find the charge that flows across a cross-section of the wire in first 10 seconds

Solution:

Current $I = dq/dt = I_O + \alpha t$

$dq = (I_O + \alpha t)dt$

Integrating on both sides

$$\int dq = \int_{t=0}^{t=10} (I_0 + \alpha t)\, dt$$

$$q = \left[I_0 t + \frac{\alpha t^2}{2} \right]_{t=0}^{t=10}$$

$q = 10I_O + 50\alpha$

substituting I_O = 10 and α = 4

q= 10(10) + 50(4) = 300C

Ohm's Law:

The current which flows in conductor is proportional to the potential difference which causes its flow at constant temperature and pressure.

Thus V = IR.

where the constant R is the resistance of conductor.

Unit of resistance is Ohm denoted as Ω .

Resistance:

Formula for resistance is $R = \rho \frac{l}{A}$

Here l is length of conductor ., A is area of cross-section of conductor and

ρ is resistivity of conductor .

Resistivity of conductor is independent of size and shape of conductor, it depends on material of conductor and temperature and pressure.

Unit of resistivity is Ohm-m.

Inverse of resistivity is conductivity denoted by σ unit is $(\text{Ohm-m})^{-1}$

Solved Numerical

Q) Calculate the electrical resistivity of the material of wire of length 200cm, area of cross-section2cm^2 and of resistance $5\times10^{-4}\Omega$

Solution

$$R = \rho \frac{l}{A}$$

$$\rho = \frac{RA}{l} = \frac{5 \times 10^{-4} \times 2 \times 10^{-4}}{2} = 5 \times 10^{-8} \Omega m$$

Q) A wire of resistance 5Ω is drawn out so that its new length is three times its original length. Find the resistance of the longer wire. What would be the effect on resistivity?

Solution:

Resistivity will not change, as it do not depend on length and cross-sectional area

Resistance of longer wire can be calculated as follows

Since volume of wire is not changed therefore

Al = A'l'

Now l' = 3l

Al = (3l)l' thus A' = A/3

New resistance

$$R' = \rho \frac{l'}{A'} = \rho \frac{3l}{A/3} = \rho \frac{l}{A} 9 = 9R$$

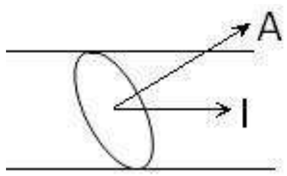
Old length resistance R = 5 Ω, thus new resistance = 45 Ω

Current density:

From equation V = IR

$$V = I\rho\frac{l}{A} = \rho J l$$

Here J is current density defined as current per unit area (taken normal to the current) J = I/A. SI unit of current density is A/m^2.



If there is angle between direction of current and area vector of cross section is θ Then current density J = I / Acosθ

Relation between Current density and Electric field .

For formula V = IR

$$V = I\rho\frac{l}{A} = \rho J l$$

$$\frac{V}{l} = \rho J$$

But $E = \frac{V}{l}$

$$E = \rho J \quad \text{or} \quad J = \sigma E$$

Solved Numerical

Q) An electron beam has an aperture of 10^{-6} m^2. The total number of electrons moving through any perpendicular cross-section per second is 6.0×10^{16}. Calculate the current density of beam.

Solution:

$$J = \frac{I}{A} = \frac{Q}{At} = \frac{ne}{At}$$

$$J = \frac{6\times10^{16}\times1.6\times10^{-19}}{10^{-6}\times1} = 9.6\times10^{3}Am^{-2}$$

Q) A current of 4.8 ampere is flowing in a copper wire of cross sectional area 3×10^{-4}m^2. Find the current density of in the wire

Solution:

$$J = \frac{4.8}{3\times10^{*4}} = 1.6\times10^{4}Am^{-2}$$

Origin of resistivity:

In metallic conductors, the electrons in the outer shells are less bounded with the nucleus. Due to thermal energy at room temperature, such valence electrons are liberated from atom leaving behind positively charged ions. These ions are arranged in a regular geometric arrangement on lattice points. These liberated electrons collide with the ions. Or constantly gets scattered from its path causing resistivity of metallic conductors.

Drift of electrons:

In absence of electric field electrons move in randomly after colliding with ions and direction and velocities of electrons after collisions is such that sum of velocities is zero and net charge passing through any cross section is zero causing no electric current.
In presence of electric field (E) electrons experiences electric force of magnitude Ee in the direction opposite to the direction of electric field thus the acceleration of electron is opposite to direction of electric field. Now F = ma thus acceleration of electron a = Ee/m this acceleration is momentary and becomes zero after collision, since electrons are continuously colliding with ions. And electrons get accelerated again and process goes of repeating.
As a result electrons are dragged in the opposite to the electric field. Now average time period between two successive collisions is known as relaxation time τ. And corresponding average velocity of electrons is known as drift velocity v_d.
Now $v_d = a\tau$

$$v_d = \frac{Ee}{m}\tau$$

Relation between drift velocity and current density

Let us consider a cylindrical conductor of cross section A . Let E be the electric filed exists in conductor.
If v_d is drift velocity of electrons then volume of the electrons passing through a cross-section in one second = v_dA .
If n is the number of electrons per unit volume then nv_dA is the number of electron in passing through a cross-section in one second.
Net charge passing through a cros-section in one second = $neAv_d$ = I .
Now I = $neAv_d$.
Thus J = nev_d

Relation between resistivity and relaxation time.

We know that J = σ E, here σ is conductivity
And J =nev_d
nev_d = σ E
Substituting value of $v_d = \frac{Ee}{m}\tau$ in above equation we get

$$\sigma E = ne\frac{Ee}{m}\tau$$

$$\sigma = \frac{ne^2}{m}\tau$$

Since σ=1/ρ

$$\rho = \frac{m}{ne^2\tau}$$

Here we have assumed that τ and n are constant.

On increasing temperature of conductor n does not change appreciable. The oscillations of ions increases with temperature and becomes more erratic. As a result, the relaxation time (τ) decreases and resistivity of conductor increases with increase in temperature

Solved Numerical

Q) Estimate the average drift speed of conduction electrons in a copper wire of cross sectional area 1.0×10^{-7} m^2 carrying a current of 1.5A. Assume that each copper atom contributes roughly one conduction electron. The density of copper is 9.0×10^3 kgm^{-3} and its atomic mass is 63.5u.

Solution

Given : weight of 1m^3 volume of copper is 9.0×10^3 kg or 9.0×10^6 gm:

Now 63.5 gm of copper contains N_A (Avogadro's number 6.23×10^{23}) of atoms Thus 9.0×10^6 gm of copper contains

$$\frac{9.0\times10^6\times6.23\times10^{23}}{63.5} = 8.8\times10^{28} atoms/m^3$$

each copper atom is assumed to contribute one atom one electron .

So number density n = 8.8×10^{28} /m^3

A = 1.0×10^{-7} m^2, I = 1.5A , e = 1.6×10^{-19}C

from formula for drift velocity

$$v_d = \frac{I}{neA}$$

$$v_d = \frac{1.5}{8.8\times10^{28}\times1.6\times10^{-19}\times1.0\times10^{-7}} = 1.065\times10^{-3} m/s$$

V_d = 1.065 mm/s

Q) A potential difference of 100V is applied to the ends of a copper wire one metre long. Calculate the average drift velocity of the electrons. Compare it with thermal velocity at 27^{O}C. conductivity of copper = 5.81×10^7 Ω^{-1}m^{-1}, n = 8.5×10^{28}

Solution :

E = V/l = 100/1 = 100V/m

J =σE

Now J = neV_d

σE = neV_d

$$v_d = \frac{\sigma E}{ne} = \frac{5.81\times10^7\times100}{8.5\times10^{28}\times1.6\times10^{-19}} = 0.43 m/s$$

Thermal velocity

$$v_{rms} = \sqrt{\frac{3k_BT}{m}}$$

Mass of electron is 9.1×10^{-31}kg, k_B = 1.38×10^{-23}J/K

$$v_{rms} = \sqrt{\frac{3\times1.398\times10^{-23}\times300}{9.1\times10^{-31}}} = 1.17\times10^5 m/s$$

$$\frac{v_d}{v_{rms}} = \frac{0.43}{1.17 \times 10^5} = 0.37 \times 10^{-5}$$

Q) Aluminum wire of diameter 0.25cm is connected in series with a copper wire of diameter 0.16cm. A current of 10Amp is passed through them.
Find (a) current density in aluminum wire (b) drift velocity of electrons in copper wire.
Given number of free electrons per unit volume of copper wire = 10^{29}
Solution:

a) Radius of aluminum wire = 1.25×10^{-3} m

Cross sectional area of aluminum = $\pi r^2 = (3.14) \times (1.25\times10^{-3})^2 = 4.9\times10^{-6} m^2$
Current density in Aluminum J= I/A

$J = \frac{10}{4.9\times10^{-6}} = 2.04 \times 10^6 Am^{-2}$

b) Radius of copper = 0.8×10^{-3} m
Cross sectional area = $(3.14) \times (0.8\times10^{-3})^2 = 2.01\times10^{-6}$ m^2

$$v_d = \frac{I}{neA} = \frac{10}{10^{29} \times 1.6 \times 10^{-19} \times 2.01 \times 10^{-6}} = 3.1 \times 10^{-4} m/s$$

Mobility

In case of conductors free electrons are mobile charge carriers. In case of electrolyte positive and negative ions are mobile charge carriers. In case of semiconductors holes and electrons are mobile charge carriers.
Mobility is defined as drift velocity of charge per unit electric field

$$\mu = \frac{|v_d|}{E}$$

S.I. unit of mobility is m^2/Vs and is 10^4 of mobility in practical units (cm^2 /Vs)
By substituting $v_d = \frac{Ee}{m}\tau$ in above equation we get

$$\mu = \frac{e\tau}{m}$$

Limitations of Ohm's Law

(1) V-I relations are not linear Example: diode transistor
(2) The relation between V and I depends on the sign of V. If we change the polarity of supply voltage magnitude of current changes
(3) The relation between V and I is not unique. There may be more values of potential for same current I. example tunnel diode, material Ga,As

Classification of materials based on resistivity :

Materials are classified as conductors, semiconductors and insulators depending on their resistivity.

Conductors have resistivity of order $10^{-8}\Omega m$ to $10^{-6}\Omega m$

Insulator have resistivity 10^{18} times greater than metals or more

Resistivity of semiconductors decreases with increase in temperature because covalent bond between adjacent atoms breaks which creates free electrons and holes causing decrease in resistivity. Or its conductivity increases.

Commercially produced resistors for domestic use or in laboratory are of two major types: wire bound resistors and carbon resistors

Wire bound resistors are made by winding the wires of an alloy viz. manganin, constantan, nichrome or similar ones. These alloys are relatively insensitive to temperature. These resistances are typically in the range of a fraction of an ohm or to a few hundred ohms

Resistors, in the higher range are made mostly from carbon. Carbon resistors are compact, inexpensive and thus find extensive use in electronic circuit. Carbon resistors are small in size and hence their values are given using colour code

Temperature dependence of resistivity:

The resistivity of materials is found to be dependent on temperature. Over a limited range of temperatures, that is not too large, the resistivity of metallic conductor is approximately given by

$$\rho_T = \rho_0[1 + \alpha(T - T_0)]$$

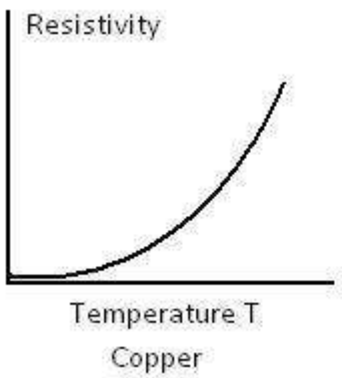
Where ρ_T is the resistivity at temperature T and ρ_0 is resistivity at temperature T_0, α is called the temperature co-efficient of resistivity, dimension of α is $[\text{temperature}]^{-1}$ units are $(^oC)^{-1}$ or $(K)^{-1}$

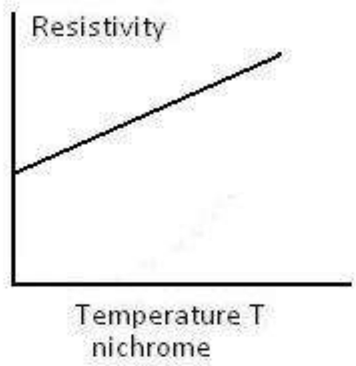
Note that temperature coefficient of Carbon and, semiconductors germanium and silicon are negative, indicating with increase in temperature resistivity decreases.

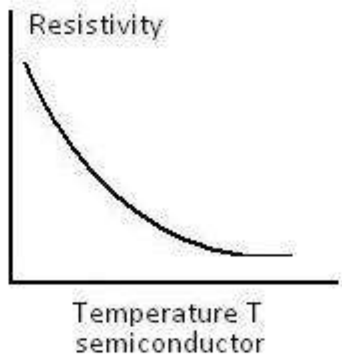
Equation of resistivity shows a linear relation between temperature and resistivity however graph of resistivity – temperature is not linear for copper.

Since alloys like manganin, constantan, nichrome are relatively insensitive to temperature there graph is straight line intercepting on Y axis

Resistivity of semiconductors decreases with temperature there graph is non linear and have negative slope indicating resistivity decreases with increase in temperature







Solved Numerical

Q) A metal wire of diameter 2mm and of length 100m has a resistance of 0.5475 ohm at 20°C and 0.805 ohm at 150 °C. Find the values of (i) temperature coefficient of resistance, (ii) its resistance at 0 °C (iii) its resistivity at 0 °C and 20 °C

Solution:

(i) If R_{20} and R_{150} be the resistance at temperature 20 °C and 150 °C respectively and α be the temperature coefficient of resistance

$R_{20} = 0.5475 = R_0 (1 +\alpha\times 20)$ – eq(1)

$R_{150} = 0.805 = R_0 (1 +\alpha\times 150)$ –eq(2)

Taking ratio of above equations

$$\frac{0.5475}{0.805} = \frac{1 + \alpha \times 20}{1 + \alpha \times 150}$$
$$0.5475(1 + \alpha \times 150) = 0.805(1 + \alpha \times 20)$$
$$0.805 - 0.5475 = 0.5475(\alpha \times 150) - 0.805(\alpha \times 20)$$
$$\alpha = \frac{0.805 - 0.5475}{0.5475 \times 150 - 0.805 \times 20} = 3.9 \times 10^{-3}$$

$\alpha = 3.9\times 10^{-3}\ {}^{o}C^{-1}$

(ii) By substituting value of α in equation (1) we get $R_0 = 0.5079$ ohm

(iii) $R_0 = \frac{\rho_0 L}{A}$

$$0.509 = \frac{\rho_0(100)}{\pi(1 \times 10^{-3})^2}$$

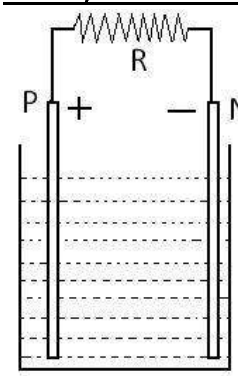
$\rho_0 = 1.596\times 10^{-8}$ ohm m

$\rho_{20} = \rho_0 (1 + \alpha\times 20)$

$\rho_{20} = 1.596 \times 10^{-8} [1 +(3.9\times 10^{-3}\times 20)]$

$\rho_{20} = 1.720\times 10^{-8}$ ohm.m

Cells, emf and internal resistance



Cell is a simple device which maintains a steady current in an electrical circuit. Cell consists of electrolyte and electrodes. Electrodes are metallic plate dipped in electrolyte. Due to exchange of electrodes between electrolyte and electrode, one electrode develop positive potential deference (V_+) between electrolyte and electrode and such electrode acts as positive terminal (P). While other electrode develop negative potential difference (V_-) and acts as negative terminal (N). When there is no current, the electrolyte has the same potential throughout, so that the potential difference between P and N is $V_+ -(-V_-) = V_+ + V_-$.

This difference is called electromotive force (emf) of the cell and is denoted by ε. Thus $\varepsilon = V_+ + V_- > 0$

emf is work done by non-electrical force in moving a positive charge from negative terminal to positive terminal of battery.

Working of cell:

Consider a wire of resistance R connected across the two terminals of the battery as shown in figure. The electric field is established in wire. As a result positive charge will move from higher potential (P) to lower potential (N) through external resistance R.
The energy of the positive charge is consumed to overcome the resistance of wire. As it reaches the negative terminal N, its energy becomes zero, it is as per law of conservation of energy.
Now due to non-electrical force positive charge on the negative electrode is moved towards the positive electrode inside the cell. Thus work is done by the non-electrical force and potential energy of positive charge increases when it reaches positive electrode. Again it flows through the external resistance R and process goes on repeating
Now if external resistance is not connected then positive charge gets accumulated on the positive electrode and produces an electric field in the direction from positive electrode to negative electrode so that direction of electric force is opposite to non-electric force. When force due to electric field becomes equal to non-electric force flow of positive charge from negative terminal to positive terminal stops and potential across terminal is ε
Now during the discharge of cell positive charge has to overcome the resistance of electrolyte, such resistance is called internal resistance denoted by r.

Terminal voltage

When positive charge flows in electrolyte they have to overcome internal resistance t. if I is the current then energy lost in electrolyte is Ir. As a result voltage across P and N is less than ε in the open circuit condition. The net energy per unit charge will be (ε – Ir) .
Thus , during the flow of current potential across between two terminal P and N is
V = ε – Ir. This potential difference is called terminal voltage.
If ε >> Ir , internal resistance is neglected
We also observe that since V is the potential difference across resistance R, from Ohm's law V = IR
Thus IR = ε – Ir

$$I = \frac{\varepsilon}{R + r}$$

Maximum current can be drown from cell is R = 0. However, in most of the cells maximum allowed current is much lower to prevent permanent damage to cell.
Internal resistance of electrolyte cell is very small. Thus electrolyte cell gives large value of current. Internal resistance of dry cell is higher thus it gives low current.

Electrical Energy, Power

Consider a conductor with endpoints A and B, in which a current I is flowing from A and B. The electric potential at A and B are denoted V(A) and V(B) respectively.
Since current is flowing from A to B and potential difference across AB is
V = V(A) – V(B) >0
In a time interval Δt, an amount of charge $\Delta Q = I\,\Delta t$ travels from A to B.
The potential energy of charge at A = ΔQV(A) and

The potential energy of charge at B = $\Delta QV(B)$
Thus change in potential energy
ΔU = Final potential energy – Initial potential energy
$\Delta U = \Delta QV(B) - \Delta QV(A) = \Delta Q[V(B) - V(A)]$
$\Delta U = -\Delta QV = -IV\ \Delta t < 0$
If charges moved without collisions through the conductor, the kinetic energy would also change so that the total energy is unchanged. Conservation of total energy would then imply that.
$\Delta K = -\Delta U$
That is $\Delta K = IV\ \Delta t > 0$
But when charges flow through the conductor they move with the steady drift velocity, because of collision with ions and atoms during transit. During collisions, energy gained by the charges thus is shared with the atoms. The atoms vibrate more vigorously, as a result conductor heats up. The amount of energy dissipated as heat in the conductor during the time interval is

$$\Delta W = IV\ \Delta t$$

The energy dissipated per unit time is the power $P = \Delta W/\Delta t$ and we have

$$P = IV$$

Using Ohm's law $V = IR$

$$P = I^2R = V^2/R$$

Power loss is also called as "Ohmic loss"

Power loss in transmission lines

Consider a device of resistance R, to which power is to be delivered via transmission station. Power of device is $P = VI$ or $I = P/V$
Let resistance of transmission cable is R_C. The power dissipated in the connecting wires, which is wasted is $P_C = I^2R_C$

$$P_C = \left(\frac{P}{V}\right)^2 R_C$$

Transmission cables from power stations are hundreds of miles and there resistance R_C is considerable. To reduce P_C voltage V is increased to very large value. Using such high voltage is dangerous, thus at user end voltage is reduced this can be achieved by transformers.

Solved Numerical

Q) What is the resistance of the filament of bulb rated at (100W – 250V)? What is the current through it when connected to 250V line? What will be power if it is connected to a 200V line?
Solution

$$P = \frac{V^2}{P}$$

$$R = \frac{V^2}{P} = \frac{250 \times 250}{100} = 625\Omega$$

The current through the lamp = P/V = 100/250 = 0.4A
The power of the lamp when it is connected to a 200V line is

$$P = \frac{V^2}{R} = \frac{200 \times 200}{625} = 64W$$

Q) Forty electric bulb are connected in series across a 220V supply. After one bulb is fused the remaining 39 are connected again in series across same supply. In which case will there be more illumination and why?

Solution

Let r be the resistance of each bulb and 40 bulb is series have a resistance of 40rΩ. When connected across a supply voltage V, the power of the system with 40 bulb will be

$$P_{40} = \frac{V^2}{40r}$$

When one bulb is fused, the resistance of the remaining 39 bulb in series =39r and the power of the system when connected to the same supply

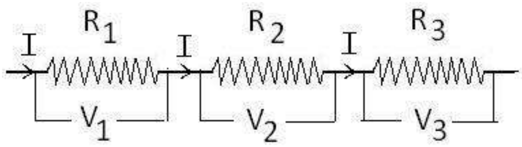
$$P_{39} = \frac{V^2}{39r}$$

It is clear that $\frac{V^2}{39r} > \frac{V^2}{40r}$

Therefore power of 39 bulb in series is greater.

Series Combination of resistors

Two resistors are said to be connected in series, if only one end of their ends points are joined . In series connection current flowing through each resistor remains same, while sum of potential drop across resistors is equal to potential drop across combination



Consider three resistor R_1R_2 and R_3 are connected in series as shown in figure.

Potential difference across R_1 is $V_1 = IR_1$

Potential difference across R_2 is $V_2 = IR_2$

Potential difference across R_3 is $V_3 = IR_3$

The potential difference across combination is $V = V_1 + V_2 + V_3$

$V = I (R_1 + R_2 + R_3)$

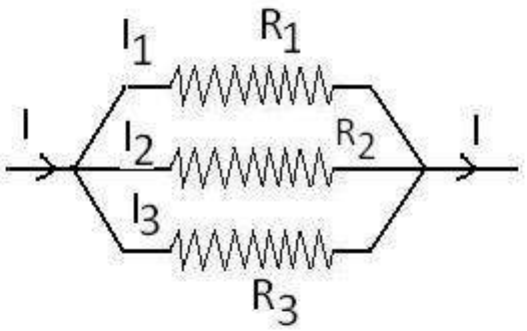
If R_{eq} is equivalent resistance and V is potential difference across combination then

$V = I (R_{eq})$

Thus $R_{eq} = R_1 + R_2 + R_3$

IF n resistance are connected in series the $R_{eq} = R_1 + R_2 + R_3 + \ldots\ldots.. + R_n$

Parallel combination of resistors



Two or more resistors are said to be connected in parallel if one end of all the resistors is joined together and similarly the other ends are joined together as shown in figure.

In parallel combination current gets divided depending up on value of resistor, sum of current passing through the resistor is equal to current passing through combination, but potential difference across each resistor is equal to potential difference across combination

Here I_1, I_2 and I_3 are the currents passing through the resistors as shown in figure

If I is the current passing through the combination then $I = I_1 + I_2 + I_3$

By applying Ohms law we get voltage across R_1 as $V = I_1R_1$ or $I_1 = V/R_1$

Similarly $I_2 = V/R_2$ and $I_3 = V/R_3$

By substituting values of current in above equation we get

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$I = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

If combination is replaced by equivalent resistance R_{eq}, we would have from Ohm's law

$$I = V/R_{eq}$$

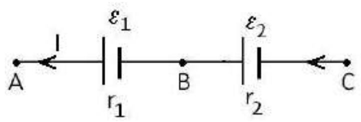
From above equations we have

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

IF n resistors are connected in parallel the equivalent resistance

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots . + \frac{1}{R_n}$$

Cells in series



Consider first two cells in series, where one terminal of the two cells is joined together leaving the other terminal in either cell free. Let ε_1 and ε_2 be the emf of the two cell having internal resistance r_1 and r_2 respectively

Let V (A), V (B), V (C) be the potentials at points A, B and C shown in figure.

Then V (A) – V (B) is the potential difference between the positive and negative terminals of the first cell.

We know that terminal voltage of first cell $V_{AB} = V(A) - V(B) = \varepsilon_1 - I r_1$

Similarly for second cell $V_{BC} = V(B) - V(C) = \varepsilon_2 - I r_2$

Hence, the potential difference between the terminals A and C of the combination is

$V_{Ac} = V(A) - V(C) = V(A) - V(B) + [V(B) - V(C)] = \varepsilon_1 - I r_1 + (\varepsilon_2 - I r_2)$

$V_{Ac} = (\varepsilon_1 + \varepsilon_2) - I(r_1 + r_2)$

If we wish to replace the combination by a single cell between A and C of emf ε_{eq} and internal resistance r_{eq}, we would have

$V_{Ac} = \varepsilon_{eq} - I r_{eq}$

Comparing the last two equations, we get $\varepsilon_{eq} = \varepsilon_1 + \varepsilon_2$ and $r_{eq} = r_1 + r_2$

If polarity of second cell is reverse then $\varepsilon_{eq} = \varepsilon_1 - \varepsilon_2$ ($\varepsilon_1 > \varepsilon_2$)

Note:

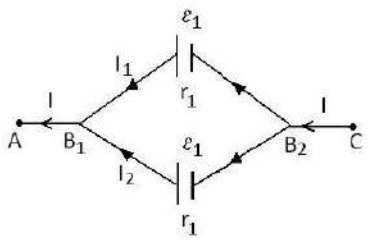
(i)The equivalent emf of a series combination of n cells is just the sum of their individual emf's, and

(ii) The equivalent internal resistance of a series combination of n cells is just the sum of their internal resistances

If n cells of emf ε having internal resistance r each connected in series. Total current I if resistance R connected across combination

$$I = \frac{n\varepsilon}{R + nr}$$

Cells in parallel



Consider a parallel combination of the cells as shown in figure I_1 and I_2 are the currents leaving the positive electrodes of the cells. At the point B_1, I_1 and I_2 flow out whereas the current I flows in B_2. Since as much charge flows in as out, we have $I = I_1 + I_2$

Let V (B_1) and V (B_2) be the potentials at B_1 and B_2, respectively. Then, considering the first cell, the potential difference across its terminals is V (B1) – V (B2). Hence

$V = V(B_1) - V(B_2) = \varepsilon_1 - I r_1$

$$I_1 = \frac{\varepsilon_1 - V}{r_1}$$

Points B_1 and B_2 are connected exactly similarly to the first cell. Hence considering the second cell, we also have

$V = V(B1) - V(B2) = \varepsilon_2 - I r_2$

$$I_2 = \frac{\varepsilon_2 - V}{r_2}$$

Combining the last three equations we get

$$I = \frac{\varepsilon_1 - V}{r_1} + \frac{\varepsilon_2 - V}{r_2} = \left(\frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2}\right) - V\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

If we want to replace the combination by a single cell, between B_1 and B_2, of emf ε_{eq} and internal resistance r_{eq}, we would have

$V = \varepsilon_{eq} - I r_{eq}$. Thus

$$\varepsilon_{eq} = \frac{\varepsilon_1 r_2 + \varepsilon_2 r_1}{r_1 + r_2}$$

$$r_{eq} = \frac{r_1 r_2}{r_1 + r_2}$$

In simpler way

$$\frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\frac{\varepsilon_{eq}}{r_{eq}} = \frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2}$$

General formula for n cells

$$\frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots . + \frac{1}{r_n}$$

$$\frac{\varepsilon_{eq}}{r_{eq}} = \frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2} + \cdots . + \frac{\varepsilon_n}{r_n}$$

If n cells of emf ε having internal resistance r each connected in parallel. Total current I if resistance R connected across combination

$$I = \frac{\varepsilon}{R + \frac{r}{m}} = \frac{m\varepsilon}{mR + r}$$

Mix grouping of cell

Let n identical cells be arranged in series and let m such rows be connected in parallel.
Total numbers of cells are nm
Emf of system = nε
Internal resistance of the system = nr/m
The current through the external resistance R

$$I = \frac{n\varepsilon}{R + \frac{nr}{m}} = \frac{mn\varepsilon}{mR + nr}$$

Solved Numerical

Q) Six cells are connected (a) in series (b) in parallel (c) in 2 rows each containing 3 cells. The emf of each cell is 1.08 V and its internal resistance is 1 ohm . Calculate the current that would flow through an external resistance of 5ohm in the three cases
Solution

(a) The cells are in series

Total emf = nε = 6×1.08 = 6.48
Total internal resistance = nr = 6×1=6 ohm
The current in circuit = Total potential / total resistance

$$I = \frac{\text{n}\varepsilon}{R + nr} = \frac{4.68}{5 + 6} = 0.589A$$

(b) The cells are in parallel
Here ε=1.08V,
internal resistances are in parallel total internal resistance is r' =r/m = r/6=1/6 ohm
The current in circuit = Total potential / total resistance

$$I = \frac{\varepsilon}{R + r/m} = \frac{1.08}{5 + \frac{1}{6}} = 0.209A$$

(c) The cells in multiple arc with n = 3 and m = 2

$$I = \frac{mn\varepsilon}{mR + nr} = \frac{6 \times 1.08}{(2 \times 5) + (3 \times 1)}$$

$$I = \frac{6.48}{13} = 0.498A$$

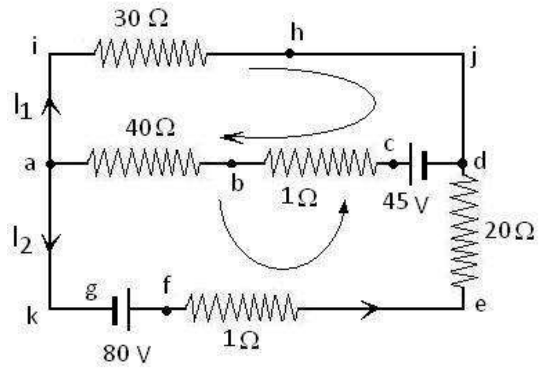
Arrangements of cells for maximum current

$$I = \frac{n\varepsilon}{R + \frac{nr}{m}} = \frac{mn\varepsilon}{mR + nr}$$

Current will be maximum if (mr+nr) should be minimum.
This happens if mR= nr
Or R = nr/m
Hence the current through the external resistance R is a maximum when it is equal to internal resistance of the battery (nr/m)

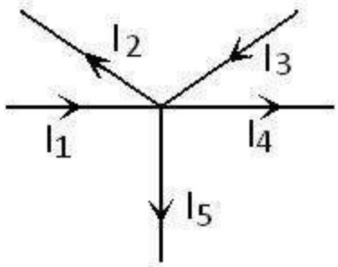
Kirchhoff's rules

Electric circuits generally consist of a number of resistors and cells interconnected sometimes in a complicated way.



Junction or Branch point: The point in network at which more than two conductors meet is called a junction or a branch point . Point a and d shown in figure are junction point .
Loop: A closed circuit formed by conductors is known as loop. As shown in figure ihjdcbai forms a closed loop

Kirchhoff's first rule or junction rule:
At any junction, the sum of the currents entering the junction is equal to the sum of currents leaving the junction.
The proof of this rule follows from the fact that when currents are steady, there is no accumulation of charges at any junction or at any point in a line. Thus, the total current flowing in, (which is the rate at which charge flows into the junction), must equal the total current flowing out.
Proof : Let Q_1, Q_2 …Q_5 be electrical charges flowing through the cross-sectional area of the respective conductors in time interval t which constitute current I_1, I_2, … I_5



Hence $Q_1 = I_1t$, $Q_2 = I_2t$, …. $Q_5 = I_5t$
It is evident from figure that the total electric charge entering the junction is $Q_1 + Q_3$, while $Q_2+ Q_4 + Q_5$ amount of charge is leaving the junction in the same interval of time
Thus $Q_1 + Q_3 = Q_2+ Q_4 + Q_5$

$\therefore$ $I_1t + I_3t = I_2t + I_4t + I_5t$
$I_1 + I_3 - I_2 - I_4 - I_5 = 0$
$\therefore \sum I = 0.$

Kirchhoff's second rule or Loop rule: *The algebraic sum of changes in potential around any closed loop involving resistors and cells in the loop is zero*
Using law of conservation of energy and the concept of electric potential any closed circuit can be analyzed. In electric circuit, the electric potential at any point is a steady circuit does not change with time.
Following are the sign convention has to be followed:

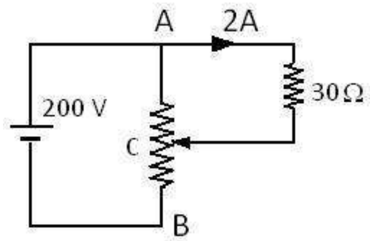
(i) If our selected path is in the direction of current then potential drop across resistor should be taken as negative else it should be taken as positive

(ii) The emf of a battery should be considered negative while moving from negative terminal of a battery to the positive terminal. The emf of battery is taken as positive while moving from positive terminal while moving from positive to negative terminal of battery

While analyzing the circuit we may get negative value of current indicates direction of current which arbitrarily chosen is opposite to the actual direction of current

Solved Numerical

Q) A potential divider of resistance 500 ohm is used to obtain variable voltages from a supply main of 200V. Determine the position of the tapping point C to get a current of 2A through a resistance of 30 ohm connected across A and C as shown



Solution:

Let the resistance of the potential divider between A and C be R ohm

The potential difference across the 30ohm resistor = $2\times 20 = 60V$

$\therefore$ The voltage drop across AC of the potential divider = 60V

Current flowing through R $I = 60/R$

Now the voltage drop across BC = 3200-60 = 140V

The current through BC is I' = 140 /(500-R)

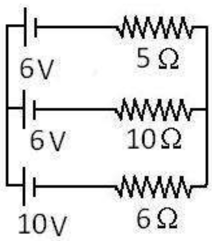
Now

$$\frac{140}{500-R} = \frac{60}{R} + 2$$

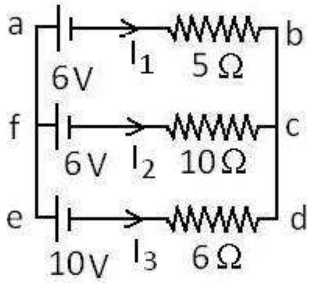
Solving R = 434.7 ohm

Hence the tapping point C lies in such position that the length AC is $\frac{434.7}{500} = 0.8694$ of the length AB

Q) Find the current in the resistors of the circuit given. The internal resistance of the battery are included in the external resistances.



Solution:



From adjacent figure

Taking loop abcfa applying kirchhoff's second law

$-6 - 5I_1 + 10I_2 + 6 = 0$

$-5I_1 + 10I_2 = 0$

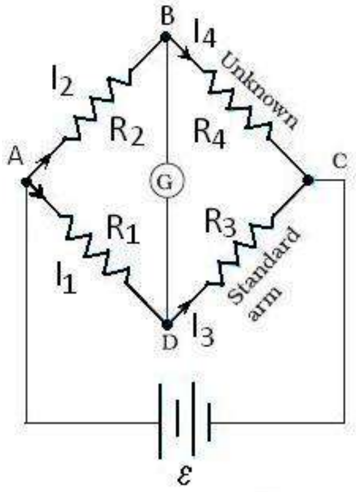
$I_1 = 2I_2$ –eq(1)

Taking loop fcbef

$-6 - 10I_2 + 6I_3 + 10 = 0$

$-10I_2 + 6I_3 + 4 = 0$
$-5I_2 + 3I_3 + 2 = 0$ --eq(2)
Applying the junction rule to junction a $I_1 + I_2 + I_3 = 0$ –eq(3)
On simplifying eq(1), (2) and (3) we get
$I_1 = (2/7)A$, $I_2 = (1/7)A$, $I_3 = (-3/7)A$
The direction of flow of I_3 is opposite to that marked in the circuit

Wheatstone bridge



The circuit shown in figure which is called the Wheatstone bridge. The bridge has four resistors R_1, R_2, R_3 and R_4 .
Across one pair of diagonally opposite points (A and C in the figure) a source is connected. This (i.e., AC) is called the battery arm.
Between the other two vertices, B and D, a galvanometer G (which is a device to detect currents) is connected. This line, shown as BD in the figure, is called the galvanometer arm.
For simplicity, we assume that the cell has no internal resistance. In general there will be currents flowing across all the resistors as well as a current I_g through G. Of special interest, is the case of a balanced bridge where the resistors are such that $I_g = 0$.
We can easily get the balance condition, such that there is no current through G. In this case, the Kirchhoff's junction rule applied to junctions D and B (see the figure) immediately gives us the relations $I_1 = I_3$ and $I_2 = I_4$. Next, we apply Kirchhoff's loop rule to closed loops ADBA and CBDC.
The first loop gives ADBA
$-I_1R_1 + 0 + I_2R_2 = 0$ ($I_g = 0$)

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}$$

and the second loop CBDC. And using $I_1 = I_3$ and $I_2 = I_4$

$I_2R_4 + 0 - I_1R_3 = 0$

$$\frac{I_1}{I_2} = \frac{R_4}{R_3}$$

Hence at balanced condition from above equation

$$\frac{R_2}{R_1} = \frac{R_4}{R_3}$$

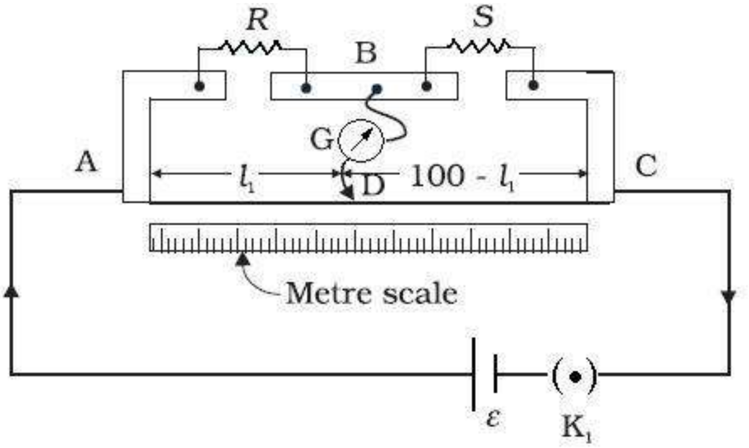
This last equation relating the four resistors is called the balance condition for the galvanometer to give zero or null deflection.
The Wheatstone bridge and its balance condition provide a practical method for determination of an unknown resistance. Let us suppose we have an unknown resistance, which we insert in the fourth arm; R_4 is thus not known. Keeping known resistances R_1

and R_2 in the first and second arm of the bridge, we go on varying R_3 till the galvanometer shows a null deflection. The bridge then is balanced, and from the balance condition the value of the unknown resistance R_4 is given by,

$$R_4 = R_3 \frac{R_2}{R_1}$$

Meter bridge

The meter bridge is shown in Figure



It consists of a wire of length 1m and of uniform cross sectional area stretched taut and clamped between two thick metallic strips bent at right angles, as shown.
The metallic strip has two gaps across which resistors can be connected. The end points where the wire is clamped are connected to a cell through a key.
One end of a galvanometer is connected to the metallic strip midway between the two gaps. The other end of the galvanometer is connected to a 'jockey'. The jockey is essentially a metallic rod whose one end has a knife-edge which can slide over the wire to make electrical connection.
R is an unknown resistance whose value we want to determine. It is connected across one of the gaps. Across the other gap, we connect a standard known resistance *S*. The jockey is connected to some point D on the wire, If the jockey is moved along the wire, then there will be one position where the galvanometer will show no current.

Let the distance of the jockey from the end A at the balance point be $l = l_1$.
The four resistances of the bridge at the balance point then are R, S, $\rho\, l_1$ and $\rho\,(100 - l_1)$.
The balance condition,

$$\frac{R}{S} = \frac{\rho l_1}{\rho(100 - l_1)} = \frac{l_1}{100 - l_1}$$

Thus, once we have found out l_1, the unknown resistance *R* is known in terms of the standard known resistance *S* by

$$R = S\frac{l_1}{100 - l_1}$$

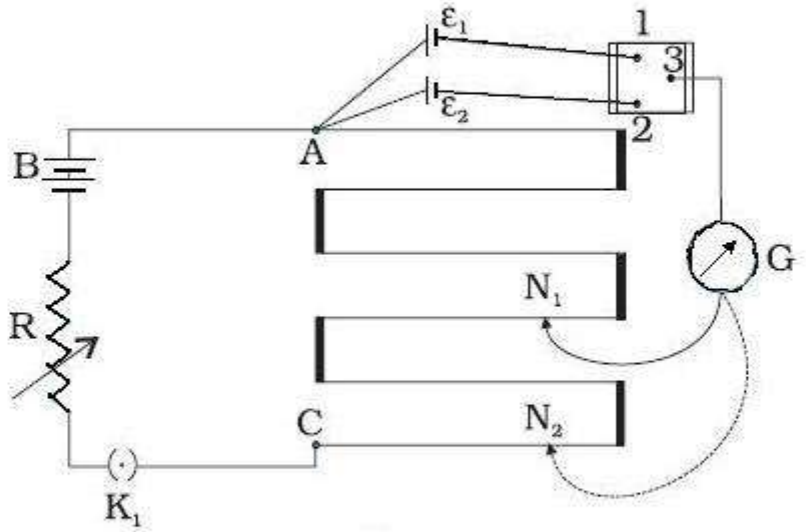
By choosing various values of *S*, we would get various values of l_1, and calculate *R* each time. An error in measurement of l_1 would naturally result in an error in *R.* It can be shown that the percentage error in *R* can be minimized by adjusting the balance point near the middle of the bridge, i.e., when l_1 is close to 50 cm.

Potentiometer

Construction

It is basically a long piece of uniform wire, sometimes a few meters in length across which a standard cell is connected. In actual design, the wire is sometimes cut in several pieces placed side by side and connected at the ends by thick metal strip.
In the figure, the wires run from A to C. The small vertical portions are the thick metal strips connecting the various sections of the wire. A current I flows through the wire which can be varied by a variable resistance (rheostat, *R*) in the circuit.

Use to compare emf

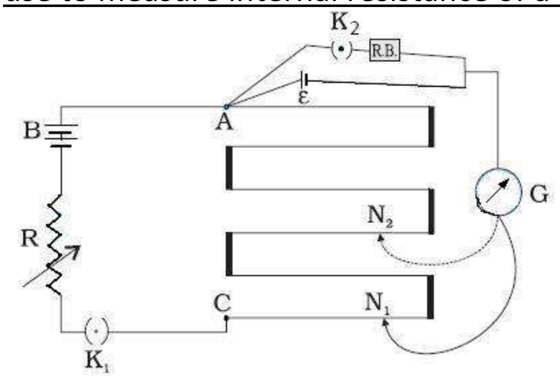


Since the wire is uniform, the potential difference between A and any point at a distance l from A is ε (l)=ϕ l (where ϕ is the potential drop per unit length.
Figure shows an application of the potentiometer to compare the emf of two cells of emf ε_1 and ε_2 . The points marked 1, 2, 3 form a two way key.
Consider first a position of the key where 1 and 3 are connected so that the galvanometer is connected to ε_1. The jockey is moved along the wire till at a point N_1, at a distance l_1 from A, there is no deflection in the galvanometer. We can apply Kirchhoff's loop rule to the closed loop AN_1G31A and get,
$\phi\, l_1 + 0 - \varepsilon_1 = 0$ Similarly, if another emf ε_2 is balanced against l_2 (AN_2)
$\phi\, l_2 + 0 - \varepsilon_2 = 0$ From the last two equations

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{l_1}{l_2}$$

Thus we can compare the emf's of any two sources. In practice one of the cells
is chosen as a standard cell whose emf is known to a high degree of accuracy. The emf of the other cell is then easily calculated from above equation

use to measure internal resistance of a cell



For this the cell (emf ε) whose internal resistance (r) is to be determined is connected across a resistance box through a key K_2, as shown in the figure. With key K_2 open, balance is obtained at length l_1 (AN_1). Then,

$\varepsilon = \phi\, l_1$

When key K_2 is closed, the cell sends a current (I) through the resistance box (R). If V is the terminal potential difference of the cell and balance is obtained at length l_2 (AN_2),

$V = \phi\, l_2$

So, we have

$$\frac{\varepsilon}{V} = \frac{l_1}{l_2}$$

But, ε = I (r + R) and V = IR. This gives

$$\frac{\varepsilon}{V} = \frac{(\mathrm{r} + \mathrm{R})}{\mathrm{R}}$$

From above equations we have

$$\frac{(\mathrm{r} + \mathrm{R})}{\mathrm{R}} = \frac{l_1}{l_2}$$

$$r = R\left(\frac{l_1}{l_2} - 1\right)$$

Using above we can find the internal resistance of a given cell.

The potentiometer has the advantage that it draws no current from the voltage source being measured. As such it is unaffected by the internal.

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Magnetic Effect of Current

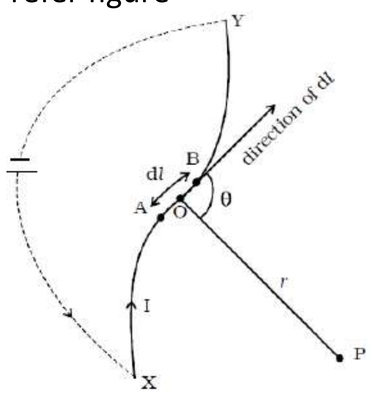
SECTION I

The magnetic field:

Magnetic field is the region around the moving charge in which magnetic force is experienced by the magnetic substances.
Magnetic field is a vector quantity and also known as magnetic induction vector. It is represented by **B**
Lines of magnetic induction may be drawn in the same way as lines of electric field.
The number of lines per unit area crossing a small area perpendicular to the direction of the induction being numerically equal to **B**.
The number of lines of **B** crossing a given area is referred as the magnetic flux linked with that area.
For this reason, **B** is also called magnetic flux density.
The unit of magnetic field is weber/m^2 and also known as tesla (T) in SI system

BIOT-SAVART LAW:

Biot and Savart conducted many experiments to determine the factors on which the magnetic field due to current in a conductor depends. The results of the experiments are summarized as Biot-Savart law. Let us consider a conductor XY carrying a current I refer figure



AB = dl is a small element of the conductor. P is a point at a distance r from the midpoint O of AB. According to Biot and Savart, the magnetic induction dB at P due to the element of length dl is
(i) directly proportional to the current (I)
(ii) directly proportional to the length of the element (dl)
(iii) directly proportional to the sine of the angle between dl and the line joining element dl and the point P (sin θ)
(iv) inversely proportional to the square of the distance of the point from the element (1/r^2)

$$dB \propto \frac{I\, dl \sin\theta}{r^2}$$

$$dB = K\frac{I\, dl \sin\theta}{r^2}$$

K is the constant of proportionality; its value is μ /4π.

Here μ is the permeability of the medium. Value of K for vacuum is 10^{-7} wb/amp m.

$$dB = \frac{\mu}{4\pi}\frac{I\, dl \sin\theta}{r^2}$$

$\mu = \mu_r \mu_o$ where μ_r is the relative permeability of the medium and μ_0 is the permeability of free space. $\mu_o = 4\pi \times 10–7$ henry/metre. For air $\mu_r = 1$.

So, in air

$$dB = \frac{\mu_0}{4\pi}\frac{I\, dl \sin\theta}{r^2}$$

In vector form,

$$dB = \frac{\mu_0}{4\pi}\frac{\overrightarrow{I\, dl} \times \vec{r}}{r^3}$$

or

$$dB = \frac{\mu_0}{4\pi}\frac{\overrightarrow{I\, dl} \times \hat{r}}{r^2}$$

The direction of dB is perpendicular to the plane containing current element Idl and r (i.e plane of the paper) and acts inwards.

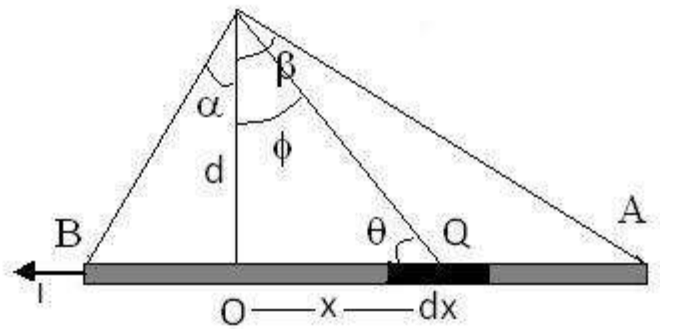
The unit of magnetic induction is tesla (or) weber $m^{-2.}$

Field due to a Straight current carrying wire

(i) When the wire is of finite length

Consider a straight wire segment carrying a current I and there is a point P at which magnetic field to be calculated as shown in figure.

This wire makes an angle of α and β at that point with normal OP. Consider an element of length dx at a distance y from O and distance of this element from point P is r and line joining P and Q makes an angle θ with the direction of current as shown in figure.



Using Biot-Savart Law magnetic field at point due to small current element is given by

$$dB = \frac{\mu_0 I}{4\pi}\left(\frac{dx \sin\theta}{r^2}\right)$$

As every element of the wire contributes to **B** in the same direction, we have

$$B = \frac{\mu_0 I}{4\pi}\int_A^B \left(\frac{dx\sin\theta}{r^2}\right)$$ .....eq(1)

From the triangle OPQ as shown in figure, we have

x = d tanϕ

Or dx = dsec^{2}ϕ dϕ

And in same triangle r = d sec ϕ and θ = (90^o – ϕ)

Where ϕ is angle between line OP and PQ

Now equation (1) can be written as

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\sec^2\varphi d\varphi\sin(90-\varphi)}{(d\sec\varphi)^2}\right)$$

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\sec^2\varphi d\varphi\sin(90-\varphi)}{(d\sec\varphi)^2}\right)$$

$$B = \frac{\mu_0 I}{4\pi}\int_{-\beta}^{\alpha} \left(\frac{d\varphi\cos\varphi}{d}\right)$$

$$B = \frac{\mu_0 I}{4\pi d}\int_{-\beta}^{\alpha} (\cos\varphi d\varphi)$$

$$B = \frac{\mu_0 I}{4\pi d}[\sin\varphi]_{-\beta}^{\alpha}$$

$$B = \frac{\mu_0 I}{4\pi d}[\sin\alpha + \sin\beta]$$

Direction of **B**: the direction of magnetic field in determined by the cross product of the vector Idl with vector r

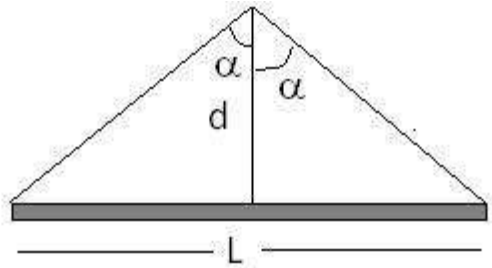
Therefore at point P the direction of the magnetic field due to the whole conductor will be perpendicular to the plane containing wire and point P or perpendicular to plane of paper and going into the plane

Case (I) when point P is on perpendicular bisector

In this case α = β using equation

$$B = \frac{\mu_0 I}{4\pi d}[\sin\alpha + \sin\beta]$$

$$B = \frac{\mu_0 I}{2\pi d}[\sin\alpha]$$



From figure

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4d^2}}$$

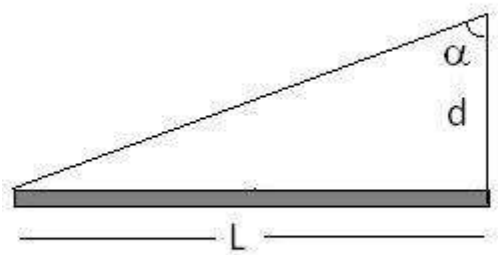
Case (II) when point P is at one end of conductor

In this case α =0 or β = 0

From equation

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha\right]$$



From figure

$$\sin\alpha = \frac{L}{\sqrt{L^2 + d^2}}$$

Case(III) When wire is of infinite length

In this case α = β = 90^O

From equation

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin 90 + \sin 90\right]$$

$$B = \frac{\mu_0 I}{2\pi d}$$

Case(IV) When the point P lies along the length of wire (but not on it)

If the point is along the length of wire (but not on it) , then as vector dl and vector r will either be parallel or antiparallel i.e θ=0 or π,

From equation

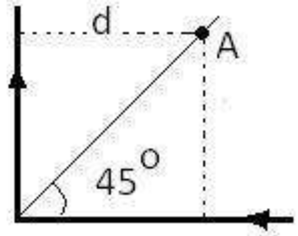
$$dB = \frac{\mu}{4\pi}\frac{\mathrm{I\ dl\ sin}\ \theta}{r^2}$$

$$dB = \frac{\mu}{4\pi}\frac{\mathrm{I\ dl\ sin\ 0}}{r^2}$$

$$dB = 0$$

Solved Problem

Q) A along straight conductor is bent at an angle of 90^O as shown in figure. Calculate the magnetic field induction at A



Solution:

For each portion α = 45 and β =90 .

From formula for magnetic field at a point

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin\alpha + \sin\beta\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\sin 45 + \sin 90\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\frac{1}{\sqrt{2}} + 1\right]$$

$$B = \frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

Each horizontal and vertical wires will produce same magnetic field at A and there directions are also same thus total field at A is

$$B = 2\times\frac{\mu_0 I}{4\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

$$B = \frac{\mu_0 I}{2\pi d}\left[\frac{\sqrt{2}+1}{\sqrt{2}}\right]$$

Q) A long straight wire carrying current produces a magnetic induction of 4 × 10^{-6}T at a point, 15 cm from the wire. Calculate the current through the wire.

Solution:

B =4 × 10^{-6}T , d = 15 cm = 0.15m

From formula

$$B = \frac{\mu_0 I}{2\pi d}$$

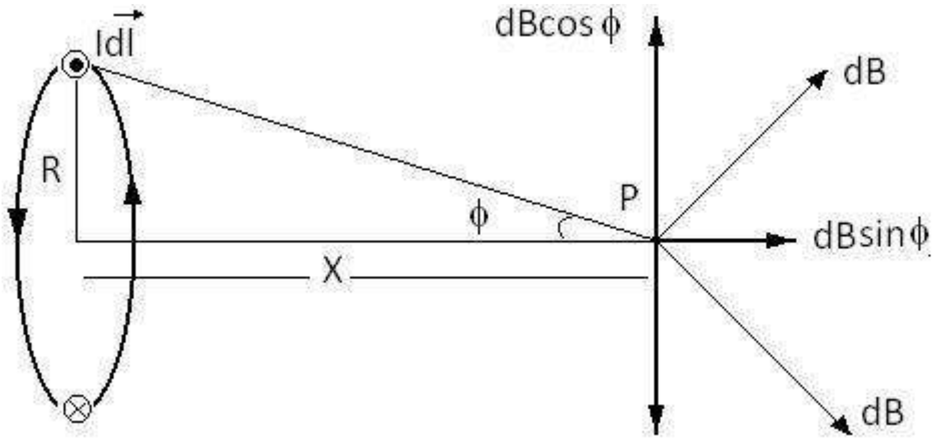
$$I = \frac{B(2\pi d)}{\mu_0}$$

$$I = \frac{4\times10^{-6}\times2\pi\times0.15}{4\pi\times10^{-7}}$$

$$I = 3A$$

Magnetic field at an axial point of a circular coil

Consider a circular loop of radius R and carrying a steady current I. We have to find out magnetic field at the axial point P, which is at distance x from the centre of the loop
X-axis is taken as along the axis of the ring.
Let the position vector of point P with respect to an element **dl** be **r** . The magnetic field dB at point due to current element I**dl** is in a direction perpendicular to the plane formed by **dl** and **r.**



Magnetic field at point P due to current element **Idl** is given by

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{i\vec{dl}\times\vec{r}}{r^3}$$

Since angle between Idl and r is 90^o we get

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{idl}{r^2}\hat{n}$$

Direction of magnetic field is perpendicular to plane containing dl and r as shown in figure
If ϕ is the angle between r and X, then from geometry of figure, component of **dB** along Y-axis will be dBcosϕ and dBsinϕ will be along X axis.
For all point on the circular coil there exists a diametrically opposite point such that magnetic field produced at point P cancels Y-component of first one thus resultant magnetic field at P is summation of X-component at P

$$B = \int dB \sin\varphi$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{r^2}\sin\varphi$$

$$\sin\varphi = \frac{R}{r}$$

$$\sin\varphi = \frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{r^2}\frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl}{R^2 + x^2}\frac{R}{\sqrt{R^2 + x^2}}$$

$$B = \frac{\mu_0}{4\pi}\int \frac{IRdl}{\left(R^2 + x^2\right)^{3/2}}$$

$$B = \frac{\mu_0}{4\pi}\frac{IR}{\left(R^2 + x^2\right)^{3/2}}\int_0^{2\pi R} dl$$

$$B = \frac{\mu_0}{4\pi}\frac{IR}{\left(R^2 + x^2\right)^{3/2}}\left(2\pi R\right)$$

$$B = \frac{\mu_0}{2}\frac{IR^2}{\left(R^2 + x^2\right)^{3/2}}$$

Case (I) If the coil has N turns then

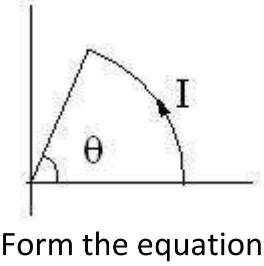
$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + x^2\right)^{3/2}}$$

Case (II) Field at the centre of ring
In above equation x = 0

$$B = \frac{\mu_0}{2}\frac{NIR^2}{R^3}$$

$$B = \frac{\mu_0 NI}{2R}$$

Case (III) Magnetic field at the centre of a current arc



Form the equation for magnetic field at centre of coil. N is number of turns

2π = 1 turn

∴ θ = θ/2π turn s replacing value of N we get

$$B = \frac{\mu_0 I}{2R}\frac{\theta}{2\pi}$$

If l is the length of arc then l = θR above equation becomes

$$B = \frac{\mu_0 I}{2R}\frac{1}{2\pi}\frac{l}{R}$$

$$B = \frac{\mu_0 Il}{4\pi R^2}$$

Direction of magnetic field **B** for circular loop

Direction of magnetic field at a point the axis of a circular coil is along the axis and its orientation can be obtained using the right-hand thumb rule. If the fingers curled along the current then stretched thumb shows direction of magnetic field.

Magnetic field will be out of the page for anticlockwise current while into the page for clockwise current

Solved Problem

Q) A circular coil of 200 turns and of radius 20 cm carries a current of 5A. Calculate the magnetic induction at a point along its axis, at a distance three times the radius of the coil from its centre

Solution:

N=200, R = 0.2 m, I =5A , x = 3R

From the formula

$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + x^2\right)^{3/2}}$$

$$B = \frac{\mu_0}{2}\frac{NIR^2}{\left(R^2 + 9R^2\right)^{3/2}}$$

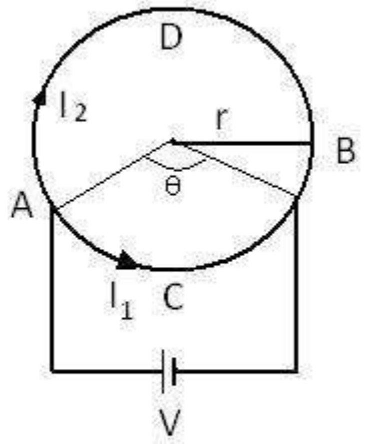
$$B = \frac{\mu_0}{2}\frac{NI}{10^{3/2}R}$$

$$B = \frac{\mu_0 NI}{20\sqrt{10}R}$$

$$B = \frac{4\pi\times 10^{-7}\times 200\times 5}{20\sqrt{10}\times 0.2}$$

$$B = 9.9\times 10^{-5}T$$

Q) A circular loop is prepared from a wire of uniform cross section. A battery is connected between any two points on its circumference. Show that the magnetic induction at the centre of the loop is zero



<u>Solution:</u>

Magnetic field at centre due to arc is given by

$$B_1 = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi}$$

and

$$B_2 = \frac{\mu_0 I_2}{2R}\frac{(2\pi-\theta)}{2\pi}$$

Both magnetic fields B_1 and B_2 are in opposite directions

Let R_1 be the resistance of arc ACB and R_2 be the resistance of arc ADB since potential across both the resistance is same thus

$I_1R_1 = I_2R_2$ eq(1)

We also know that resistance ∝ length of wire

And lenth of wire ACB = rθ

Length of arc ADB = r(2π –θ)

Thus if ρ is resistance per unit length then

$R_1 = \rho r\theta$ and $R_2 = \rho r(2\pi - \theta)$

Thus equation (1) becomes

$I_1 \rho r\theta = I_2 \rho r(2\pi - \theta)$

$I_1 \theta = I_2 (2\pi - \theta)$ eq(2)

Now total magnetic field at centre

$$B = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} - \frac{\mu_0 I_2}{2R}\frac{(2\pi-\theta)}{2\pi}$$

from eq (3)

$$B = \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} - \frac{\mu_0 I_1}{2R}\frac{\theta}{2\pi} = 0$$

Q) A charge Q is uniformly spread over a disc of radius R made from non conducting material. This disc is rotated about its geometrical axis with frequency f. Find the magnetic field produced at the centre of the disc.

Solution:

Suppose disc with radius R is divided into the concentric rings with various radii, Consider one such ring with radius r and thickness dr.

Total charge on disc = Q , charge per unit area ρ = $Q/\pi R^2$

∴ The charge on the ring with radius r = (area of the ring) (charge per unit area)

q = $(2\pi r dr)(Q/\pi R^2)$

If the ring is rotating with frequency f, then current produced I

$$I = \frac{Q}{\pi R^2}(2\pi r dr) f$$

This ring can be considered as circular loops carrying current I

Magnetic field at the centre due to this current will be

$$dB = \frac{\mu_0}{2r} I$$

$$dB = \frac{\mu_0}{2r}\frac{Q}{\pi R^2}(2\pi r dr) f$$

$$dB = \frac{\mu_0 Q f}{R^2}(dr)$$

∴ Magnetic field B produced at the centre due to the whole disc

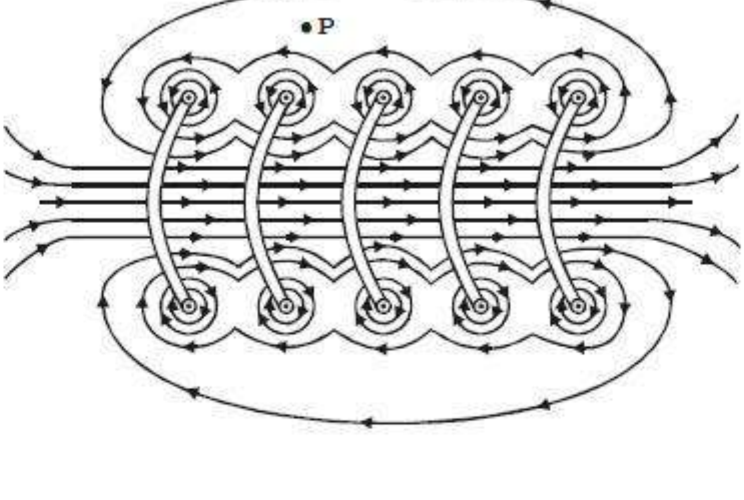
$$B = \int dB = \int_0^R \frac{\mu_0 Q f}{R^2} dr$$

$$B = \frac{\mu_0 Q f}{R}$$

Solenoid:

When two identical rings carrying current in same direction are placed closed to each other co-axially. It is obvious that the magnetic field produced due to the rings is in same direction on the common axis. Moreover the lines close to the axis are almost parallel to the axis and in the same direction.

Thus if a number of such rings are kept very closed to each other and current is passed in the same direction.



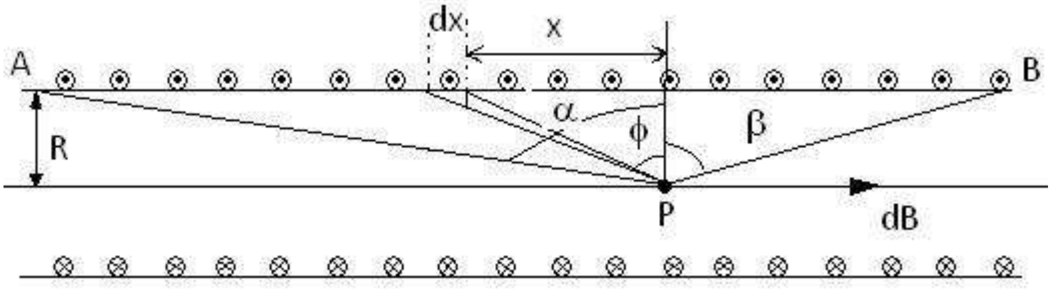
The magnetic fields associated with each single turn are almost concentric circles and hence tend to cancel between the turns. At the interior midpoint, the field is strong and along the axis (i.e) the field is parallel to the axis. For a point such as P, the field due to the upper part of the solenoid turns tends to cancel the field due to the lower part of the turns, acting in opposite directions. Hence the field outside the circular coil is very less

Solenoid is a device in which this situation is realized.

A helical coil consisting of closely wound turns of insulated conducting wire is called solenoid.

When length of a solenoid is very large compared to its radius, the solenoid is called long solenoid. For long solenoid magnetic field outside is practically zero.

Magnetic field at a point on the axis of the a SHORT solenoid



Consider a solenoid of length L and radius R containing N closely spaced turns and carrying steady current I. let number of turns per unit length be n

The field at point P on the axis of a solenoid can be obtained by superposition of fields due to large number of turns all having their centre on the axis of the solenoid as shown in figure

Consider a coil of width dx at a distance x from the point P on the axis as shown in figure

The field at P due to ndx turns is

$$dB = \frac{\mu_0}{2}\frac{(ndx)IR^2}{\left(R^2+x^2\right)^{3/2}}$$

From figure x = R tanϕ

dx = $R\sec^2\phi\, d\phi$

On substituting values in above equation we get

$$dB = \frac{\mu_0}{2}\frac{\left(nR\sec^2\varphi d\varphi\right)IR^2}{\left(R^2 + R^2\tan^2\varphi\right)^{3/2}}$$

$$dB = \frac{\mu_0}{2}\frac{nI\sec^2\varphi}{\sec^3\varphi}d\varphi$$

$$dB = \frac{\mu_0}{2}nI\cos\varphi d\varphi$$

$$B = \int_{-\alpha}^{\beta}\frac{\mu_0}{2}nI\cos\varphi d\varphi$$

$$B = \frac{\mu_0}{2}nI\int_{-\alpha}^{\beta}\cos\varphi d\varphi$$

$$B = \frac{\mu_0}{2}nI\left[\sin\alpha + \sin\beta\right]$$

Case (I) If the solenoid is of infinite length and the point is well inside the solenoid
In this case α = β = π/2 then B is

$$B = \frac{\mu_0}{2}nI\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right]$$

$$B = \mu_0 nI$$

Case(II) If the solenoid is of INFINITE length and the point is near one end
In this case α = 0 and β = π/2

$$B = \frac{\mu_0}{2}nI\left[\sin 0 + \sin\frac{\pi}{2}\right]$$

$$B = \frac{\mu_0}{2}nI$$

Case (III) If the solenoid is of FINITE length and the point is on the perpendicular bisector of its axis
In this case α = β

$$B = \frac{\mu_0}{2}nI\left[\sin\alpha + \sin\alpha\right]$$

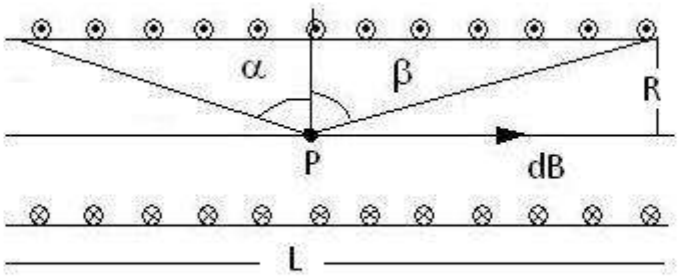
$$B = \mu_0 nI\sin\alpha$$

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4R^2}}$$

Solved problem

Q) A solenoid of length 0.4m and diameter 0.6m consists of a single layer of 1000turns of fine wire carrying a current of 5 $\times 10^{-3}$ampere. Calculate the magnetic field on the axis of the middle and at the end of the solenoid

Solution:



In case of a finite solenoid, the field at the point on the axis is given by

$$B = \frac{\mu_0}{2} nI [\sin\alpha + \sin\beta]$$

$$n = \frac{N}{L} = \frac{1000}{0.4} = 2.5\times10^3$$

$$B = 2.5\pi\times10^{-6}[\sin\alpha + \sin\beta]$$

a) Middle point α = β thus
   $B = 2.5\pi\times10^{-6}(2\sin\alpha)$ and

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4R^2}}$$

$$\sin\alpha = \frac{L}{\sqrt{L^2 + d^2}}$$

$$\sin\alpha = \frac{0.4}{\sqrt{(0.4)^2 + (0.6)^2}}$$

$$\sin\alpha = \frac{4}{7.2}$$

   $B = 2.5\pi\times10^{-6}\times2\times(4/7.2) = 8.72\times10^{-6}$ T

b) When the points is at the end on axis

$$\sin\beta = \frac{L}{\sqrt{L^2 + R^2}}$$

$$\sin\beta = \frac{0.4}{\sqrt{(0.4)^2 + (0.3)^2}}$$

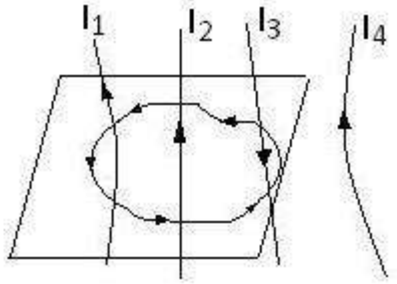
$$\sin\beta = \frac{4}{5}$$

   $B = 2.5\pi\times10^{-6}\times2\times(4/5) = 6.28\times10^{-6}$ T

Ampere’s Law

Consider electric currents I_1,I_2, I_3, I_4 as shown in figure. All these current produce magnetic field in the region around electric current.

A plane which is not necessarily horizontal is shown in figure. An arbitrary closed curve is also shown in figure



Sign convention for current:

Arrange right hand screw perpendicular to plane containing closed loop and rotate in the direction of vector element taken for line integration. Electric current in the direction of advancement of screw is considered positive and current in opposite direction are considered negative.

Now from sign convention I_1 and I_2 are positive while I_3 is negative.

Hence algebraic sum $\sum I = I_1 + I_2 - I_3$

Here we don’t worry about current not enclosed by the loop

The statement of the Ampere’s Law is as under:

The line integral of magnetic induction over a closed loop in a magnetic field is equal to the product of algebraic sum of electric current enclosed by loop and the magnetic permeability”

The law can be represented mathematically as

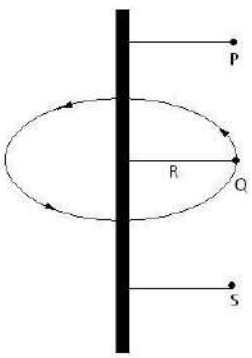
$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \sum I$$

The magnetic induction in the above equation is due to all current. Whereas algebraic sum of current on right hand side is only of those currents which are enclosed by the loop

This law is true for steady current.

Application of Ampere’s Law

(A) To find Magnetic field Due to a very long straight conductor carrying electric current



Let I be the current flowing through a very long conductor. Now consider points like P, Q, S located at same perpendicular distance R from wire. Since the two ends of wire at infinity and due to symmetry of wire magnetic field at pints P, Q and S is same.

Thus magnetic field at all point on the circumference of circle of radius R must be same. Or B is constant

Consider a small segment of length dl along the circumference.

Now by applying Ampere’s Law we get

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 \sum I$$

$$\oint B\,dl\cos\theta = \mu_0 I$$

$\vec{B}$ *and dl are in same direction*

$$\oint B\,dl = \mu_0 I$$

$$B\oint dl = \mu_0 I$$

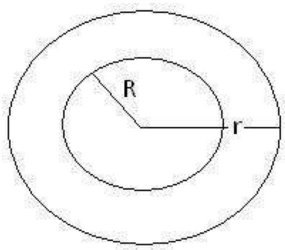
$$B(2\pi R) = \mu_0 I$$

$$B = \frac{\mu_0}{2\pi}\frac{I}{R}$$

Thus Outside the conductor B ∝ (1/R)

Magnetic field inside the conductor

Consider a top view of conductor of radius r . We want to find magnetic field at a distance R < r. Consider a loop of radius R as shown in figure



Let conductor carries a current I thus current through conductor of radius R is

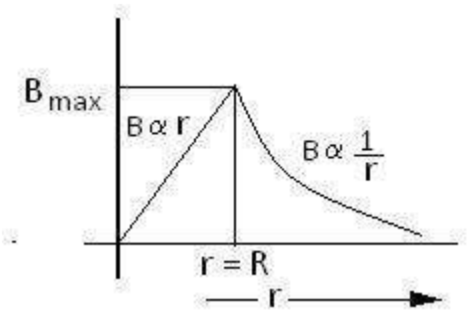
$$i = \left(\frac{I}{\pi r^2}\right)\pi R^2 = I\frac{R^2}{r^2}$$

From Ampere's Law

$$B(2\pi R) = \mu_0 i$$

$$B(2\pi R) = \mu_0 I\frac{R^2}{r^2}$$

$$B = \left(\frac{\mu_0 I}{2\pi r^2}\right)R$$

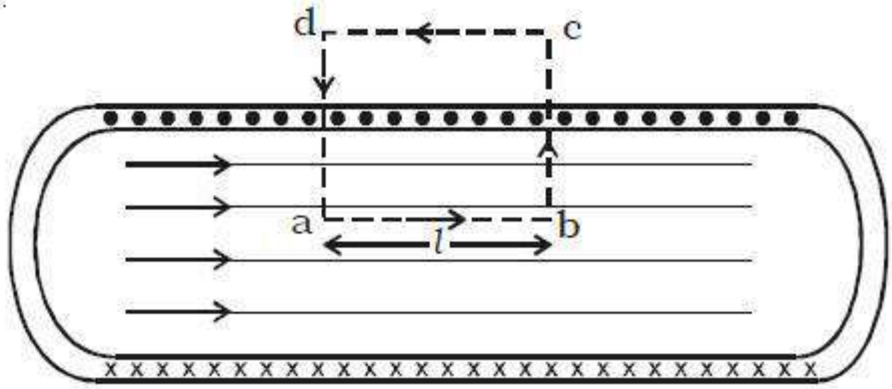


For inside the conductor B∝ R

Hence for magnetic field

(i) Outside the conductor B ∝ (1/R)
(ii) Inside the conductor B ∝ R
(iii) On the conductor Maximum
(iv) At end points outside conductor = 0

(B) Magnetic field inside a LONG solenoid using Ampere's Circuital Law



A solenoid is a wire wound closely in the form of a helix, such that adjacent turns are electrically insulated

The magnetic field inside a very tightly wound long solenoid is uniform everywhere along the axis of the solenoid and is zero outside it.

To calculate the magnetic field at point 'a', let us draw rectangle abcd as shown in figure.

The line ab is parallel to the solenoid axis and hence parallel to magnetic field **B** inside the solenoid thus **B.dl** = B(dl)

Line da and bc are perpendicular thus **B.dl** = 0

Line cd is outside the solenoid here B = 0 thus **B.dl** = 0

If i is the current and n is the number of turns per unit length then current enclosed by the loop = ni l

From Ampere's Law

$$\int_a^b Bdl = \mu_0 nil$$

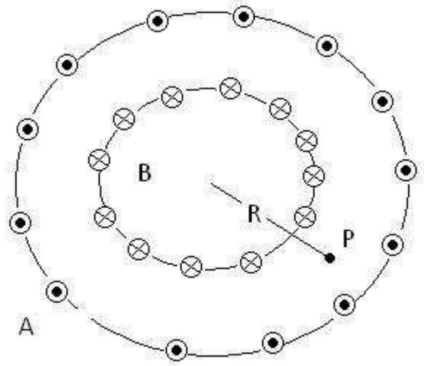
$$Bl = \mu_0 nil$$

$$B = \mu_0 ni$$

Toroid:

If a solenoid is bent in the form of a circle and its two ends are joined with each other the device is called a toroid.

The toroid is a hollow circular ring on which a large number of turns of a wire are closely wound. It can be viewed as a solenoid which has been bent into a circular shape to close itself. It is shown in figure carrying a current i



Magnetic field at point A and B is zero as points are outside the toroid
Magnetic field at point P inside the toroid which is at distance R from its centre as shown in figure. Clearly magnetic field at point on the circle of radius R is constant. And directing towards the tangent to the circle. Therefore

$$\oint \vec{B} \cdot \vec{dl} = B(2\pi R)$$

If total number of turns is N and current passing is I, the total current passing through said loop must be NI
From Ampere's Law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 NI$$

$$B(2\pi R) = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{2\pi R}$$

$$B = \mu_0 nI$$

Here n = N/2πR the number of turns per unit length of toroid
This magnetic field is uniform at each point inside toroid
In an ideal toroid, the turns are completely circular. In such toroid magnetic field inside the toroid is uniform and outside is zero.
But toroid used in practice turns are helical and hence small magnetic field is produced outside the toroid
Toroid is used for nuclear fusion devise for confinement of plasma.

SECTION II

Force on a charged particle in a magnetic field

When a charged ‘q’ moving in a magnetic field **B** with velocity **v** then force experienced by the charge is given by

$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

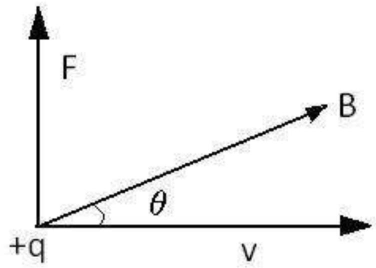
The magnitude is given by

$F = qvB\sin\theta$ Here θ is angle between direction of **B** and direction of velocity **v**

Direction of force is perpendicular to both **B** and **v**

The right hand thumb rule:

For determining the direction of cross product **v** × **B**, you point the four fingers of your right hand along the direction of v, and palm in the direction of magnetic field B the curl the fingers. The thumb is then points in the direction of **v** × **B**



If q is negative then direction of F will be opposite to direction of **v** × **B**

Important points:

(1) The magnetic force will be maximum when $\sin\theta = 1 \Rightarrow \theta = 90^o$
Change is moving perpendicular to magnetic field $F_{max} = qvB$
In this situation F, v, B are mutually perpendicular to each other.

(2) The magnetic force will be minimum when $\sin\theta = 0 \Rightarrow \theta = 0$ or 180^o
It means charge is moving parallel to magnetic field $F_{min} = 0$

(3) Magnetic force is zero when charge is stationary

(4) In case of motion of charged particle in a magnetic field, as the force is always perpendicular to direction of charge work done is zero. Or magnetic force cannot change kinetic energy of charge and speed remains constant

Difference between Electric and Magnetic field

(1) Magnetic force is always perpendicular to the field while electric force is collinear with the field

(2) Magnetic force is velocity dependent i.e. acts only when charged particle is in motion while electric force is independent of the state of rest or motion of the charge.

(3) Magnetic force does not work when the charged particle is displaced while electric force does work in displacing the charged particle.

(4) Magnetic force is always non-central while the electric force may or may not be.

Non-central force: A force between two particles that is not directed along the line connecting them.

Motion of a charged particle in a uniform magnetic field

(A) When the charged particle is given velocity perpendicular to the field:

Let a particle of charge q and mass m is moving with a velocity 'v' and enters at right angles to uniform magnetic field N as shown in figure

The force on the particle is qvB and this force will always act in a direction perpendicular to v. Hence, the particle will always act in a direction perpendicular to v. Hence the particle will move on a circular path. If the radius of the path is r then

$$\frac{mv^2}{r} = Bqv$$

$$r = \frac{mv}{qB}$$

Thus radius of the path is proportional to the momentum mv of the particle and inversely proportional to the magnitude of magnetic field

Time period:

The time period is the time taken by the charged particle to complete on rotation of the circular path which is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}$$

The time period is independent of the speed

Frequency:

The frequency is the number of revolution of charged particle in one second which is given by

$$f = \frac{1}{T} = \frac{qB}{2\pi m}$$

SOLVED NUMERICAL

Q) Two particles of mass M and m and having equal electric charge are accelerated through equal potential difference and then move inside a uniform magnetic field, normal to it. If the radii of the circular paths are R and r respectively find the ratio of their masses

<u>Solution:</u>

Since charge is same on both the particles and are accelerated through equal potential both particles will have same kinetic energy. Let p_1 be the momentum of particle of mass M and p_2 be the momentum of particle of mass 'm' thus

$$\frac{p_1^2}{2M} = \frac{p_2^2}{2m}$$

$$\frac{p_1^2}{p_2^2} = \frac{M}{m}$$

From the equation for radius r ∝ momentum

Thus

$$\frac{M}{m} = \left(\frac{R}{r}\right)^2$$

(B) When a charged particle is moving at an angle to the field

In this case the charged particle having charge q and mass m is moving with velocity v and it enters the magnetic field B at angle θ as shown in figure. Velocity can be resolved in two component one along magnetic field and the other perpendicular to it. Let these components are $V_{||}$ and $V_{\perp}$

$V_{||} = V\cos\theta$ and $V_{\perp} = V\sin\theta$

The parallel component $V_{||}$ of velocity remains unchanged as it is parallel to B.

Due to perpendicular component $V_{\perp}$ the particle will move on a circular path.

So resultant path will be combination of straight line motion and circular motion, which will be helical path

The radius of path:

$$r = \frac{mv\sin\theta}{qB}$$

Time period:

$$T = \frac{2\pi r}{v_{\perp}}$$

$$T = \frac{2\pi mv\sin\theta}{v\sin\theta qB}$$

$$T = \frac{2\pi m}{qB}$$

Frequency (f)

$$f = \frac{qB}{2\pi m}$$

Pitch :

Pitch of helix described by charged particle is defined as the distance moved by the centre of circular path in the time in which particle completes one revolution

Pitch = $V_{||}$ (time period)

$$pitch = v\cos\theta\frac{2\pi m}{Bq}$$

$$pitch = \frac{2\pi mv\cos\theta}{Bq}$$

Motion of charged particle in combined electric and magnetic field

When the moving charged particle is subjected simultaneously to both electric field and magnetic field B, the moving charged particle will experience electric force $\mathbf{F_e} = q\mathbf{E}$ and magnetic force $\mathbf{F_m} = q(\mathbf{v} \times \mathbf{B})$ **,** so the net force on it will be

$\mathbf{F} = q[\,E + (\mathbf{v} \times \mathbf{B})]$ which is called Lorentz force equation**:**

Case (I) when v, B, and E all three are collinear

In this situation as the particle is moving parallel or antiparallel to the field, the magnetic force on it will be zero and only electric force will act so

$\vec{a} = \frac{q\vec{E}}{m}$

Hence particle will flow straight path with changing speed and hence kinetic energy, momentum will also change

Case(II) **v, E** and **B** are mutually perpendicular

v, E and **B** are mutually perpendicular.

In case situation of E and B are such that

$F = \mathbf{F_e} + \mathbf{F_m} = 0$

Then a = 0, particle will move in its original without change in velocity in this situation

qE = qvB

or v = E/B

This principle is used in velocity selector to get a charged beam having a specific velocity

SOLVED PROBLEM

Q) A particle of mass 1×10^{-26} kg and charge $+1.6\times10^{-19}$ C travelling with velocity 1.28×10^{6} m/s in +x direction enters a region in which a uniform electric field E and a uniform magnetic field b are present such that $E_x = E_y = 0$; $E_z = 102.4$ kV/m and $B_x = B_z = 0$, $B_z = 8\times10^{-2}$ wb/m^2. The particle enters in a region at the origin at time t = 0. Find the location (x, y, z) of the particle at $t = 5\times10^{-6}$s

Solution

From Lorentz equation

$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

$F = q[102.4 \times10^{3}\ \mathbf{i} + (1.28\times10^{6}\ \mathbf{i} \times 8\times10^{-2}\ \mathbf{k})]$

$F = q[102.4 \times10^{3}\ \mathbf{i} + (-102.4 \times10^{3}\ \mathbf{i})]$

$\mathbf{F} = 0$

Hence, the particle will move along + x axis with constant velocity 1.28×10^{6} m/s

X=vt = 6.40 m

Location is (6.4, 0, 0)

Cyclotron

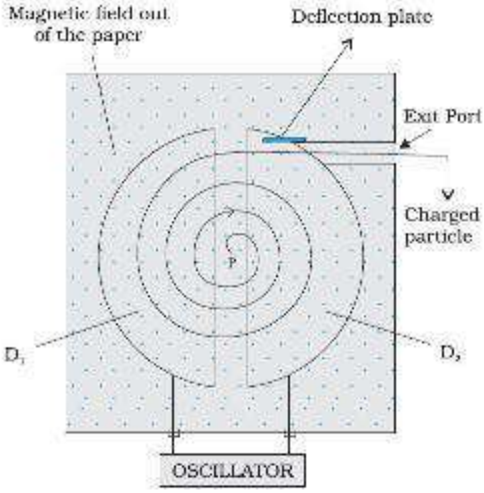
Cyclotron is a device used to accelerate charged particles to high energies. It was devised by Lawrence.

Principle

Cyclotron works on the principle that a charged particle moving normal to a magnetic field experiences magnetic Lorentz force due to which the particle moves in a circular path.

Construction

It consists of a hollow metal cylinder divided into two sections D_1 and D_2 called Dees, enclosed in an evacuated chamber. The Dees are kept separated and a source of ions is placed at the centre in the gap between the Dees. They are placed between the pole pieces of a strong electromagnet. The magnetic field acts perpendicular to the plane of the Dees. The Dees are connected to a high frequency oscillator. The whole assembly is evacuated to minimize collisions between the ions and the air molecules. A high frequency alternating voltage is applied to the Dees. In the sketch shown in Fig. positive ions or positively charged particles (e.g., protons) are released at the centre P.



Working

When a positive ion of charge *q* and mass *m* is emitted from the source, it is accelerated towards the Dee having a negative potential at that instant of time. Due to the normal magnetic field, the ion experiences magnetic Lorentz force and moves in a circular path. By the time the ion arrives at the gap between the Dees, the polarity of the Dees gets reversed. Hence the particle is once again accelerated and moves into the other Dee with a greater velocity along a circle of greater radius. Thus the particle moves in a spiral path of increasing radius and when it comes near the edge, it is taken out with the help of a deflector plate (D.P). The particle with high energy is now allowed to hit the target. When the particle moves along a circle of radius *r* with a velocity *v*, the magnetic Lorentz force provides the necessary centripetal force. They move in a semi-circular path in one of the dees and arrive in the gap between the dees in a time interval *T*/2; where *T,* the period of revolution

$$Bqv = \frac{mv^2}{r}$$

$$r = \frac{mv}{Bq}$$

The time taken to describe a circle

$$r = \frac{mv}{Bq}$$

$$r = \frac{m\omega r}{Bq}$$

$$\omega = \frac{Bq}{m}$$

$$But\ \omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{T} = \frac{Bq}{m}$$

$$T = \frac{2\pi m}{Bq}$$

$$f = \frac{Bq}{2\pi m}$$

It is clear from equation that the time taken by the ion to describe a circle is independent of (i) the radius (r) of the path and (ii) the velocity (*v*) of the particle
So, in a uniform magnetic field, the ion traverses all the circles in exactly the same time.
If the high frequency oscillator is adjusted to produce oscillations of frequency as given in equation resonance occurs. Cyclotron is used to accelerate protons, deutrons and
α - particles.
Limitations
(i) Maintaining a uniform magnetic field over a large area of the Dees is difficult.
(ii) At high velocities, relativistic variation of mass of the particle upsets the resonance condition.
(iii) At high frequencies, relativistic variation of mass of the electron is appreciable and hence electrons cannot be accelerated by cyclotron.

Solved numerical

Q) A particle having 2C charge passes through magnetic field of 4 k T and some uniform electric field with velocity 25j. IF Lorentz force acting on it is 400 I N. find the electric field in this region
Solution
Lorentz force
F =q[**E** +(**v** ×**B**)]
400 **i** = 2 [E + 25(4) (**j** ×**k**)]
400 **i** = 2E +200**i**
E = 100 **i** V/m

Force on current carrying conductor in magnetic field

Let L be the length of the straight conductor carrying current I and placed perpendicular to a uniform magnetic induction B

A current in a conductor is due to flow of charge

If v_d is drift velocity of charge A is cross section of conductor n is density of charge per unit volume then from equation

$I = nqv_dA$

Now number of charges in conductor of length L is N = nAL

The force on N charges $F = BNqv_d$

Total force $F = NnALqv_d$

From equation for current

F = BIL

In vector form $\mathbf{F} = I(\mathbf{L} \times \mathbf{B})$

Where L is a vector in the direction of the current, magnitude of **L** is L for the segment of wire in a uniform magnetic field

$F = ILB\sin\theta$ here θ is the angle between vector IL and **B**

Magnitude of the force

The magnitude of the force is $F = BIL \sin\theta$

(i) If the conductor is placed along the direction of the magnetic field, $\theta = 0^o$, Therefore force F = 0.

(ii) If the conductor is placed perpendicular to the magnetic field,

$\theta = 90^o$, $F = BIl$. Therefore the conductor experiences maximum force.

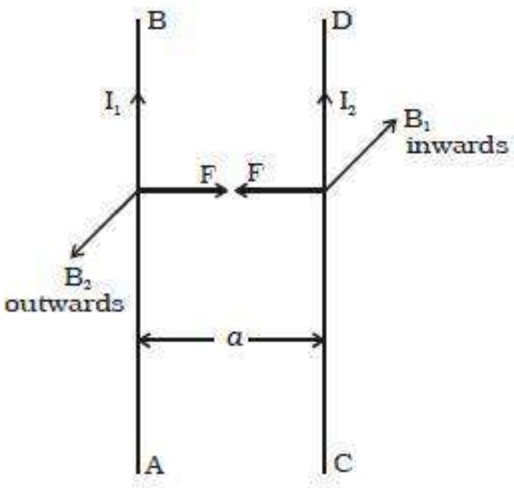
(iii) Force on a closed loop of an arbitrarily shaped conductor is zero

The direction of the force on a current carrying conductor placed in a magnetic field is given by Fleming's Left Hand Rule.

The forefinger, the middle finger and the thumb of the left hand are stretched in mutually perpendicular directions. If the forefinger points in the direction of the magnetic field, the middle finger points in the direction of the current, then the thumb points in the direction of the force on the conductor.

Force between two long straight parallel current carrying conductors

AB and CD are two straight very long parallel conductors placed in air at a distance *a*. They carry currents I_1 and I_2 respectively.



The magnetic induction due to current I_1 in AB at a distance *a* is

$$B_1 = \frac{\mu_0 I_1}{2\pi a} \quad \text{...eq(1)}$$

This magnetic field acts perpendicular to the plane of the paper and inwards. The conductor CD with current I_2 is situated in this magnetic field. Hence, force on a segment of length L of CD due to magnetic field B_1 is

$F = B_1 I_2 L$

substituting equation (1)

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

By Fleming's Left Hand Rule, F acts towards left. Similarly, the magnetic induction due to current I_2 flowing in CD at a distance *a* is

$$B_2 = \frac{\mu_0 I_2}{2\pi a} \quad \text{.....eq(3)}$$

This magnetic field acts perpendicular to the plane of the paper and outwards. The conductor AB with current I1, is situated in this field. Hence force on a segment of length *l* of AB due to magnetic field B2 is

$$F = B_2 I_1 L$$

substituting equation (3)

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

By Fleming's left hand rule, this force acts towards right. These two forces given in equations (2) and (4) attract each other. Hence, two parallel wires carrying currents in the same direction attract each other and if they carry currents in the opposite direction, repel each other.

Definition of ampere

The force between two parallel wires carrying currents on a segment of length L is

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

Force per unit length of the conductor is

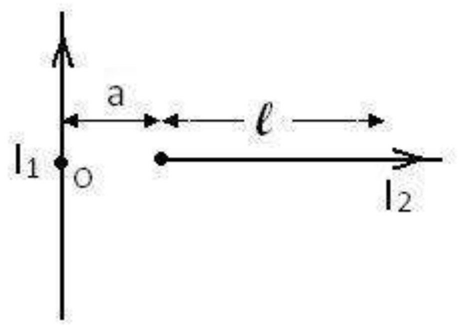
$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

If $I_1 = I_2 = 1$ Amp , a = 1 m Then $F/L = 2 \times 10^{-7}$

The above conditions lead the following definition of ampere. Ampere is defined as that constant current which when flowing through two parallel infinitely long straight conductors of negligible cross section and placed in air or vacuum at a distance of one metre apart, experience a force of 2×10^{-7} newton per unit length of the conductor.

Solved Numerical

Q) As shown in figure very long conductor wire carrying current I_1 is arranged in y direction another conducting wire of length l carrying current I_2 is placed along X-axis at a distance a from this wire. Find the torque acting on this wire with respect to point O



Solution:

We can consider wire of current I_2 is in the magnetic field produced current I_1

The force acting on a current element I_2dx located at a distance x from O is,

$d\mathbf{F} = I_2 dx\, \mathbf{I} \times \mathbf{B}$

Here B is

$$B = \frac{\mu_0 I_1}{2\pi x}\left(-\hat{k}\right)$$

Thus

$$d\vec{F} = I_2 dx\hat{i} \times \frac{\mu_0 I_1}{2\pi x}\left(-\hat{k}\right)$$

$$d\vec{F} = \frac{\mu_0 I_1 I_2 dx}{2\pi x}\hat{j}$$

The torque acting on this element with respect to O is

$d\tau = x\mathbf{i} \times d\mathbf{F}$

$$d\vec{\tau} = xi \times \frac{\mu_0 I_1 I_2 dx}{2\pi x} j$$

$$d\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi} dx\hat{k}$$

Total torque acting on this coil can be obtained by taking integration of this equation between x = 0 to x = a + l

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi}\int_a^{a+l} dx\hat{k}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi}\left[x\right]_a^{a+l}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2}{2\pi}\left[a+l-a\right]\hat{k}$$

$$\vec{\tau} = \frac{\mu_0 I_1 I_2 l}{2\pi}\hat{k}$$

Q) A straight wire of length 30cm and mass 60mg lies in a direction 30^o east of north. The earth's magnetic field at this is in horizontal and has a magnitude of 0.8 ×10^{-4}T. What current must be passes through the wire so that it may float in air? [g = 10 m/s^2]

Solution:

$$F = BILsin\theta$$

This force will act upward should be equal to downward force of gravitation = mg thus

$$mg = BILsin\theta$$

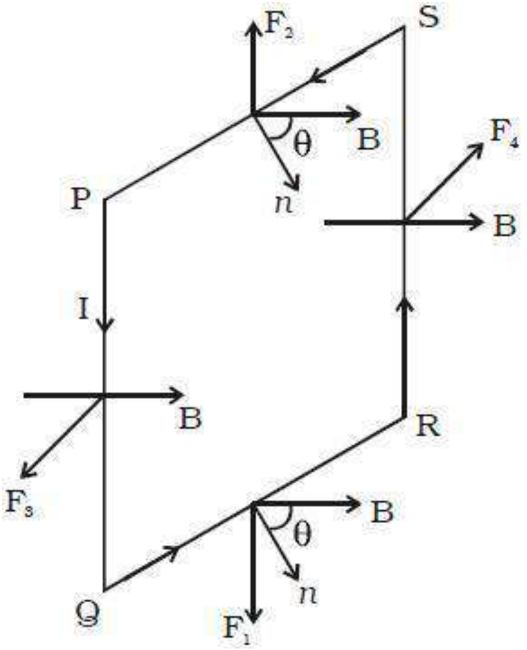
m = 60×10^{-6}Kg , B= 0.8 ×10^{-4}T, L = 30×10^{-2}, θ = 30^o, g = 10 m/s^2

60×10^{-6}×10 = 0.8 ×10^{-4} × (I) × 30×10^{-2}× (1/2)

I = 50 A

Current Loop in uniform Magnetic field

Let us consider a rectangular loop PQRS of length *l* and breadth *b* (Fig 3.24). It carries a current of I along PQRS. The loop is placed in a uniform magnetic field of induction B. Let θ be the angle between the normal to the plane of the loop and the direction of the magnetic field. Force on the arm QR,



Force on the arm QR,

$$\vec{F_1} = \overline{I(\overrightarrow{QR})} \times \overline{B}$$

Since the angle between **I(QR)** and **B** is (90^o – θ),

Magnitude of the force F_1 = BIb sin (90^o – θ)

ie. F_1 = BIb cos θ

Force on the arm SP,

$$\vec{F_2} = \overline{I(\overrightarrow{SP})} \times \overline{B}$$

Since the angle between **I(SP)** and **B** is (90^o + θ),

Magnitude of the force F_2 = BIb cos θ

The forces F_1 and F_2 are equal in magnitude, opposite in direction and have the same line of action. Hence their resultant effect on the loop is zero.

Force on the arm PQ,

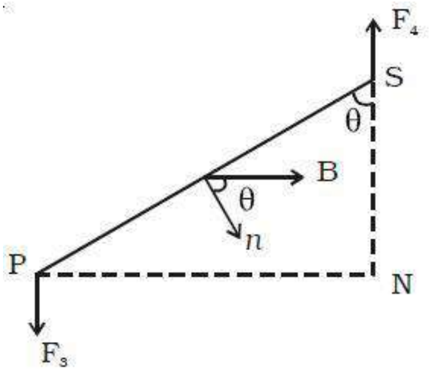
$$\vec{F} = \overline{I(\overrightarrow{PQ})} \times \overline{B}$$

Since the angle between I(PQ) and B is 90^o,
Magnitude of the force F_3 = BIL sin 90^o = BIL
F_3 acts perpendicular to the plane of the paper and outwards.

Force on the arm RS,

$$\overrightarrow{F_4} = \overrightarrow{I(RS)} \times \overline{B}$$

Since the angle between I(RS) and B is 90^o,
Magnitude of the force F_4 = BIL sin 90o = BIL



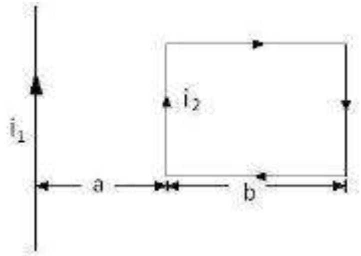
F_4 acts perpendicular to the plane of the paper and inwards.

The forces F_3 and F_4 are equal in magnitude, opposite in direction and have different lines of action. So, they constitute a couple. Hence,
Torque = BIL × PN = BIL × PS × sin θ = BIL × b sin θ = BIA sin θ
If the coil contains N turns, τ = NBIA sin θ
So, the torque is maximum when the coil is parallel to the magnetic field and zero when the coil is perpendicular to the magnetic field.
The torques can be expressed as vector product of the magnetic moment of the coil and the magnetic field. We define the magnetic moment of the current loop as, **m** =N I **A** where the direction of the area vector **A** is given by the right-hand thumb rule and is directed into the plane of the paper in Figure. Then as the angle between **m** and **B** is θ
τ = **m** × **B**
The dimensions of the magnetic moment are [A] [L^2] and its unit is Am^2.
From equation we see

(i) the torque τ vanishes when **m** is either parallel or antiparallel to the magnetic field **B**.
(ii) This indicates a state of equilibrium as there is no torque on the coil (this also applies to any object with a magnetic moment **m**).
(iii) When **m** and **B** are parallel the equilibrium is a stable one. Any small rotation of the coil produces a torque which brings it back to its original position.
(iv) When they are antiparallel, the equilibrium is unstable as any rotation produces a torque which increases with the amount of rotation.
(v) The presence of this torque is also the reason why a small magnet or any magnetic dipole aligns itself with the external magnetic field.

Solved Numerical

Q) The arrangement is as shown below
(a) Find the potential energy of the loop
(b) Find the work done to increase the spacing between the wire and the loop a to 2a

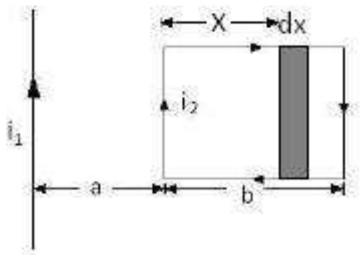


Solution:

(a) Magnetic field produced due to wire carrying current I_1 is inversely proportional to distance thus magnetic field associated with loop is not uniform. Consider small area of width dx and height L

Magnetic moment of a small element of the loop dM = i_2Ldx

The direction of the magnetic moment is perpendicular to the plane of the paper



Potential energy dU = -**dM . B**

Where B is the magnetic field at the position of this element

$$B = \frac{\mu_0}{4\pi}\frac{2I_1}{a+x}$$

$$dU = -\frac{\mu_0}{4\pi}2I_1I_2L\left(\frac{dx}{a+x}\right)$$

$$U = -\frac{\mu_0}{4\pi}2I_1I_2L\int_0^L\left(\frac{dx}{a+x}\right)$$

$$U = \frac{\mu_0}{4\pi}2I_1I_2L\log\left(\frac{a+b}{a}\right)$$

(b) Work done to increase the spacing between the wire and the loop from a to 2a

W = ΔU

$$U_i = -\frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{a+b}{a}\right)$$

$$U_f = \frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{2a+b}{2a}\right)$$

$$\Delta U = U_f - U_i = \frac{\mu_0}{4\pi} 2I_1I_2L\log\left(\frac{2a+2b}{2a+b}\right)$$

Q) A rectangular coil of area 20 cm × 10 cm with 100 turns of wire is suspended in a radial magnetic field of induction 5 × 10-3 T. If the galvanometer shows an angular deflection of 150 for a current of 1mA, find the torsional constant of the suspension wire.

Solution:

n = 100, A = 20 cm × 10 cm = 2 × 10^{-1} × 10^{-1} m^2 , B = 5 × 10^{-3} T, , I = 1mA = 10^{-3}A,

θ = 15^0 = 15 (π/180) = π/12 , C = ?

nBIA = Cθ

$$C = \frac{nBIA}{\theta}$$

$$C = \frac{10\times5\times10^{-3}\times10^{-3}\times2\times10^{-1}\times10^{-1}}{\left(\frac{\pi}{12}\right)}$$

$$C = 3.82\times10^{-5}\,Nm\,rad^{-1}$$

Moving coil galvanometer

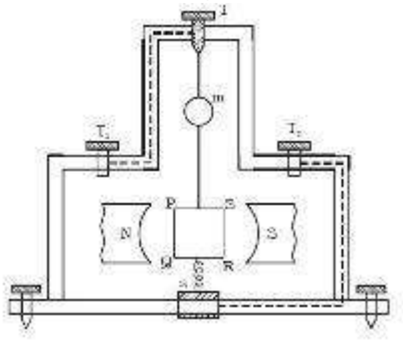
Moving coil galvanometer is a device used for measuring the current in a circuit.

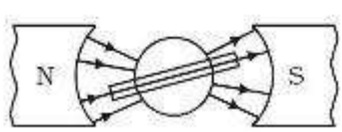
Principle

Moving coil galvanometer works on the principle that a current carrying coil placed in a magnetic field experiences a torque.

Construction

It consists of a rectangular coil of a large number of turns of thin insulated copper wire wound over a light metallic frame shown in figure

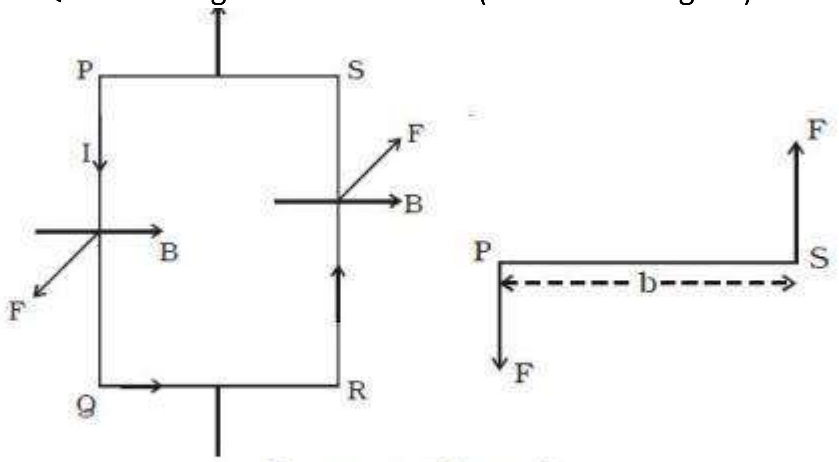




The coil is suspended between the pole pieces of a horse-shoe magnet by a fine phosphor – bronze strip from a movable torsion head. The lower end of the coil is connected to a hair spring (HS) of phosphor bronze having only a few turns. The other end of the spring is connected to a binding screw. A soft iron cylinder is placed symmetrically inside the coil. The hemispherical magnetic poles produce a radial magnetic field in which the plane of the coil is parallel to the magnetic field in all its positions as shown in figure. A small plane mirror (m) attached to the suspension wire is used along with a lamp and scale arrangement to measure the deflection of the coil.

Theory

Let PQRS be a single turn of the coil (as shown in figure)



Torque on the coil

A current I flows through the coil. In a radial magnetic field, the plane of the coil is always parallel to the magnetic field. Hence the sides QR and SP are always parallel to the field. So, they do not experience any force. The sides PQ and RS are always perpendicular to the field. PQ = RS = L, length of the coil and PS = QR = b, breadth of the coil

Force on PQ, F = BI (PQ) = BIL. According to Fleming's left hand rule, this force is normal to the plane of the coil and acts outwards.

Force on RS, F = BI (RS) = BIL.

This force is normal to the plane of the coil and acts inwards.

These two equal, oppositely directed parallel forces having different lines of action constitute a couple and deflect the coil. If there are *n* turns in the coil,

Torque of the deflecting couple = *N* BIL × b = *N*BIA

When the coil deflects, the suspension wire is twisted. On account of elasticity, a restoring couple is set up in the wire. This couple is proportional to the twist. If θ is the angular twist, then, moment of the restoring couple = Cθ

where C is the restoring couple per unit twist

At equilibrium, deflecting couple = restoring couple

$$NBIA = C\theta$$

$$I = \frac{C}{NBA}\theta$$

$$I = K\theta$$

Here K is the galvanometer constant.

$I \propto \theta$.

Since the deflection is directly proportional to the current flowing through the coil, the scale is linear and is calibrated to give directly the value of the current

We define the current sensitivity of the galvanometer as the deflection per unit current.
current sensitivity is,

$$\frac{\theta}{I} = \frac{NAB}{C}$$

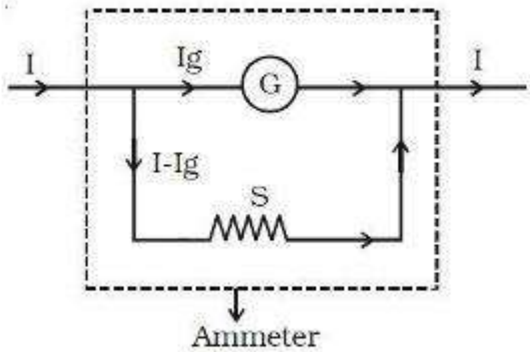
C is restoring couple per unit twist
Note current sensitivity is proportional to number of turns (N)

Conversion of galvanometer into an ammeter

A galvanometer is a device used to detect the flow of current in an electrical circuit. To measure current in circuit Galvanometer is connected in series. Because of following reasons Galvanometer cannot be used as Ammeter

(i) Being a very sensitive instrument, a large current cannot be passed through the galvanometer, as it may damage the coil.
(ii) Galvanometer have resistance in few kilo-ohm resistance which get added to resistance of circuit as a result current in circuit changes.

However, a galvanometer is converted into an ammeter by connecting a low resistance in parallel with it. As a result, when large current flows in a circuit, only a small fraction of the current passes through the galvanometer and the remaining larger portion of the current passes through the low resistance.
The low resistance connected in parallel with the galvanometer is called shunt resistance. The scale is marked in ampere. The value of shunt resistance depends on the fraction of the total current required to be passed through the galvanometer.
Let I_g be the maximum current that can be passed through the galvanometer.



The current I_g will give full scale deflection in the galvanometer.
Galvanometer resistance = G. Shunt resistance = S
Current in the circuit = I
Current through the shunt resistance $I_s = (I – I_g)$

Since the galvanometer and shunt resistance are parallel, potential is common.
$I_g \cdot G = (I - I_g)S$

$$S = G\frac{I_g}{I - I_g}$$

The shunt resistance is very small because Ig is only a fraction of I. The effective resistance of the ammeter R_a is (G in parallel with S)

$$\frac{1}{R_a} = \frac{1}{G} + \frac{1}{S}$$

$$R_a = \frac{GS}{G+S}$$

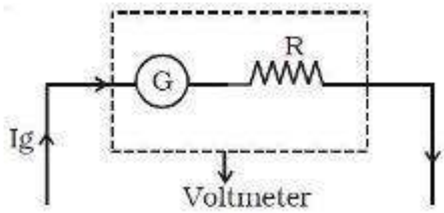
R_a is very low and this explains why an ammeter should be connected in series. When connected in series, the ammeter does not appreciably change the resistance and current in the circuit. Hence an ideal ammeter is one which has zero resistance.

Conversion of galvanometer into a voltmeter

Voltmeter is an instrument used to measure potential difference between the two ends of a current carrying conductor. A galvanometer can be converted into a voltmeter by connecting a high resistance in series with it. The scale is calibrated in volt. The value of the resistance connected in series decides the range of the voltmeter.
Galvanometer resistance = G
The current required to produce full scale deflection in the galvanometer = Ig
Range of voltmeter = V Resistance to be connected in series = R
Since R is connected in series with the galvanometer, the current through the galvanometer,

$$I_g = \frac{V}{R+G}$$

$$R = \frac{V}{I_g} - G$$



From the equation the resistance to be connected in series with the galvanometer is calculated.
The effective resistance of the voltmeter is $R_v = G + R$
R_v is very large, and hence a voltmeter is connected in parallel in a circuit as it draws the least current from the circuit.
In other words, the resistance of the voltmeter should be very large compared to the resistance across which the voltmeter is connected to measure the potential difference. Otherwise, the voltmeter will draw a large current from the circuit and hence the current through the remaining part of the circuit decreases. In such a case the potential difference measured by the voltmeter is very much less than the actual potential difference. The error is eliminated only when the voltmeter has a high resistance. An ideal voltmeter is one which has infinite resistance.

Current loop as a magnetic dipole

Ampere found that the distribution of magnetic lines of force around a finite current carrying solenoid is similar to that produced by a bar magnet. This is evident from the fact that a compass needle when similar deflections moved around these two bodies show. After noting the close resemblance between these two, Ampere demonstrated that a simple current loop behaves like a bar magnet and put forward that all the magnetic phenomena is due to circulating electric current. This is Ampere's hypothesis.
The magnetic induction at a point along the axis of a circular coil carrying current is

$$B = \frac{\mu_0 N I a^2}{2\left(a^2 + x^2\right)^{3/2}}$$

The direction of this magnetic field is along the axis and is given by right hand rule. For points which are far away from the centre of the coil, *x>>a, a*^{2} is small and it is neglected. Hence for such points,

$$B = \frac{\mu_0 N I a^2}{2x^3}$$

If we consider a circular loop, n = 1, its area A = πa^2

$$B = \frac{\mu_0 I A}{2\pi x^3} \quad \text{....eq(1)}$$

The magnetic induction at a point along the axial line of a short bar magnet is

$$B = \frac{\mu_0}{4\pi}\frac{2M}{x^3}$$

$$B = \frac{\mu_0}{2\pi}\frac{M}{x^3} \quad \text{...eq(2)}$$

Comparing equations (1) and (2), we find that
M = IA ...(3)
Hence a current loop is equivalent to a magnetic dipole of moment M = IA
The magnetic moment of a current loop is defined as the product of the current and the loop area. Its direction is perpendicular to the plane of the loop.

The magnetic dipole moment of a revolving electron

According to Neil Bohr's atom model, the negatively charged electron is revolving around a positively charged nucleus in a circular orbit of radius *r*. The revolving electron in a closed path constitutes an electric current. The motion of the electron in anticlockwise direction produces conventional current in clockwise direction. Current, i = e/T
where T is the period of revolution of the electron. If f is the orbital velocity of the electron, then

$$T = \frac{2\pi r}{v}$$

$$i = \frac{ev}{2\pi r}$$

Due to the orbital motion of the electron, there will be orbital magnetic moment μ_l

$\mu_l = i$ A, where A is the area of the orbit.

$$\mu_l = \frac{ev}{2\pi r}\pi r^2$$

$$\mu_l = \frac{evr}{2}$$

If m is the mass of the electron

$$\mu_l = \frac{e}{2m}(mvr)$$

mvr is the angular momentum (l) of the electron about the central nucleus.

$$\mu_l = \frac{e}{2m}(l)$$

$$\frac{\mu_l}{l} = \frac{e}{2m}$$

is called gyromagnetic ratio and is a constant. Its value is 8.8×10^{10} C kg^{-1}. Bohr hypothesized that the angular momentum has only discrete set of values given by the equation.

$l = nh/2\pi$ ...(2) where n is a natural number

and h is the Planck's constant = 6.626×10^{-34} Js.

From above two equations for μ_l we get

$$\mu_l = \frac{e}{2m}\frac{nh}{2\pi} = \frac{neh}{4\pi m}$$

The minimum value of magnetic moment is

$$(\mu_l)_{min} = \frac{eh}{4\pi m}, n = 1$$

Value of (eh/4πm) is called Bohr magneton

By substituting the values of e, h and m, the value of Bohr magneton is found to be 9.27×10^{-24} Am^2

In addition to the magnetic moment due to its orbital motion, the electron possesses magnetic moment due to its spin. Hence the resultant magnetic moment of an electron is the vector sum of its orbital magnetic moment and its spin magnetic moment.

Solved Numerical

Q) A moving coil galvanometer of resistance 20 Ω produces full scale deflection for a current of 50 mA. How you will convert the galvanometer into (i) an ammeter of range 20 A and (ii) a voltmeter of range 120 V.

Solution:

G = 20 Ω ; Ig = 50 x 10^{-3} A ; I = 20 A, S = ?

V = 120 V, R = ?

(i) Ammeter

$$S = G\frac{I_g}{I - I_g}$$

$$S = 20\frac{50\times10^{-3}}{20 - 50\times10^{-3}}$$

$$S = 0.05\,\Omega$$

A shunt of 0.05 Ω should be connected in parallel

(ii) Voltmeter

$$R = \frac{V}{I_g} - G$$

$$R = \frac{120}{50\times10^{-3}} - 20$$

$$R = 2380\,\Omega$$

A resistance of 2380 Ω should be connected in series with the galvanometer

Q) The deflection in a galvanometer falls from 50 divisions to 10 divisions when 12 Ω resistance is connected across the galvanometer. Calculate the galvanometer resistance.

Solution:

θ_1 = 50 divs, θ_g = 10 divs, S = 12Ω G = ?

I α θ_1

Ig α θ_g

In a parallel circuit potential is common.

$$G\cdot I_g = S\left(I - I_g\right)$$

$$G = \frac{S\left(I - I_g\right)}{I_g}$$

$$G = \frac{12\left(50 - 10\right)}{10}$$

$$G = 48\,\Omega$$

Q) In a hydrogen atom electron moves in an orbit of radius 0.5 Å making 1016 revolutions per second. Determine the magnetic moment associated with orbital motion of the electron.

<u>Solution:</u>

r = 0.5 Å = 0.5×10^{-10} m, n = 1016 s^{-1}

Orbital magnetic moment $\mu_l = i.A$ ...(1)

i = e/T

I = e.f ...(2)

$A = \pi r^2$ ...(3)

substituting equation (2), (3) in (1)

$\mu_l = e.n.\ \pi r^2$

$= 1.6 \times 10\text{-}19 \times 1016 \times 3.14\ (0.5 \times 10^{-10})^2$

$= 1.256 \times 10^{-23}$

$\therefore \mu_l = 1.256 \times 10^{-23}\ Am^2$

----------------END--------------

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Bar magnet

The iron ore magnetite which attracts small pieces of iron, cobalt, nickel etc. is a natural magnet. The natural magnets have irregular shape and they are weak. A piece of iron or steel acquires magnetic properties when it is rubbed with a magnet. Such magnets made out of iron or steel are artificial magnets. Artificial magnets can have desired Shape and desired strength. If the artificial magnet is in the form of a rectangular or cylindrical bar, it is called a bar magnet.

Basic properties of magnets

(i) When the magnet is dipped in iron filings, they cling to the ends of the magnet. The attraction is maximum at the two ends of the magnet. These ends are called poles of the magnet.
(ii) When a magnet is freely suspended, it always points along north-south direction. The pole pointing towards geographic north is called North Pole *N* and the pole which points towards geographic south is called South Pole *S.*
(iii) Magnetic poles always exist in pairs. (i.e) isolated magnetic pole does not exist.
(iv) The magnetic length of a magnet is always less than its geometric length, because the poles are situated a little inwards from the free ends of the magnet. (But for the purpose of calculation the geometric length is always taken as magnetic length.)
(v) Like poles repel each other and unlike poles attract each other.
North Pole of a magnet when brought near North Pole of another magnet, We can observe repulsion, but when the north pole of one magnet is brought near South Pole of another magnet, we observe attraction.
(vi) The force of attraction or repulsion between two magnetic poles is given by Coulomb's inverse square law.

Magnetic field

Magnetic field is the space in which a magnetic pole experiences a force or it is the space around a magnet in which the influence of the magnet is felt.

Magnetic lines of force

A magnetic field is better studied by drawing as many numbers of magnetic lines of force as possible. A magnetic line of force is a line along which a free isolated north pole would travel when it is placed in the magnetic field.

Properties of magnetic lines of force

(i) Magnetic lines of forces are closed continuous curves, extending through the body of the magnet.
(ii) The direction of line of force is from North Pole to South Pole outside the magnet. While it is from South Pole to North Pole inside the magnet.
(iii) The tangent to the magnetic line of force at any point gives the direction of magnetic field at that point. (i.e) it gives the direction of magnetic induction ($\vec{B}$) at that point.
(iv) They never intersect each other.
(v) They crowd where the magnetic field is strong and thin out where the field is weak.

Magnetic moment

Since any magnet has two poles, it is also called a magnetic dipole. The magnetic moment of a magnet is defined **as the product of the pole strength and the distance between the two poles.** If m is the pole strength of each pole and 2l is the distance
between the poles, the magnetic moment Magnetic moment is a vector quantity. It is denoted by M. Its unit is A m^2. Its direction is from south pole to north pole

$$\vec{M} = m(2\vec{l})$$

Bar magnet as an equivalent solenoid

The magnetic dipole moment m associated with a current loop was defined to be m=NIA where N is the number of turns in the loop, I the current and A the area vector. The direction of magnetic moment m of a loop can be found by using right hand rule, curl fingers in the direction of current then thumb gives the direction of magnetic moment. The resemblance of magnetic field lines for a bar magnet and a solenoid suggest that a bar magnet may be thought of as a large number of circulating currents in analogy with a solenoid.
Let the solenoid consists of n turns per unit length. Let its length be 2l and radius a. We can evaluate the axial field at a point P, at a distance r from the centre O of the solenoid. To do this, consider a circular element of thickness dx of the solenoid at a distance x from its centre. It consists of n d x turns. Let I be the current in the solenoid. The magnetic field on the axis of a circular current loop at point P due to the circular element is

$$dB = \frac{\mu_0 n dx I a^2}{2[(r-x)^2 + a^2]^{3/2}}$$

The magnitude of the total field is obtained by summing over all the elements — in other words by integrating from x = – l to x = + l . Thus,

$$B = \frac{\mu_0 n I a^2}{2} \int_{-l}^{+l} \frac{dx}{2[(r-x)^2 + a^2]^{3/2}}$$

Consider the far axial field of the solenoid, i.e., r >> a and r >> l . Then the denominator is approximated by

$$2[(r-x)^2 + a^2]^{3/2} = r^3$$

Then

$$B = \frac{\mu_0 n I a^2}{2r^3} \int_{-l}^{+l} dx$$

$$B = \frac{\mu_0 n I a^2}{2r^3}(2l)$$

Note that the magnitude of the magnetic moment of the solenoid is,
m = n (2l) I (πa^2) = (total number of turns × current × cross-sectional area).
Thus,

$$B = \frac{\mu_0}{4\pi}\frac{2m}{r^3}$$

This is also the far axial magnetic field of a bar magnet which one may obtain experimentally. Thus, a bar magnet and a solenoid produce similar magnetic fields. The magnetic moment of a bar magnet is thus equal to the magnetic moment of an equivalent solenoid that produces the same magnetic field.

Magnetic induction
Magnetic induction is the fundamental character of a magnetic field at a point. Magnetic induction at a point in a magnetic field is the force experienced by unit north pole placed at that point. It is denoted by B. Its unit is N/Am.
It is a vector quantity. It is also called as magnetic flux density. If a magnetic pole of strength m placed at a point in a magnetic field experiences a force F, the magnetic induction at that point is

$$\vec{B} = \frac{\vec{F}}{m}$$

Magnetic flux and magnetic flux density
The number of magnetic lines of force passing through an area A is called magnetic flux. It is denoted by φ. Its unit is Weber. It is a scalar quantity.

The number of magnetic lines of force crossing unit area kept normal to the direction of line of force is magnetic flux density. Its unit is Wb m^{-2} or Tesla.
Magnetic flux $\phi = \vec{B} \cdot \vec{A}$

Uniform and non-uniform magnetic field
Magnetic field is said to be uniform if the magnetic induction has the same magnitude and the same direction at all the points in the region. It is represented by drawing parallel lines
If the magnetic induction varies in magnitude and direction at different points in a region, the magnetic field is said to be non-uniform. The magnetic field due to a bar magnet is non-uniform. It is represented by convergent or divergent lines

Force between two magnetic poles
In 1785, Coulomb made use of his torsion balance and discovered the law governing the force between the two magnetic poles.

Coulomb's inverse square law
Coulomb's inverse square law states that the force of attraction or repulsion between the two magnetic poles is directly proportional to the product of their pole strengths and inversely proportional to the square of the distance between them.
If m_1 and m_2 are the pole strengths of two magnetic poles separated by a distance of d in a medium, then
F $\propto m_1m_2$ and F$\propto\frac{1}{d^2}$

$$F \propto \frac{m_1m_2}{d^2}$$

$$F = K\frac{m_1m_2}{d^2}$$

where *K* is the constant of proportionality and

$$K = \frac{\mu}{4\pi}$$

where μ is the permeability of the medium. But $\mu = \mu_0 \times \mu_r$
μ_0 - permeability of free space or vacuum.
μ_r - relative permeability of the medium
Let $m_1 = m_2 = 1$, and d = 1 m

$$K = \frac{\mu}{4\pi}$$

In free space, $\mu_o = 4\pi \times 10^{-7}$ H m^{-1}

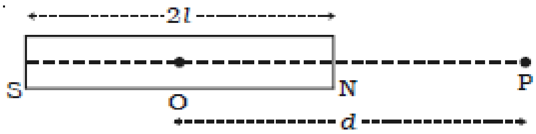
$$F = 10^{-7}\ \frac{m_1m_2}{d^2}$$

$$F = 10^{-7}\ \frac{1 \times 1}{1}$$

F = 10-7 N

Therefore, unit pole is defined as that pole which when placed at a distance of 1 metre in free space or air from an equal and similar pole, repels it with a force of 10^{-7} N.

Magnetic induction at a point along the axial line due to a magnetic dipole (Bar magnet)
NS is the bar magnet of length 2l and of pole strength m. P is a point on the axial line at a distance d from its midpoint O



According to inverse square law,

$$F = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{d^2}$$

Magnetic induction (B_1) at P due to north pole of the magnet,
Along NP

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(NP)^2}$$

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(d-l)^2}$$

Magnetic induction (B_2) at P due to south pole of the magnet,
Along PS

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(PS)^2}$$

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(d+l)^2}$$

Magnetic induction at *P* due to the bar magnet,

$$B = B_1 - B_2$$

$$B = \frac{\mu_0}{4\pi}\frac{m}{(d-l)^2} - \frac{\mu_0}{4\pi}\frac{m}{(d+l)^2}$$

$$B = \frac{\mu_0 m}{4\pi}\left(\frac{4ld}{(d^2-l^2)^2}\right)$$

$$B = \frac{\mu_0 m}{4\pi}\left(\frac{2l \times 2d}{(d^2-l^2)^2}\right)$$

$$B = \frac{\mu_0}{4\pi}\left(\frac{2l \times M}{(d^2-l^2)^2}\right)$$

As $M = m \times 2l$

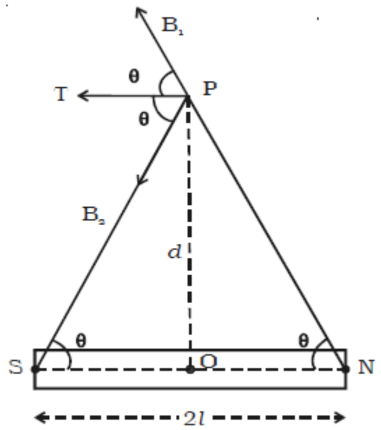
For a short bar magnet, l is very small compared to *d*, hence l^2 is neglected

$$B = \frac{\mu_0}{4\pi}\frac{2M}{d^3}$$

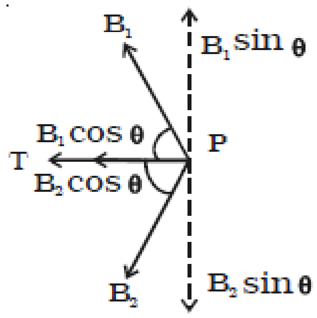
The direction of B is along the axial line away from the north pole.

Magnetic induction at a point along the equatorial line of a bar magnet

NS is the bar magnet of length 2*l* and pole strength m. P is a point on the equatorial line at a distance d from its midpoint O



Components magnetic field at point P are as follows



Magnetic induction (B_1) at P due to north pole of the magnet,
Along NP

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(NP)^2}$$

$$B_1 = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}$$

(AS $NP^2 = NO^2 + OP^2$)
Magnetic induction ($B2$) at P due to south pole of the magnet,
Along PS

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(PS)^2}$$

$$B_2 = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}$$

Resolving B_1 and B_2 into their horizontal and vertical components. Vertical components B_1 sin θ and B_2 *sin* θ are equal and opposite and therefore cancel each other
The horizontal components B_1 cos θ and B_2 cos θ will get added along PT.
Resultant magnetic induction at P due to the bar magnet is
$B = B_1 \cos\theta + B_2 \cos\theta$. (along PT)

$$B = \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}\frac{l}{\sqrt{d^2+l^2}} + \frac{\mu_0}{4\pi}\frac{m}{(d^2+l^2)}\frac{l}{\sqrt{d^2+l^2}}$$

$$cos\theta = \frac{SO}{PS} = \frac{NO}{NP}$$

As M =2lm

$$B = \frac{\mu_0}{4\pi}\frac{M}{(d^2+l^2)^{3/2}}$$

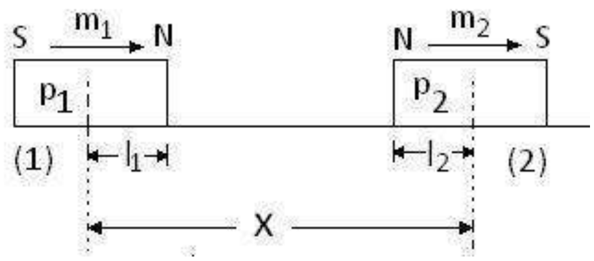
For a short bar magnet, l^2 is neglected.

$$B = \frac{\mu_0}{4\pi}\frac{M}{d^3}$$

The direction of 'B' is along PT parallel to NS.

Solved numerical

Q) Find the force between two small bar magnets of magnetic moments m_1 and m_2 lying on the axis, as shown in figure (p_1 and p_2 are the pole strength of magnet (1) and (2) respectively, X is far greater than l_1 and l_2)



Solution :
To calculate force on magnet (2) due to magnet (1)
We will calculate magnetic field due to magnet (1) at the poles of the magnet(2).
Magnet (2) is on the axis of magnet (1).
Magnetic field at North pole of magnet (2), magnetic moment of magnet(1) is m_1

$$B_N = \frac{\mu_0}{4\pi}\frac{2m_1}{(x-l_2)^3}$$

The repulsive force F_N acting on the north pole of magnet(2) having pole strength p_2

$$F_N = p_2 B_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(x-l_2)^3}$$

Similarly magnetic field at South pole of magnet(2), is

$$B_S = \frac{\mu_0}{4\pi}\frac{2m_1}{(x+l_2)^3}$$

The attractive force F_S acting on the north pole of magnet(2) having pole strength p_2

$$F_S = p_2 B_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(x+l_2)^3}$$

Hence resultant force on magnet(2) is
$F=F_N - F_S$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{1}{(x-l_2)^3} - \frac{1}{(x+l_2)^3}\right]$$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{6x^2 l_2}{(x^2-l_2^2)^3}\right]$$

$$F = \frac{\mu_0}{4\pi} 2p_2 m_1 \left[\frac{6x^2 l_2}{(x^2)^3}\right]$$

$$F = \frac{\mu_0 m_1}{2\pi}\left[\frac{(2p_2 l_2)(3x^2)}{x^6}\right]$$

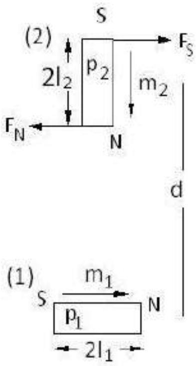
As $p_2 l_2 = m_2$

$$F = \frac{\mu_0 m_1}{2\pi}\left[\frac{(m_2)(3)}{x^4}\right]$$

$$F = \frac{3\mu_0 m_1 m_2}{2\pi x^4}$$

Resultant force is repulsive and acts on magnet (2) in a direction away from magnet(1)

Q) Find the torque on small bar magnet(2) due to small bar magnet(1), when they are placed perpendicular to each other as shown in figure l_2 and l_1 are far less than d



Solution :

From the figure Magnetic field at north pole of second magnet due to magnet(1) is

$$B_N = \frac{\mu_0}{4\pi}\frac{2m_1}{(d-l_2)^3}$$

Force on north pole is towards left is

$$F_N = p_2 B_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d-l_2)^3}$$

As X >> l_2

$$F_N = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d)^3}$$

Magnetic field at south pole of second magnet due to magnet(1) is

$$B_S = \frac{\mu_0}{4\pi}\frac{2m_1}{(d)^3}$$

Force on south pole is towards right is

$$F_S = p_2 B_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d+l_2)^3}$$

As X >> l_2

$$F_S = \frac{\mu_0}{4\pi}\frac{2m_1 p_2}{(d)^3}$$

Since $F_N = F_S$ are non collinear, equal and opposite in direction, they form a couple. Hence the torque is produced

$$\vec{\tau} = \vec{F_N} \times 2\vec{l_2} = \vec{F_S} \times 2\vec{l_2}$$

Since F_N and F_S are perpendicular to l_2 magnitude of torque with respect to centre of magnet (2)

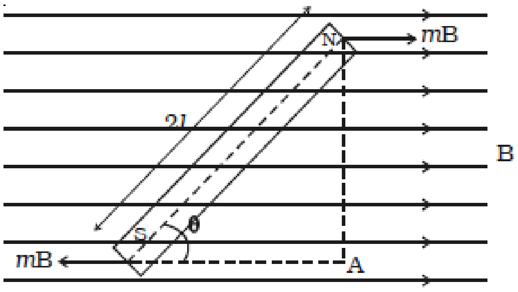
$$\tau = 2F_N l_2 = \frac{\mu_0}{4\pi}\frac{m_1 2l_2 p_2}{d^3}$$

as $2l_2p_2 = m_2$ magnetic dipole moment of magnet(2)

$$\tau = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{d^3}$$

Torque on a bar magnet placed in a uniform magnetic field

Consider a bar magnet *NS* of length 2*l* and pole strength m placed in a uniform magnetic field of induction *B* at an angle θ with the direction of the field



Due to the magnetic field B, a force mB acts on the north pole along the direction of the field and a force mB acts on the south pole along the direction opposite to the magnetic field.
These two forces are equal and opposite, hence constitute a couple. The torque τ due to the couple is
τ= one of the forces × perpendicular distance between them
τ = F × NA
τ = mB × NA ...(1)
τ = mB × 2l sin θ
τ = MB sin θ ...(2)
Vectorially,

$$\vec{\tau} = \vec{M} \times \vec{B}$$

The direction of τ is perpendicular to the plane containing $\vec{M}$ and $\vec{B}$
If B = 1 and θ = 90°
Then from equation (2), τ = M
Hence, moment of the magnet M is equal to the torque necessary to keep the agent at right angles to a magnetic field of unit magnetic induction.

Periodic time of bar magnet
Here τ is restoring torque and θ is the angle between $\vec{M}$ and $\vec{B}$.
Now Newton's second law

$$\tau = I\frac{d^2\theta}{dt^2}$$

Here I is moment of inertia of bar magnet
Therefore, in equilibrium

$$I\frac{d^2\theta}{dt^2} = -mBsin\theta$$

Negative sign with mB sinθ implies that restoring torque is in opposition to deflecting torque. For small values of θ in radians, we approximate sin θ ≈ θ and get

$$I\frac{d^2\theta}{dt^2} = -mB\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{mB}{I}\theta$$

This represents a simple harmonic motion. The square of the angular frequency is $\omega^2 = mB/I$ and the time period is

$$T = 2\pi\sqrt{\frac{I}{mB}}$$

The magnetic potential energy U_m
The magnetic potential energy U_m is given by

$$U_m = \int \tau d\theta$$

$$U_m = \int mBsin\theta d\theta$$

$$U_m = -mBcos\theta$$

$$U_m = \vec{m} \cdot \vec{B}$$

When the needle is perpendicular to the field, Equation shows that potential energy is minimum (= –mB) at θ = 0° (most stable position) and maximum (= +mB) at θ = 180° (most unstable position).

Solved numerical

Q) Work done in moving a magnet of magnetic moment m from most stable to most unstable position
Solution:
Most stable position is θ = 0° and most unstable position is θ = 180° hence work done
W = U_B(θ =180°) - U_B(θ = 0°) = mB –(-mB) = 2mB

Q) A bar magnet is suspended horizontally by a torsion less wire in magnetic meridian. In order to deflect the magnet through 30° from the magnetic meridian, the upper end of the wire has to be rotated by 270°. Now this magnet is replaced by another magnet. In order to deflect the second magnet through the same angle from the magnetic meridian, the upper end of the wire has to be rotated by 180°. What is the ratio of the magnetic moments of the two bar magnets.

Solution

Let C be the deflecting torque per unit twist and M_1 and M_2 be the magnetic moments of the two magnets.

The deflecting torque is $\tau = C\theta$

The restoring torque is $\tau = MB \sin\theta$

In equilibrium,

deflecting torque = restoring torque

For the Magnet – I

$C(270^o - 30^o) = M_1 B_H \sin\theta$ ... (1)

For the magnet – II

$C(180^o - 30^o) = M_2 B_H \sin\theta$ ... (2)

Dividing (1) by (2)

$$\frac{M_1}{M_2} = \frac{240^o}{150^o} = \frac{8}{5}$$

Q) A magnetic needle placed in uniform magnetic field has magnetic moment $6.7\times10^{-2} Am^2$ and moment of inertia of 15×10^{-6} kgm^2. It performs 10 complete oscillations in 6.7 s. What is the magnitude of the magnetic field

Solution:

The periodic time of oscillation is

$$T = 2\pi\sqrt{\frac{I}{mB}}$$

$$B = 4\pi^2\frac{I}{mT^2}$$

$$B = \frac{4\pi^2(3.13)^2 \times 15 \times 10^{-6}}{6.7 \times 10^2 \times (0.67)^2} = 0.02T$$

Gauss's Law for Magnetic Field

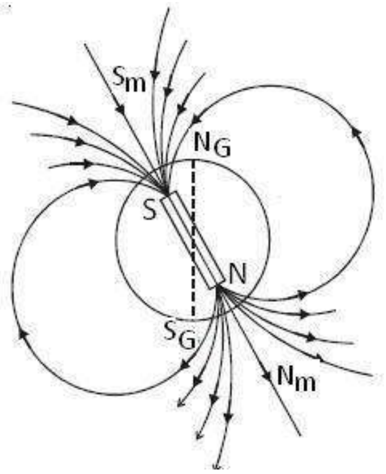
Magnetic field lines always forms a closed loops, the magnetic flux associated with any closed surface is always zero

$$\oint_{\substack{Closed\\Sufrace}} \vec{B}\cdot\overrightarrow{da} = 0$$

Where B is the magnetic field and ds is an infinitesimal area vector of the closed surface "The net magnetic flux passing through any closed surface is zero" This statement is called Gauss's law for magnetic field.

Earth's magnetic field and magnetic elements

A freely suspended magnetic needle at a point on Earth comes to rest approximately along the geographical north - south direction.

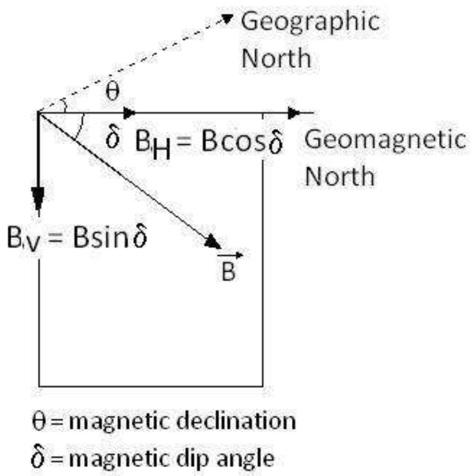


This shows that the Earth behaves like a huge magnetic dipole with its magnetic poles near its geographical poles. Since the north pole of the magnetic needle approximately points towards geographic north (NG) it is appropriate to call the magnetic pole near NG as the magnetic south pole of Earth Sm. Also, the pole near SG is the magnetic north pole of the Earth (Nm).

The Earth's magnetic field at any point on the Earth can be completely defined in terms of certain quantities called magnetic elements of the Earth, namely

(i) Declination or the magnetic variation θ.

The angle between the magnetic meridian and geographic meridian at a place on the surface of the earth is called magnetic declination at that place



(ii) Dip or inclination δ

Magnetic dip or angle of inclination is the angle δ (up or down) that the magnetic field of earth makes with the horizontal at a place in magnetic meridian

(iii) The horizontal and vertical component of the Earth's magnetic field.

$B_V = B\sin\delta$ and $B_H = B\cos\delta$

$$tan\delta = \frac{B_V}{B_H}$$

$$B = \sqrt{B_V^2 + B_H^2}$$

Causes of the Earth's magnetism

The exact cause of the Earth's magnetism is not known even today. However, some important factors which may be the cause of Earth's magnetism are:

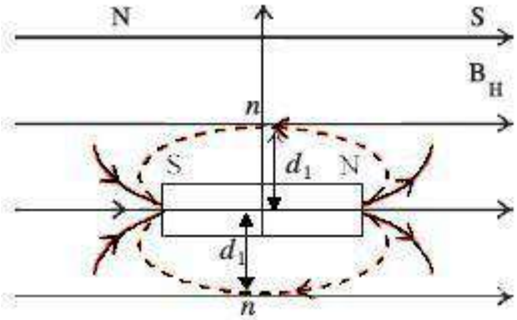
(i) Magnetic masses in the Earth.
(ii) Electric currents in the Earth.
(iii) Electric currents in the upper regions of the atmosphere.
(iv) Radiations from the Sun.
(v) Action of moon etc.
However, it is believed that the Earth's magnetic field is due to the molten charged metallic fluid inside the Earth's surface with a core of radius about 3500 km compared to the Earth's radius of 6400 km.

Solved Numerical

Q) A short bar magnet is placed with its north pole pointing north. The neutral point is 10 cm away from the centre of the magnet. If B = 4×10^{-5} T, calculate the magnetic moment of the magnet.

Solution:

When we keep North pole pointing north pole it means, it is in the direction of field lines of earth is opposite to magnetic field lines of magnet.
As shown in figure let neutral point (where effective magnetic field becomes zero) be at point n, at distance d_1 = 20 cm



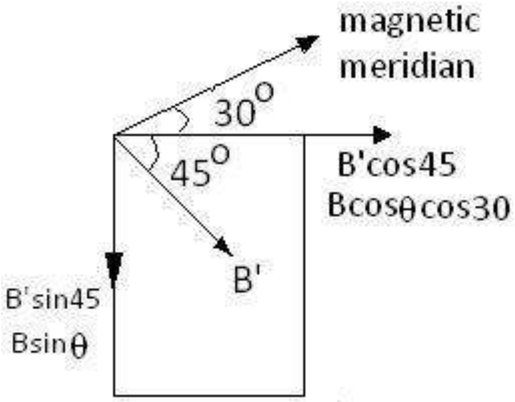
Now magnetic field due to bar magnet = Horizontal component of earth

$$\frac{\mu_0}{4\pi}\frac{m}{d_1^3} = B_H$$

$$10^{-7}\frac{m}{(0.1)^3} = 4 \times 10^5$$

m=0.4 A m^2

Q) A magnet makes an angle of 45° with the horizontal in a plane making an angle of 30° with the magnetic meridian. Find the true value of the dip angle at the place.
Solution:



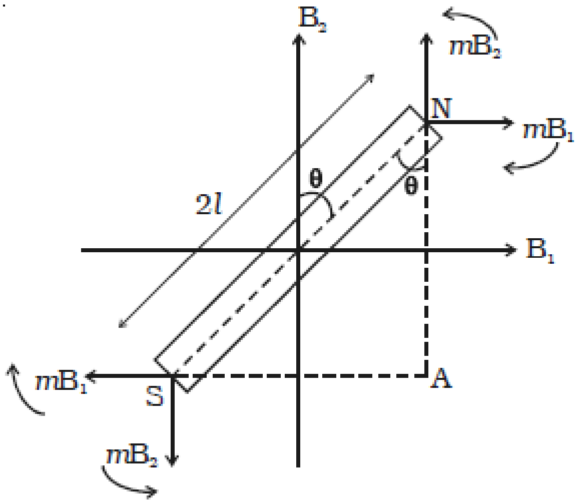
Let B be the magnetic field in magnetic meridian, making an angle of θ with horizontal.
Thus Horizontal component is B_H = Bcos θ and
vertical component is B_V = Bsinθ
Component of Horizontal component of magnetic field in magnetic meridian along plane = Bcosθcos30
Let magnetic field in plane be B'. Thus Horizontal component B'_H = B'cos45 and Vertical component B'_V = B'sin45
From above

$B\sin\theta = B'\sin45$ eq(1)
And
$B\cos\theta\cos30 = B'\cos45$ eq(2)
Taking ratio of eq(1) and eq(2) we get

$$\frac{B\sin\theta}{B\cos\theta\cos30} = \frac{B'\sin45}{B'\cos45}$$
$$\tan\theta = \cos30$$
$$\tan\theta = \frac{\sqrt{3}}{2} = 0.866$$
$$\theta = \tan^{-1}(0.866)$$

Tangent law

A magnetic needle suspended, at a point where there are two crossed magnetic fields acting at right angles to each other, will come to rest in the direction of the resultant of the two fields.



B_1 and B_2 are two uniform magnetic fields acting at right angles to each other. A magnetic needle placed in these two fields will be subjected to two torques tending to rotate the magnet in opposite directions. The torque τ_1 due to the two equal and opposite parallel forces mB_1 and mB_1 tend to set the magnet parallel to B_1. Similarly the torque τ_2 due to the two equal and opposite parallel forces mB_2 and mB_2 tends to set the magnet parallel to B_2. In a position where the torques balances each other, the magnet comes to rest. Now the magnet makes an angle θ with B_2 as shown in the Fig.
The deflecting torque due to the forces mB_1 and mB_1
$\tau_1 = mB_1 \times NA$
$= mB_1 \times NS\cos\theta$
$= mB_1 \times 2l\cos\theta$
$= 2l\, mB_1\cos\theta$
$\therefore \tau_1 = MB_1\cos\theta$
Similarly the restoring torque due to the forces mB_2 and mB_2
$\tau_2 = mB_2 \times SA$
$= mB_2 \times 2l\sin\theta$
$= 2lm \times B2\sin\theta$
$\tau_2 = MB2\sin\theta$
At equilibrium,
$\tau_1 = \tau_2$
$\therefore MB_1\cos\theta = MB_2\sin\theta$
$\therefore B_1 = B_2\tan\theta$
This is called Tangent law
Invariably, in the applications of tangent law, the restoring magnetic

field B_2 is the horizontal component of Earth's magnetic field B_H.

Solved Numerical

A short bar magnet of magnetic moment 5.25×10^{-2} A m^2 is placed with its axis perpendicular to the Earth's field direction. At what distance from the centre of the magnet on (i) its equatorial line and (ii) its axial line, is the resultant field inclined at 45° with the Earth's field. Magnitude of the Earth's field at the place is 0.42×10^{-4} T.

Solution

From Tangent Law

$$\frac{B}{B_H} = tan\theta$$

$$B = B_H \tan\theta = 0.42 \times 10^{-4} \times \tan 45^\circ$$

$$B = 0.42 \times 10^{-4}\ T$$

(i) For the point on the equatorial line

$$B = \frac{\mu_0}{4\pi}\frac{m}{d^3}$$

$$d^3 = \frac{\mu_0}{4\pi}\frac{m}{B}$$

$$d^3 = 10^{-7} \times \frac{5.25 \times 10^{-2}}{0.42 \times 10^{-4}}$$

$$d = 5\times10^{-2}\ m$$

(ii) For the point on the axial line

$$B = \frac{\mu_0}{4\pi}\frac{2m}{d^3}$$

$$d^3 = \frac{\mu_0}{4\pi}\frac{2m}{B}$$

$$d^3 = 10^{-7} \times \frac{2 \times 5.25 \times 10^{-2}}{0.42 \times 10^{-4}}$$

$$d = 6.3 \times 10^{-2}\ m.$$

Magnetic properties of materials

The study of magnetic properties of materials assumes significance since these properties decide whether the material is suitable for permanent magnets or electromagnets or cores of transformers etc.

Before classifying the materials depending on their magnetic behavior, the following important terms are defined.

(i) Magnetizing field or magnetic intensity

The magnetic field used to magnetize a material is called the magnetizing field. It is denoted by H and its unit is A m^{-1}.

(Note : Since the origin of magnetism is linked to the current, the magnetizing field is usually defined in terms of ampere turn)

(ii) Magnetic permeability

Magnetic permeability is the ability of the material to allow the passage of magnetic lines of force through it. *Relative permeability μ_r of a material is defined as the ratio of number of magnetic lines of force per unit area B inside the material to the number of lines of force per unit area in vacuum B_o produced by the same magnetizing field.*

∴ Relative permeability $\mu_r = B / B_O$

$$\mu_r = \frac{\mu H}{\mu_0 H} = \frac{\mu}{\mu_0}$$

(since μ_r is the ratio of two identical quantities, it has no unit.)

∴ The magnetic permeability of the medium $\mu = \mu_o\mu_r$ where μ_o is the

permeability of free space.
Magnetic permeability μ of a medium is also defined as the ratio of magnetic induction B inside the medium to the magnetizing field H inside the same medium.

$$\mu = \frac{B}{H}$$

(iii) Intensity of magnetization
Intensity of magnetization represents the extent to which a material has been magnetized under the influence of magnetizing field H. Intensity of magnetization of a magnetic material is defined as the magnetic moment per unit volume of the material.

$$M = \frac{m}{V}$$

Its unit is A m^{-1}.
For a specimen of length 2l, area A and pole strength m,

$$M = \frac{2lm}{2lA}$$

$$M = \frac{m}{A}$$

Hence, intensity of magnetization (M) is also defined as the pole strength per unit area of the cross section of the material.

(iv) Magnetic induction
When a soft iron bar is placed in a uniform magnetizing field H, the magnetic induction inside the specimen B is equal to the sum of the magnetic induction B_o produced in vacuum due to the magnetizing field and the magnetic induction B_m due to the induced magnetization of the specimen.
$B = B_o + B_m$
But $B_o = \mu_o H$ and $B_m = \mu_o M$
$B = \mu_o H + \mu_o M$
$\therefore B = \mu_o (H + M)$

$$H = \frac{B}{\mu o} - M$$

where **H** has the same dimensions as **M** and is measured in units of A m^{-1}.
Thus, the total magnetic field **B**

(v) Magnetic susceptibility
Magnetic susceptibility χ_m is a property which determines how easily and how strongly a specimen can be magnetized.
Susceptibility of a magnetic material is defined as the ratio of intensity of magnetization induced in the material to the magnetizing field H in which the material is placed.
Thus

$$\chi_m = \frac{M}{H}$$

Since *I* and *H* are of the same dimensions, χ_m has no unit and is dimensionless.

Relation between χ_m and μ_r

$$\chi_m = \frac{M}{H}$$

$$M = \chi_m H$$

We know $B = \mu_o (H + M)$

$$B = \mu_o (H + \chi_m H)$$

$$B = \mu_o H (1 + \chi_m)$$

If μ is the permeability, we know that $B = \mu H$.

$$\therefore \mu H = \mu_o H (1 + \chi_m)$$

$$\frac{\mu}{\mu_0} = (1 + \chi_m)$$

$$\mu_r = 1 + \chi_m$$

Solved Numerical

A bar magnet of mass 90 g has magnetic moment 3 A m^2. If the intensity of magnetization of the magnet is 2.7×10^5 A m^{-1}, find the density of the material of the magnet.

Solution

Intensity of magnetization, $M = \frac{m}{V}$

volume V =mass/ ρ

$$M = \frac{m\rho}{mass}$$

$$\rho = \frac{M \times mass}{m} = \frac{2.7 \times 10^5 \times 0.090}{3}$$

$$\rho = 8100\ \frac{kg}{m^3}$$

Q) A magnetizing field of 50 A m^{-1} produces a magnetic field of induction 0.024 T in a bar of length 8 cm and area of cross section 1.5 cm^2. Calculate (i) the magnetic permeability (ii) the magnetic susceptibility

Solution

Permeability

$$\mu = \frac{B}{H} = \frac{2.4 \times 10^{-2}}{50} = 4.8 \times 10^{-4} Hm^{-1}$$

susceptibility

$$\chi_m = \frac{\mu}{\mu_0} - 1$$

$$\chi_m = \frac{4.8 \times 10^{-4}}{4\pi \times 10^{-7}} - 1 = 381.16$$

Q) A solenoid has a core of material with relative permeability of 400. The current passing through the wire of solenoid is 2A. If the number of turns per cm are 10, calculate the magnitude of

(a) H (b) B (c) χ_m (d) M

Solution

Here μ_r = 400, I = 2A n = 10 turns/cm = 1000 turns/m

(a) Magnetic intensity H = nI = 1000 × 2 = 2000 Am^{-1}

(b) Magnetic field B = $\mu_o \mu_r H = 4\pi \times 10^{-7} \times 400 \times 2000$ = 1.0 T

(c) Magnetic susceptibility of the core material is

$\chi_m = \mu_r - 1 = 400 - 1 = 399$

(d) Magnetization

M = χ_mH = 399 × 2000 = 8 ×10^5 A/m

Q) The region inside a current carrying toroidial winding is filled with tungsten of susceptibility 6.8×10^{-5}. What is the percentage increase in the magnetic field in the presence of the material with respect to the magnetic field without it?

Solution:

The magnetic field in the current carrying toroidial winding without tungsten is

$B_O = \mu_O H$

The magnetic field in the same current carrying toroidial winding with tungsten is

B = μH

$$\therefore \frac{B - B_o}{B_o} = \frac{\mu - \mu_o}{\mu_o}$$

But $\mu = \mu_o (1 + \chi_m)$

$$\frac{\mu}{\mu_o} = 1 + \chi_m$$

$$\chi_m = \frac{\mu - \mu_O}{\mu_0}$$

$$\therefore \frac{B-B_o}{B_o} = \chi_m$$

$$\therefore \frac{B-B_o}{B_o} \times 100 = \chi_m \times 100$$
$$\therefore \frac{B-B_o}{B_o} \times 100 = (6.8 \times 10^{-5}) \times 100$$
$$\therefore \frac{B-B_o}{B_o} \times 100 = (6.8 \times 10^{-3})\%$$

Classification of magnetic materials

On the basis of the behavior of materials in a magnetizing field, the materials are generally classified into three categories namely,
(i) Diamagnetic, (ii) Paramagnetic and (iii) Ferromagnetic

(i) Properties of diamagnetic substances

Diamagnetic substances are those in which the net magnetic moment of atoms is zero.
The susceptibility has a low negative value.
(For example, for bismuth $\chi m = -0.00017$).
2. Susceptibility is independent of temperature.
3. The relative permeability is slightly less than one.
4. When placed in a non uniform magnetic field they have a tendency to move away from the field (i.e) from the stronger part to the weaker part of the field. They get magnetized in a direction opposite to the field.
5. When suspended freely in a uniform magnetic field, they set themselves perpendicular to the direction of the magnetic field
Examples : Bi, Sb, Cu, Au, Hg, H_2O, H_2 etc.

(ii) Properties of paramagnetic substances

Paramagnetic substances are those in which each atom or molecule has a net non-zero magnetic moment of its own.
1. Susceptibility has a low positive value.
(For example : χ_m for aluminium is +0.00002).
2. Susceptibility is inversely proportional to absolute temperature. As the temperature increases susceptibility decreases.
3. The relative permeability is greater than one.
4. When placed in a non uniform magnetic field, they have a tendency to move from weaker part to the stronger part of the field. They get magnetized in the direction of the field.
5. When suspended freely in a uniform magnetic field, they set themselves parallel to the direction of magnetic field
Examples : Al, Pt, Cr, O_2, Mn, $CuSO_4$ etc.
Pierre Curie observed the magnetization M of a paramagnetic material is directly proportional to the external magnetic filed B and inversely proportional to its absolute temperature T, called Curie's law

$$M = C\frac{B}{T}$$

Where C = Curie's constant

$$M = C\frac{B}{T}\frac{\mu_0}{\mu_0}$$

$$M = CH\frac{\mu_0}{T}\left(\because H = \frac{B}{\mu_0}\right)$$

$$\frac{M}{H} = \chi_m = C\frac{\mu_0}{T}$$

(iii) Properties of ferromagnetic substances

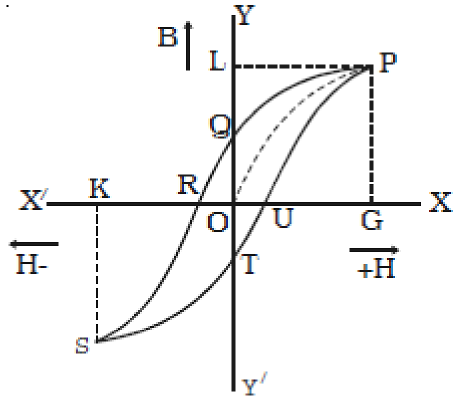
Ferromagnetic substances are those in which each atom or molecule has a strong spontaneous net magnetic moment. These substances exhibit strong paramagnetic properties.
1. The susceptibility and relative permeability are very large.
(For example : μ_r for iron = 200,000)
2. Susceptibility is inversely proportional to the absolute temperature.
As the temperature increases the value of susceptibility
decreases. *At a particular temperature, ferromagnetic become paramagnetic. This transition temperature is called Curie temperature.*
The relation between magnetic susceptibility of the substance in the acquired paramagnetic form and temperature is given by

$$\chi_m = \frac{C_1}{T - T_C}$$

C_1 is a constant
For example: Curie temperature of iron is about 1000 K.
3. When suspended freely in uniform magnetic field, they set themselves parallel to the direction of magnetic field.
4. When placed in a non uniform magnetic field, they have a tendency to move from the weaker part to the stronger part of the field. They get strongly magnetized in the direction of the field.
Examples : Fe, Ni, Co and a number of their alloys.

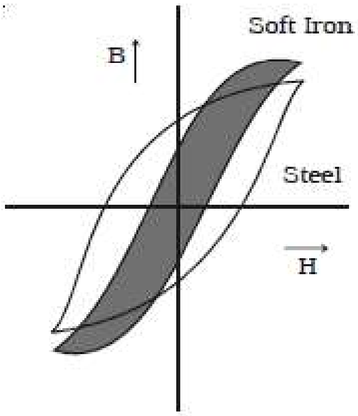
Hysteresis

Consider an iron bar being magnetized slowly by a magnetizing field H whose strength can be changed. It is found that the magnetic induction B inside the material increases with the strength of the magnetizing field and then attains a saturated level. This is depicted by the path OP in the



If the magnetizing field is now decreased slowly, then magnetic induction also decreases but it does not follow the path PO. Instead, when H = 0, B has non zero value equal to OQ. This implies that some magnetism is left in the specimen. *The value of magnetic induction of a substance, when the magnetizing field is reduced to zero, is called residual magnetic induction of the material.* OQ represents the residual magnetism of the material. Now, if we apply the magnetizing field in the reverse direction, the magnetic induction decreases along QR till it becomes zero at R. Thus to reduce the residual magnetism (remnant magnetism) to zero, we have to apply a magnetizing field OR in the opposite direction.
The value of the magnetizing field H which has to be applied to the magnetic material in the reverse direction so as to reduce its residual magnetism to zero is called its coercivity.
When the strength of the magnetizing field H is further increased in the reverse direction, the magnetic induction increases along RS till it acquires saturation at a point S (points P and S are symmetrical). If we now again change the direction of the field, the magnetic induction follows the path STUP. *This closed curve PQRSTUP is called the*

'hysteresis loop' and it represents a cycle of magnetization. The word 'hysteresis' literally means lagging behind. We have seen that magnetic induction B lags behind the magnetizing field H in a cycle of magnetization. *This phenomenon of lagging of magnetic induction behind the magnetizing field is called hysteresis.*
In the process of magnetization of a ferromagnetic substance through a cycle, there is expenditure of energy. The energy spent in magnetizing a specimen is not recoverable and there occurs a loss of energy in the form of heat. This is so because, during a cycle of magnetization, the molecular magnets in the specimen are oriented and reoriented a number of times. This molecular motion results in the production of heat. It has been found that *loss of heat energy per unit volume of the specimen in each cycle of magnetization is equal to the area of the hysteresis loop.* The shape and size of the hysteresis loop is characteristic of each material because of the differences in their retentivity, coercivity, permeability, susceptibility and energy losses etc. By studying hysteresis loops of various materials, one can select suitable materials for different purposes.
At $H = 0$, $B \neq 0$. The value of B at $H = 0$ is called retentivity or remanence.
At $H \neq 0$, $B = 0$. The value of H at $B = 0$ is called coercivity.
Hysteresis loss



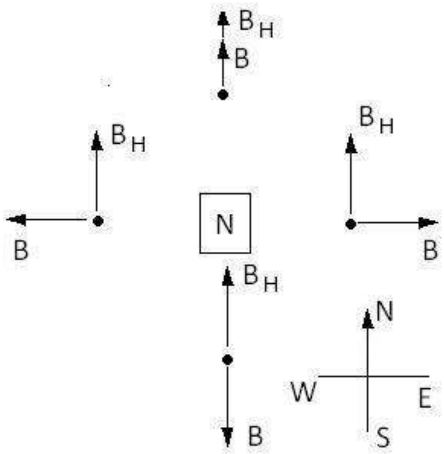
In the process of magnetization of a ferromagnetic substance through a cycle, there is expenditure of energy. The energy spent in magnetizing a specimen is not recoverable and there occurs a loss of energy in the form of heat. This is so because, during a cycle of magnetization, the molecular magnets in the specimen are oriented and reoriented a number of times. This molecular motion results in the production of heat. It has been found that *loss of heat energy per unit volume of the specimen in each cycle of magnetization is equal to the area of the hysteresis loop.* The shape and size of the hysteresis loop is characteristic of each material because of the differences in their retentivity, coercivity, permeability, susceptibility and energy losses etc. By studying hysteresis loops of various materials, one can select suitable materials for different purposes.

Uses of ferromagnetic materials
(i) Permanent magnets
The ideal material for making permanent magnets should possess high retentivity (residual magnetism) and high coercivity so that the magnetization lasts for a longer time. Examples of such substances are steel and alnico (an alloy of Al, Ni and Co).
(ii) Electromagnets
Material used for making an electromagnet has to undergo cyclic changes. Therefore, the ideal material for making an electromagnet has to be one which has the least hysteresis loss. Moreover, the material should attain high values of magnetic induction B at low values of magnetizing field H. Soft iron is preferred for making electromagnets as it has a thin hysteresis loop[small area, therefore less hysteresis loss] and low retentivity.It attains high values of B at low values of magnetizing field.
(iii) Core of the transformer

A material used for making transformer core and choke is subjected to cyclic changes very rapidly. Also, the material must have a large value of magnetic induction B. Therefore, soft iron that has thin and tall hysteresis loop is preferred. Some alloys with low hysteresis loss are: radio-metals, pern-alloy.
(iv) Magnetic tapes and memory store
Magnetization of a magnet depends not only on the magnetizing field but also on the cycle of magnetization it has undergone. Thus, the value of magnetization of the specimen is a record of the cycles of magnetization it has undergone. Therefore, such a system can act as a device for storing memory. Ferro magnetic materials are used for coating magnetic tapes in a cassette player and for building a memory store in a modern computer. Examples : Ferrites (Fe, Fe_2O, $MnFe_2O_4$ etc.).

Questions
Q) It is observed that the neutral points lie along the axis of a magnet placed on the table. What is the orientation of the magnet with respect to the earth's magnetic field
Ans. North pole of the magnet is towards the south of the earth
Q) A bar magnet is stationary in magnetic meridian. Another similar magnet is kept to it such that the centre lie on their perpendicular bisectors. If the second magnet is free to move, then what type of motion it will have - translator, rotator or both
Ans: Only translator
Q) A short bar magnet placed with its axis making an angle θ with a uniform external field B experiences a torque. What is the magnetic moment of the magnet
Q) Name the parameters needed to completely specify the earth's magnetic field at a point on the earth's surface
Ans: Declination, Dip and Horizontal component of earth's field
Q) What is geomagnetic equator
Ans: The great circle on the earth's surface whose plane is perpendicular to the magnetic axis is called magnetic equator.
Q) What is magnetic meridian
Ans: A vertical plane passing through the magnetic axis of earth is called magnetic meridian
Q) Name the physical quantity which is measured in Wb A^{-1}
Ans: The ratio of the magnetic induction and the magnetic moment is measured in Wb A^{-1}
Q) Name one property of magnetic material used for making permanent magnet
Ans: High coercivity
Q) The ratio of the horizontal component to the resultant magnetic field of earth at a given place is $(1/\sqrt{2})$. What is the angle of dip at that place
Ans : $cos\theta = \frac{B_H}{B} = \frac{1}{\sqrt{2}}$
$\theta = 45^O$
Q) Why does a paramagnetic sample display greater magnetization (for same magnetizing field) when cooled
Ans: The tendency to disrupt the alignment of dipoles with the magnetizing field arising from random thermal motion is reduced at lower temperatures. So, as the paramagnetic substance is cooled, its atomic dipoles tends to get aligned with the magnetizing field. Thus, the paramagnetic substance display a greater magnetization when cooled
Q) What is SI unit of magnetic permeability?
Ans: T m A^{-1}
Q) Why do magnetic lines of force prefer to pass through iron than air
Ans: Permeability of soft iron is greater than that of air
Q) What is the SI unit of susceptibility
Ans: It has no unit
Q) Identify a substance, which has negative magnetic susceptibility.
Ans: Diamagnetic substance. Magnetic susceptibility is positive for both para and ferromagnetic substance
Q) What is the net magnetic moment of an atom of a diamagnetic material

Ans: Zero
Q) What is the dimensional formula of magnetic flux
Ans: [$ML^2T^{-2}A^{-1}$]
Q) An iron nail is attracted by a magnet. What is the source of kinetic energy
Ans: It is the magnetic field energy which is partly converted into kinetic energy
Q) A bar magnet is cut into two equal pieces transverse to its length. What happens to its dipole moment
Ans: The magnetic moment will be halved because length will be halved
Q) What is magnet
Ans: A magnet is an arrangement of two equal and opposite magnetic poles separated by a certain distance. It has attractive and directive properties
Q) What is the SI unit of magnetic moment of a dipole
Ans: Am^2 or JT^{-1}
Q) What is Hysteresis?
Ans: Hysteresis is defined as the lagging of the magnetic induction B behind the corresponding magnetic field H
Q) Define angle of magnetic dip
Ans: It is the angle made by the direction of earth's total magnetic field with the horizontal component of the earth's magnetic field at magnetic poles
Q) What is the effect on the magnetization of diamagnetic substance when it is cooled
Ans: The magnetization of a diamagnetic substance is independent of temperature
Q) A magnet is held vertically on a horizontal plane. How many neutral points are there in the horizontal plane



Ans.) The magnetic field due to the magnet and the magnetic field of earth are shown at four different points a, b, c and d. Clearly, the two fields cancel only at the point a. So, a is the neutral point.

Q) In the stirrup of a vibration magnetometer are placed two magnets one above the other with their axes parallel. When will their time period be maximum/minimum
Ans: The time period will be maximum when opposite poles are together

$$T_{max} = 2\pi\sqrt{\frac{I_1 + I_2}{(m_1 - m_2)B_H}}$$

The time period will be minimum when like poles are together

$$T_{max} = 2\pi\sqrt{\frac{I_1 + I_2}{(m_1 + m_2)B_H}}$$

Q) Two substances A and B have their relative permeability slightly greater and less than unity respectively. What do you conclude about A and B
Ans: $\chi_m = \mu_r - 1$
Relative permeability of A is slightly greater than 1. So χ_m is small and positive. So, substance is paramagnetic.
Relative permeability of B is slightly less 1
So χ_m is small and negative. Clearly, substance is diamagnetic
Q) How does the knowledge of declination at a place help in navigation?

Ans: Declination at place gives us the angle between the geographic and the magnetic meridians. So, the knowledge of declination shall help in steering the ship in the required direction so as to reach the destination
Q) Two identical-looking iron bars A and B given, one of which is definitely known to be magnetized [We don’t know which one]. How would one ascertain whether or not both are magnetized? If only one is magnetized, how does one ascertain which one? [Use nothing but the two bars A and B]
Ans: Try to bring different ends of the magnets closer. A repulsive force in some situation establishes that both are magnetized. If it is always attractive, then one of them is not magnetized. To see which one, pick up one say A and lower one of its ends: first one of the ends of their other say b, and then on the middle part of B. A experiences no force, and then B is magnetized. If you do not notice any change from end to middle point Of B, then A is magnetized.
Q) A magnetized needle in a uniform magnetic field experiences a torque but no net force. An iron nail near a bar magnet, however, experiences a force of attraction in addition to a torque. Why?
Ans: In the case of uniform magnetic field, the forces experienced by the needle are equal in magnitude, opposite in direction and have different lines of action. So, net force is zero. But torque is not zero
The iron nail experiences a non-uniform magnetic field due to the bar magnet. The induced magnetic moment in the nail, therefore, experiences both force and torque. The net force is attractive because the induced (say) south pole in the nail is closer to the north pole of the magnet than the induced north pole
Q) Why two magnetic lines of force due to a bar magnet do not cross each other?
Ans: If two magnetic lines of force cross at a point, then this would mean that there are two directions of magnetic field at the point of crossing. This is physically absurd. Thus, two magnetic lines of force cannot cross each other
Q) What is the basic use of hysteresis curve?
Ans: Hysteresis loop gives useful information about the different properties, of materials, such as coercivity, retentivity, energy loss. This information helps us in the suitable selection of materials for different purposes.
Q) Does the magnetization of paramagnetic salt depend on temperature? Justify your answer
Ans: The atoms of a paramagnetic substance posses small magnetic dipole moments. But these atomic dipoles are oriented in a random manner. In the presence of the external magnetic field, these dipoles tend to align in the direction of the field. But the tendency for alignment is hindered by thermal agitation. So, the magnetization of paramagnetic salt decreases with increase of temperature.

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ELECTROMAGNETIC INDUCTION

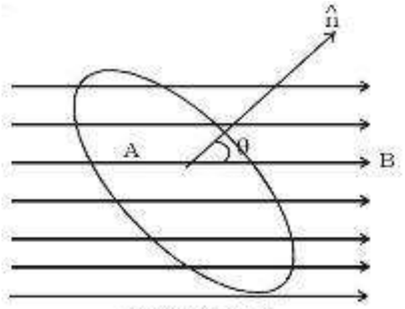
In the year 1820, Hans Christian Oersted demonstrated that a current carrying conductor is associated with a magnetic field. Thereafter, attempts were made by many to verify the reverse effect of producing an induced emf by the effect of magnetic field.

Electromagnetic induction

Michael Faraday demonstrated the reverse effect of Oersted experiment. He explained the possibility of producing emf across the ends of a conductor when the magnetic flux linked with the conductor changes. This was termed as electromagnetic induction. The discovery of this phenomenon brought about a revolution in the field of power generation.

Magnetic flux

The magnetic flux (ϕ) linked with a surface held in a magnetic field (B) is defined as the number of magnetic lines of force crossing a closed area (A) as shown in figure.



Magnetic flux

If θ is the angle between the direction of the field and normal to the area, then

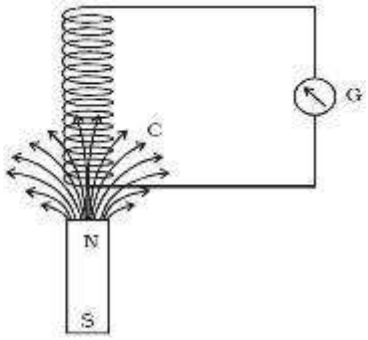
$$\phi = \vec{B} \cdot \vec{A}$$

$$\phi = BA\cos\theta$$

Induced emf and current – Electromagnetic induction.

Whenever there is a change in the magnetic flux linked with a closed circuit an emf is produced. This emf is known as the induced emf and the current that flows in the closed circuit is called induced current. The phenomenon of producing an induced emf due to the change in the magnetic flux associated with a closed circuit is known as electromagnetic induction. Faraday discovered the electromagnetic induction by conducting several experiments.

figure consists of a cylindrical coil C made up of several turns of insulated copper wire connected in series to a sensitive galvanometer G. A strong bar magnet NS with its north pole pointing towards the coil is moved up and down. The following inferences were made by Faraday



(i) Whenever there is a relative motion between the coil and the magnet, the galvanometer shows deflection indicating the flow of induced current.
(ii) The deflection is momentary. It lasts so long as there is relative motion between the coil and the magnet.
(iii) The direction of the flow of current changes if the magnet is moved towards and withdrawn from it.
(iv)The deflection is more when the magnet is moved faster, and less when the magnet is moved slowly.
(v) However, on reversing the magnet (i.e) south pole pointing towards the coil, same results are obtained, but current flows in the opposite direction.

Faraday demonstrated the electromagnetic induction by another experiment also.

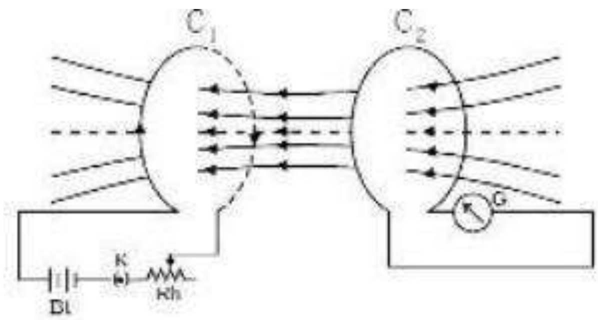


figure shows two coils C_1 and C_2 placed close to each other. The coil C_1 is connected to a battery (Bt) through a key K and a rheostat. Coil C_2 is connected to a sensitive galvanometer G and kept close to C_1. When the key K is pressed, the galvanometer connected with the coil C_2 shows a sudden momentary deflection. This indicates that a current is induced in coil C_2. This is because when the current in C_1 increases from zero to a certain steady value, the magnetic flux linked with the coil C_1 increases. Hence, the magnetic flux linked with the coil C_2 also increases. This causes the deflection in the galvanometer. On releasing K, the galvanometer shows deflection in the opposite direction. This indicates that a current is again induced in the coil C_2. This is because when the current in C_1 decreases from maximum to zero value, the magnetic flux linked with the coil C_1 decreases. Hence, the magnetic flux linked with the coil C_2 also decreases. This causes the deflection in the galvanometer in the opposite direction.

Faraday's laws of electromagnetic induction

Based on his studies on the phenomenon of electromagnetic induction, Faraday proposed the following two laws.

First law

Whenever the amount of magnetic flux linked with a closed circuit changes, an emf is induced in the circuit. The induced emf lasts so long as the change in magnetic flux continues.

Second law

The magnitude of emf induced in a closed circuit is directly proportional to the rate of change of magnetic flux linked with the circuit. Let ϕ_1 be the magnetic flux linked with the coil initially and ϕ_2 be the magnetic flux linked with the coil after a time t. Then Rate of change of magnetic flux = $(\phi_2 - \phi_1)/t$

According to Faraday's second law, the magnitude of induced emf is, $E \propto (\phi_2 - \phi_1)/t$

If $dt \rightarrow 0$, $d\phi$ is the change in magnetic flux in a time dt, then the above equation can be written as $E \propto (d\phi/dt)$

Lenz's law

The Russian scientist H.F. Lenz in 1835 discovered a simple law giving the direction of the induced current produced in a circuit. Lenz's law states that the induced current produced in a circuit always flows in such a direction that it opposes the change or cause that produces it.

If the coil has N number of turns and ϕ is the magnetic flux linked with each turn of the coil then, the total magnetic flux linked with the coil at any time is $N\phi$

$$E = -\frac{d}{dt}(N\phi)$$

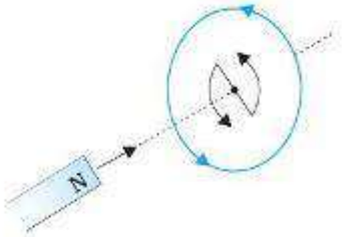
$$E = -N\frac{d\phi}{dt}$$

$$E = -\frac{N(\phi_2 - \phi_1)}{t}$$

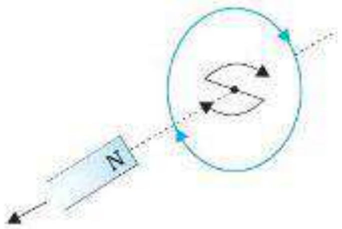
Lenz's law - a consequence of conservation of energy

Copper coils are wound on a cylindrical cardboard and the two ends of the coil are connected to a sensitive galvanometer.

A magnet is moved towards the coil as shown in figure. The upper face of the coil acquires north polarity. Consequently work has to be done to move the magnet further against the force of repulsion.



When we withdraw the magnet away from the coil, its upper face acquires south polarity. Now the work done is against the force of attraction.



When the magnet is moved, the number of magnetic lines of force linking the coil changes, which causes an induced current to flow through the coil. The direction of the induced current, according to Lenz's law is always to oppose the motion of the magnet. The work done in moving the magnet is converted into electrical energy. This energy is dissipated as heat energy in the coil.

If on the contrary, the direction of the current were to help the motion of the magnet, that is, if when north pole approaches coil south pole is produced at the face of coil due to flow of current. And when North pole goes away from coil if North pole is produced at the face of coil , magnet would start moving faster increasing the change of magnetic flux linking the coil. This results in the increase of induced current. Hence kinetic energy and electrical energy would be produced without any external work being done, but this is impossible.

Therefore, the induced current always flows in such a direction to oppose the cause. Thus it is proved that Lenz's law is the consequence of conservation of energy.

*There is another way to find the direction of current inside the loop which is described below

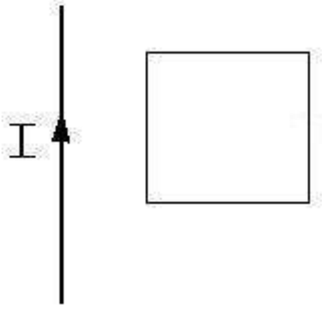


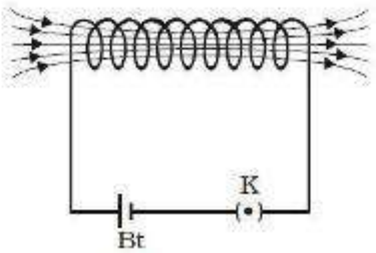
Figure shows a conducting loop placed near a long, straight wire carrying a current I as shown. The direction of magnetic field will be going inside the paper

If the current **increases** continuously, Flux linked with coil increases. Now according to Lenz's law direction of induced current will be such that it will produce magnetic field in opposite to the direction of magnetic field produced by current carrying long wire.

Thus magnetic field produced due to loop will be coming out of paper. Or direction of current will be "Anticlockwise direction"
If the current **decreases** continuously, Flux linked with coil decreases. Now according to Lenz law direction of magnetic field produced by current in loop should be along the direction of magnetic field due to wire
Thus current will flow in "Clockwise direction"

Fleming's right hand rule
The forefinger, the middle finger and the thumb of the right hand are held in the three mutually perpendicular directions. If the forefinger points along the direction of the magnetic field and the thumb is along the direction of motion of the conductor, then the middle finger points in the direction of the induced current. This rule is also called generator rule.

Self Induction
The property of a coil which enables to produce an opposing induced emf in it when the current in the coil changes is called self induction. A coil is connected in series with a battery and a key (K) as shown in figure.



On pressing the key, the current through the coil increases to a maximum value and correspondingly the magnetic flux linked with the coil also increases. An induced current flows through the coil which according to Lenz's law opposes the further growth of current in the coil.
On releasing the key, the current through the coil decreases to a zero value and the magnetic flux linked with the coil also decreases.
According to Lenz's law, the induced current will oppose the decay of current in the coil.

Coefficient of self induction
When a current I flows through a coil, the magnetic flux (ϕ) linked with the coil is proportional to the current.
$\phi \propto I$ or $\phi = LI$
where L is a constant of proportionality and is called coefficient of self induction or self inductance.
If $I = 1A$,
$\phi = L \times 1$, then $L = \phi$

Therefore, coefficient of self induction of a coil is numerically equal to the magnetic flux linked with a coil when unit current flows through it. According to laws of electromagnetic induction.

$$\mathrm{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(LI)$$

$$\mathrm{E} = -L\frac{dI}{dt}$$

$$if \quad \frac{dI}{dt} = 1As^{-1}$$

$$L = -\mathrm{E}$$

The coefficient of self induction of a coil is numerically equal to the opposing emf induced in the coil when the rate of change of current through the coil is unity. The unit of self inductance is henry (H).
One henry is defined as the self-inductance of a coil in which a change in current of one ampere per second produces an opposing emf of one volt.

Self inductance of a long solenoid

Let us consider a solenoid of N turns with length *l* and area of cross section A. It carries a current I. If B is the magnetic field at any point inside the solenoid, then

$$B = \frac{\mu_0 NI}{l}$$

Magnetic flux per turn = B × area of each turn
Thus Magnetic flux per turn =

$$\phi = \frac{\mu_0 NI}{l}A$$

Hence, the total magnetic flux (ϕ) linked with the solenoid is given by the product of flux through each turn and the total number of turns.

$$\phi = \frac{\mu_0 NI}{l}AN$$

$$\phi = \frac{\mu_0 N^2 IA}{l}$$

If L is the coefficient of self induction of the solenoid, then
We know that ϕ = LI
From above equations

$$LI = \frac{\mu_0 N^2 IA}{l}$$

$$L = \frac{\mu_0 N^2 A}{l}$$

If the core is filled with a magnetic material of permeability μ, then

$$L = \frac{\mu N^2 A}{l}$$

Energy associated with an inductor

Whenever current flows through a coil, the self−inductance opposes the growth of the current. Hence, some work has to be done by external agencies in establishing the current. If e is the induced emf then,

$$E = -L\frac{dI}{dt}$$

The small amount of work dw done in a time interval dt is

dw = E.I dt

$$dw = -L\frac{dI}{dt}I.dt$$

The total work done when the current increases from 0 to maximum value (Io) is

$$w = \int dw = \int_0^{I_0} -LIdI$$

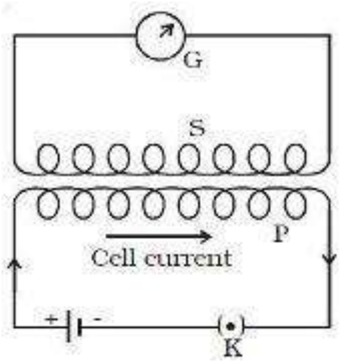
$$w = -\frac{1}{2}LI_0^2$$

This work done is stored as magnetic potential energy in the coil

Energy stored in the coil =

$$U = -\frac{1}{2}LI_0^2$$

Mutual induction



Whenever there is a change in the magnetic flux linked with a coil, there is also a change of flux linked with the neighbouring coil, producing an induced emf in the second coil. This phenomenon of producing an induced emf in a coil due to the change in current in the other coil is known as mutual induction.

P and S are two coils placed close to each other as shown in figure, P is connected to a battery through a key K. S is connected to a galvanometer G. On pressing K, current in P starts increasing from zero to a maximum value. As the flow of current increases, the magnetic flux linked with P increases. Therefore, magnetic flux linked with S also increases producing an induced emf in S. Now, the galvanometer shows the deflection.

According to Lenz's law the induced current in S would oppose the increase in current in P by flowing in a direction opposite to the current in P, thus delaying the growth of current to the maximum value.

When the key 'K' is released, current starts decreasing from maximum to zero value, consequently magnetic flux linked with P decreases. Therefore magnetic flux linked with S also decreases and hence, an emf is induced in S. According to Lenz's law, the induced current in S flows in such a direction so as to oppose the decrease in current in P thus prolonging the decay of current.

Coefficient of mutual induction

I_P is the current in coil P and ϕ_s is the magnetic flux linked with coil S due to the current in coil P.

$\therefore \phi_s \propto I_P$ or $\phi_s = M I_P$

where M is a constant of proportionality and is called the coefficient of mutual induction or mutual inductance between the two coils.

If I_P = 1A, then, $M = \phi_s$

Thus, coefficient of mutual induction of two coils is numerically equal to the magnetic flux linked with one coil when unit current flows through the neighboring coil.

If E_s is the induced emf in the coil (S) at any instant of time, then from the laws of electromagnetic induction,

$$E_S = -\frac{d\phi_s}{dt} = -\frac{d}{dt}\left(MI_p\right) = -M\frac{dI_p}{dt}$$

If $(dI_p/dt) = 1$ then $E_s = -M$

Thus, the coefficient of mutual induction of two coils is numerically equal to the emf induced in one coil when the rate of change of current through the other coil is unity.

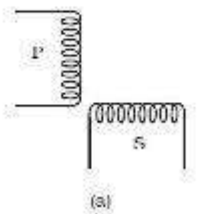
The unit of coefficient of mutual induction is henry.

One henry is defined as the coefficient of mutual induction between a pair of coils when a change of current of one ampere per second in one coil produces an induced emf of one volt in the other coil.

The coefficient of mutual induction between a pair of coils depends on the following factors

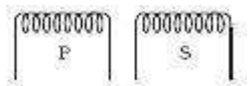
(i) Size and shape of the coils, number of turns and permeability of material on which the coils are wound.

(ii) proximity of the coils Two coils P and S have their axes perpendicular to each other as shown in figure (a) ,

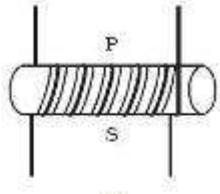


When a current is passed through coil P, the magnetic flux linked with S is small and hence, the coefficient of mutual induction between the two coils is small.

The two coils are placed in such a way that they have a common axis as shown in figure(b)



When current is passed through the coil P the magnetic flux linked with coil S is large and hence, the coefficient of mutual induction between the two coils is large.
If the two coils are wound on a soft iron core as shown in figure (c)

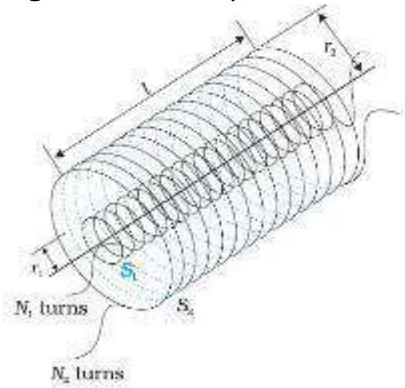


(c)

the mutual induction is very large.

Mutual induction of two long solenoids.

S_1 and S_2 are two long solenoids each of length L. The solenoid S_2 is wound closely over the solenoid S_1 as shown in figure N_1 and N_2 are the number of turns in the solenoids S_1 and S_2 respectively. Both the solenoids are considered to have the same area of cross section A as they are closely wound together. I_1 is the current flowing through the solenoid S_1. The magnetic field B_1 produced at any point inside the solenoid S_1 due to the current I_1 is



$$B_1 = \mu_0 \frac{N_1}{L} I_1 \quad \text{----eq(1)}$$

The magnetic flux linked with each turn of S_2 is equal to B_1A.
Total magnetic flux linked with solenoid S_2 having N_2 turns is
$\phi_2 = B_1AN_2$ –eq(2)
Substituting for B_1 from equation (1)

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{L}$$

$$\phi_2 = \left(\mu_0 \frac{N_1}{L} I_1\right) AN_2$$

$$\phi_2 = \frac{\mu_0 N_1 N_2 A I_1}{L}$$

But

$$\phi_2 = M_{21} I_1$$

From above equations

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{L}$$

Also from eq(1) and eq(2) $\phi_2 = M_{21}I_1$

$$\frac{d\phi_2}{dt} = M_{21}\frac{dI_1}{dt}$$

$$E_2 = -M_{21}\frac{dI_1}{dt}$$

Taking I_1 = 1 unit we get $\phi_2 = M_{21}$

The constant of proportionality M_{21} is termed as the mutual inductance.

Thus "The magnetic flux linked with one of the coil of system of two coils per unit current flowing through the other coil is called mutual inductance of the system"

Instead of coil S_1 if we setup current in coil S_2 , this produces magnetic flux ϕ_1 that links with coil S_1. If we have current I_2 flowing through coil S_2, then $\phi_1 = B_2AN_1$ and M_{12} will be

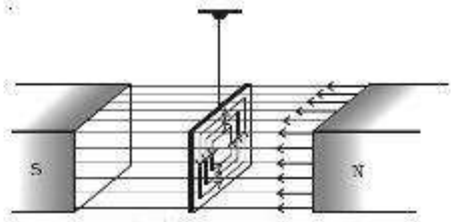
$$M_{12} = \frac{\mu_0 N_1 N_2 A}{L}$$

Unit of mutual inductance is WbA^{-1} = henry (H) or VsA^{-1}

Since mutual inductance is same in both the cases $M_{21} = M_{12} = M$. this result is called reciprocity theorem.

Eddy currents

Foucault in the year 1895 observed that when a mass of metal moves in a magnetic field or when the magnetic field through a stationary mass of metal is altered, induced current is produced in the metal. This induced current flows in the metal in the form of closed loops resembling 'eddies' or whirl pool. Hence this current is called eddy current. The direction of the eddy current is given by Lenz's law.



Eddy current

When a conductor in the form of a disc or a metallic plate as shown in Fig, swings between the poles of a magnet, electrons inside the plate experience a force [**F** = -e(**v** ×**B**)] because of the motion of plate . Under the influence of this force electrons move on the path which offers minimum resistance and constitute eddy currents are set up inside the plate. These currents, according to Lenz's law, flow in such a direction that the magnetic field produced due to them opposes the motion of the conductor. Hence plates comes to rest due to damping.

If the metallic plate with holes drilled in it is made to swing inside the magnetic field, the effect of eddy current is greatly reduced consequently the plate swings freely inside the field. Eddy current can be minimized by using thin laminated sheets instead of solid metal.

Application of Eddy currents

(i) *Magnetic braking in trains*: Strong electromagnets are situated above the rails in some electrically powered trains. When the electromagnets are activated, the eddy currents

induced in the rails oppose the motion of the train. As there are no mechanical linkages, the braking effect is smooth.
(ii) *Electromagnetic damping*: Certain galvanometers have a fixed core made of nonmagnetic metallic material. When the coil oscillates, the eddy currents generated in the core oppose the motion and bring the coil to rest quickly.
Induction furnace: Induction furnace can be used to produce high temperatures and can be utilised to prepare alloys, by melting the constituent metals. A high frequency alternating current is passed through a coil which surrounds the metals to be melted. The eddy currents generated in the metals produce high temperatures sufficient to melt it.
(iv) *Electric power meters*: The shiny metal disc in the electric power meter(analogue type) rotates due to the eddy currents. Electric currents are induced in the disc by magnetic fields produced by sinusoidally varying currents in a coil.
You can observe the rotating shiny disc in the power meter of your house.

Methods of producing induced emf

We know that the induced emf is given by the expression

$$E = -\frac{d\left(NAB\cos\theta\right)}{dt}$$

Hence, the induced emf can be produced by changing
(i) the magnetic induction (B)
(ii) area enclosed by the coil (A) and
(iii) the orientation of the coil (θ) with respect to the magnetic field.

Emf induced by changing the magnetic induction*.*

The magnetic induction can be changed by moving a magnet either towards or away from a coil and thus an induced emf is produced in the coil. The magnetic induction can also be changed in one coil by changing the current in the neighboring coil thus producing an induced emf.

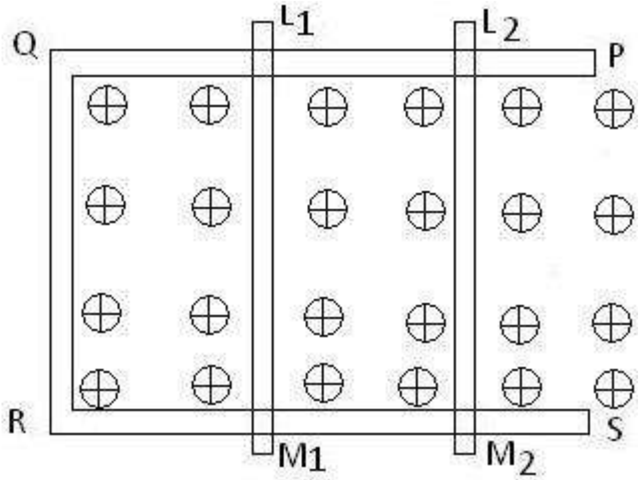
Motional emf

The magnetic flux $\phi = BA\cos\theta$ linked with a coil can be varied by many ways

(1) The magnet can be moved with respect to coil
(2) The coil can be rotated in a magnetic field (by changing angle θ between **A** and **B)**
(3) The coil can be placed inside the magnetic field in a specific position and the magnitude of the magnetic induction B can be changed with time
(4) The magnet can be moved inside a non-uniform magnetic field
(5) By changing the shape of coil (that is by shrinking or stretching it)

In all above cases mentioned above, the magnetic flux linked with coil changes and hence emf is induced in the coil
"The induced emf produced in a coil due the change in magnetic flux linked with coil due to some kind of motion is called motional emf"

PQRS is a conductor bent in the shape as shown in the figure L_1M_1 is a sliding conductor of length L resting on the arms PQ and RS. A uniform magnetic field 'B' acts perpendicular to the plane of the conductor. The closed area of the conductor is L_1QRM_1. When L_1M_1 is moved through a distance d*x* in time dt, the new area is L_2QRM_2. Due to the change in area $L_2 L_1M_1M_2$, there is a change in the flux linked with the conductor. Therefore, an induced emf is produced.



Change in area dA = Area $L_2L_1M_1M_2$

∴ dA = L d*x*

Change in the magnetic flux, dϕ = B.dA = BL d*x*

$$E = -\frac{d\phi}{dt}$$

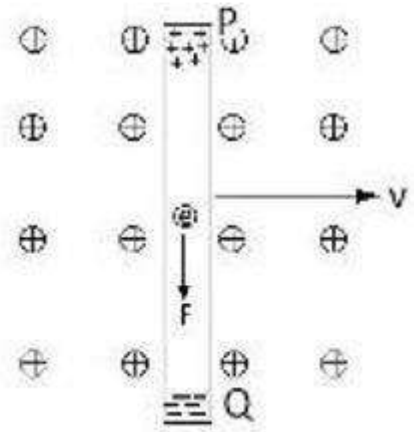
$$E = -BL\frac{dx}{dt}$$

$$E = -BLv$$

where *v* is the velocity with which the sliding conductor is moved.

The origin of the generation of induced emf

A conducting rod PQ is moving in a magnetic field with its plane perpendicular to it as shown in figure



The positive ions and electrons will also move along with it in the direction of motion of the rod.

In the present case, they move with a velocity **v** perpendicularly to the magnetic field B. Hence they experience a Lorentz force **F** = q(**v** ×**B**). Direction of this force can be found out by using right hand screw rule which is normal to plane form by v and B.
Here positive ions will experience force from Q to P but as they remains fixed at their lattice point, they will not move under the influence of this force
But electrons will flow from P to Q since electrons are free to move, they deposit at Q end of the rod and make it negatively charged.
Because of positive charge at P end of the rod and negative charge at end Q , rod behaves as battery with emf = Blv

Conversion of mechanical energy into Electrical energy

When rod moves with velocity v perpendicular to the magnetic field B pointing into plane of paper, emf induced is given by bvl.
Current induced in the conductor if resistance of value R is connected to the rod then current I = E/R = Bvl/R

Force on the conductor is given = I (**L**×**B**)
Mechanical Power P_m = Force × velocity

$$P_m = BILv$$

$$P_m = BLv\left(\frac{BLv}{R}\right)$$

$$P_m = \frac{B^2v^2L^2}{R}$$

Electrical power generated in circuit P_e = EI

$$P_e = (BvL)\left(\frac{BvL}{R}\right)$$

$$P_e = \frac{B^2v^2L^2}{R}$$

From above equations $P_m = P_e$
Thus mechanical work done in continuing the motion of the rod is obtained in the form of electrical energy.

AC generator (Dynamo) – Single phase

The ac generator is a device used for converting mechanical energy into electrical energy. The generator was originally designed by a Yugoslav scientist Nikola Tesla.
Principle
It is based on the principle of electromagnetic induction, according to which an emf is induced in a coil when it is rotated in a uniform magnetic field.

Essential parts of an AC generator

(i) Armature

Armature is a rectangular coil consisting of a large number of loops or turns of insulated copper wire wound over a laminated soft iron core or ring. The soft iron core not only increases the magnetic flux but also serves as a support for the coil

(ii) Field magnets

The necessary magnetic field is provided by permanent magnets in the case of low power dynamos. For high power dynamos, field is provided by electro magnet. Armature rotates between the magnetic poles such that the axis of rotation is perpendicular to the magnetic field.

(iii) Slip rings

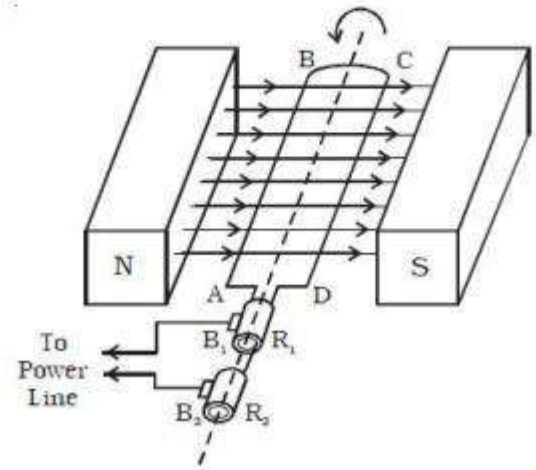
The ends of the armature coil are connected to two hollow metallic rings R_1 and R_2 called slip rings. These rings are fixed to a shaft, to which the armature is also fixed. When the shaft rotates, the slip rings along with the armature also rotate.

(iv) Brushes

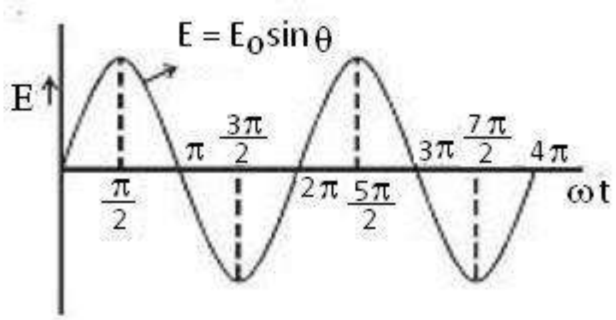
B_1 and B_2 are two flexible metallic plates or carbon brushes. They provide contact with the slip rings by keeping themselves pressed against the ring. They are used to pass on the current from the armature to the external power line through the slip rings.

Working

Whenever, there is a change in orientation of the coil, the magnetic flux linked with the coil changes, producing an induced emf in the coil. The direction of the induced current is given by Fleming's right hand rule. Suppose the armature ABCD is initially in the vertical position. It is rotated in the anticlockwise direction. The side AB of the coil moves downwards and the side DC moves upwards refer figure



Then according to Flemings right hand rule the current induced in arm AB flows from B to A and in CD it flows from D to C. Thus the current flows along DCBA in the coil. In the external circuit the current flows from B_1 to B_2. On further rotation, the arm AB of the coil moves upwards and DC moves downwards. Now the current in the coil flows along ABCD. In the external circuit the current flows from B_2 to B_1. As the rotation of the coil continues, the induced current in the external circuit keeps changing its direction for every half a rotation of the coil. Hence the induced current is alternating in nature as shown in figure



As the armature completes ν rotations in one second, alternating current of frequency ν cycles per second is produced.
Flux linked with coil
.

$$\phi = NBAcos\theta$$

Here θ is the angle between Area vector of coil and Magnetic field

Since coil is rotating with angular frequency ω we get θ = ωt

$$\phi = NBAcos\omega t$$

Now

$$e = -\frac{d\phi}{dt} = -BAN\frac{dcos\omega t}{dt}$$

$$e = BAN\omega sin\omega t$$

Here E_0 = BANω is peak value of emf induced
The induced emf at any instant is given by E= $E_o$sin ωt
The peak value of the emf, E_o = NBAω where N is the number of turns of the coil, A is the area enclosed by the coil, B is the magnetic field and ω is the angular velocity of the coil

Solved numerical

Q) A conducting circular loop of surface area 2.5×10^{-3} m^2 is placed perpendicular to a magnetic field which varies as B = 0.20 sin[50π s^{-1})t]. Find the charge flowing through any cross section during the time t = 0 to t = 40 s. Resistance of the loop is 10 Ohms

<u>Solution:</u>
Here ω = 50π, B_0 = 0.2, A =area 2.5×10^{-3} m^2
Induced emf =

$$E = -\frac{d\phi}{dt}$$

$$E = -A\frac{dB}{dt}$$

$$E = -AB_0\frac{d(\sin\omega t)}{dt}$$

$$E = -AB_0\omega\cos\omega t$$

Induced current

$$I = \frac{E}{R} = -\frac{AB_0\omega\cos\omega t}{R}$$

$$I_0 = \frac{-AB_0\omega}{R}$$

$$I = I_0\cos\omega t$$

The current changes sinusoidally with time period T = 2π/ω = 2π /50 π =40×10^{-3}s

The charge flowing through any cross section during time t = 0 to t = 0.04 s is

$$Q = \int_0^{0.04} Idt$$

$$Q = I_0\int_0^{0.04}\cos\omega t\, dt$$

$$Q = \frac{I_0}{\omega}\left[\sin\omega t\right]_0^{0.04}$$

$$Q = 0$$

Q) Two parallel, long, straight conducting conductors lie on a smooth plane surface . Two other parallel conductors rest on then at right angles so as to form a square of side a initially

A uniform magnetic field B exists at right angles to the plane formed by conductors.

Now they start moving out with a constant velocity v

a) Will the induced emf depends on time
b) Will the current be time dependent?

<u>Solution:</u>

a)Since velocity of conductor is v in outward direction displacement of each conductor is vt

Area of loop form by the loop will increase by (a+2vt)^2

Thus flux φ = B(a+2vt)^2

$$E = \frac{d\phi}{dt} = 4Bv(a+2vt)$$

Thus induced emf depends on time

b)If r is the resistance per unit length then R = 4(a+2vt)r

Current I =

$$I = \frac{E}{R} = \frac{4Bv(a+2vt)}{4(a+2vt)r} = \frac{Bv}{r}$$

Thus current I is constant

Q) A solenoid of length 1 m and 0.05 m diameter has 500 turns. If a current of 2A passes through the coil, calculate (i) the coefficient of self induction of the coil and (ii) the magnetic flux linked with a the coil.

<u>Solution:</u>

$$L = \frac{\mu_0 N^2 A}{l} = \frac{\mu_0 N^2 \pi r^2}{l}$$

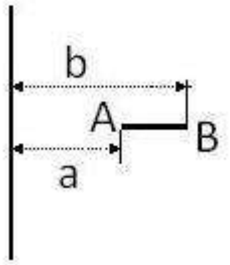
$$L = \frac{4\pi\times10^{-7}\times\left(5\times10^{2}\right)^{2}\times3.14\left(0.025\right)^{2}}{l}$$

$$L = 0.616\, mH$$

(ii) Magnetic flux φ = LI

$\phi = 0.616\times10^{-3}\times2 = 1.232\, milli\ weber$

Q) Figure shows a copper rod moving with velocity v parallel to a long straight wire carrying a current I. Calculate the induced emf. In the rod assuming v = 5 m/s, I = 100 amp, a = 10 cm b = 20 cm



Solution:

The induction at a point whose perpendicular distance from rod is x is given by

$$B_{(x)} = \frac{\mu_0 I}{2\pi x}$$

The e.m.f induced on the moving the rod = E

$$E = v\int_{x=a}^{x=b} B_{(x)}dx$$

$$E = v\int_{x=a}^{x=b} \frac{\mu_0 I}{2\pi x}dx$$

$$E = \frac{\mu_0 I v}{2\pi}\left[\ln x\right]_a^b$$

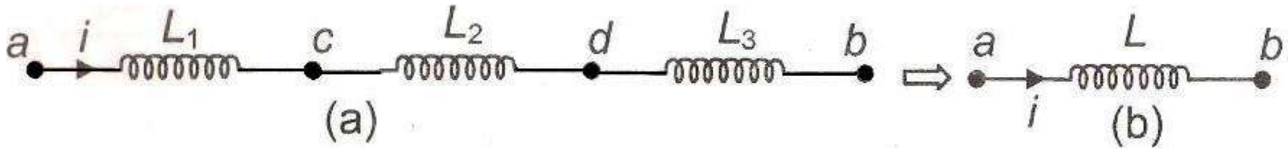
$$E = \frac{\mu_0 I v}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$E = \frac{4\pi\times10^{-7}\times100\times5}{2\pi}\ln 2$$

$$E = 0.69\times10^{-4}\, V$$

Q) Let three inductors L_1, L_2 and L_3 are the three inductors connected in series calculate the equivalent inductance of combination

Solution



Current through each inductor is same

Let potential difference across L_1 , L_2, L_3 be V_1, V_2, V_3

Thus $V = V_1 + V_2 + V_3$

Now $V_1 = L_1$ (di/dt) , $V_2 = L_2$ (di/dt), $V_3 = L_3$ (di/dt)

Thus

$$V = L_1\left(\frac{di}{dt}\right) + L_2\left(\frac{di}{dt}\right) + L_3\left(\frac{di}{dt}\right)$$

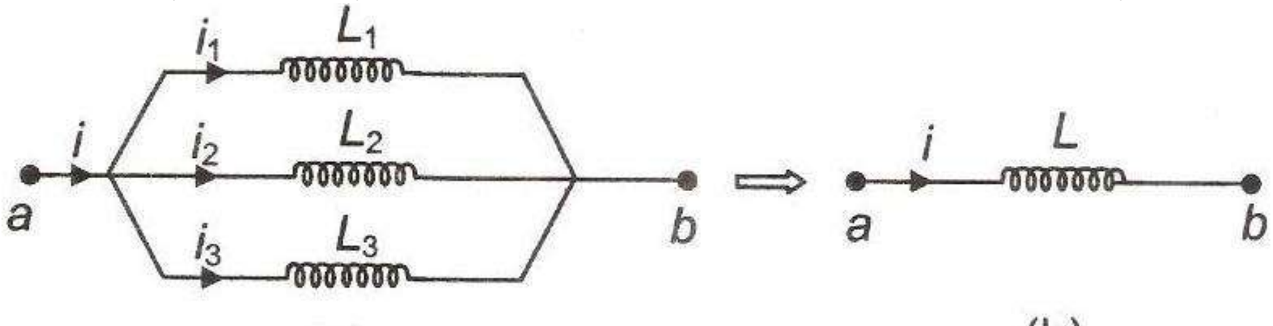
$$V = \left(L_1 + L_2 + L_3\right)\left(\frac{di}{dt}\right)$$

If equivalent inductance is L as show in figure (b)

V = L (di/dt) from above equations

$L = L_1 + L_2 + L_3$

Q) Obtain equation for equivalent inductance when inductors are connected in parallel



(a) (b)

Potential across each inductor is same

Let current through L_1 , L_2, L_3 be I_1, I_2, I_3

Thus $I = I_1 + I_2 + I_3$

$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dI_3}{dt} \quad \text{eq(i)}$$

Since $V = L_1$ (dI_1/dt)

$dI_1/dt = V/L_1$ similarly $dI_2/dt = V/L_2$ and $dI_3/dt = V/L_3$

Also if L is equivalent inductance then dI/dt = V/L

Substituting values in equation(i) we get

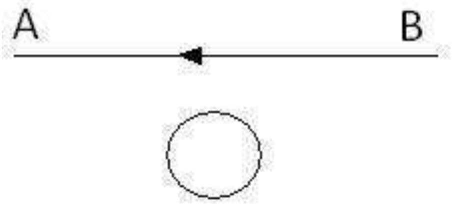
$$\frac{V}{L} = \frac{V}{L_1} + \frac{V}{L_2} + \frac{V}{L_3}$$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Questions

Q) A lamp connected in parallel with a coil of large inductance glows brilliantly before going off. Why?

Ans) This is due to large self-induced emf
Q) Can a wire act as inductor
Ans) No. This is because the magnetic flux linked with a wire of negligible cross-section area is zero
Q) A metal block and a brick of the same size area allowed to fall freely from the same height above the ground. Which of the two would reach the ground earlier and why?
Q) Two identical loops of copper and iron, are rotated with the same angular velocity in a uniform magnetic field. In which case the induced emf is more and why?
Q) A closed loop is held stationary in the magnetic field between the north and south poles of two permanent magnets held fixed. Can we hope to generate current in the loop by using very strong magnet
Q) A small piece of metal wire is dragged across the gap between the pole pieces of a magnet in 0.5 second. The magnetic flux between the pole pieces is known to be 8×10^{-4} Wb. Estimate the emf induced in the wire [Ans1.6 mV]
Q) What is the dimensional formula for mutual inductance of two coil
Q) Name the physical quantity which is measured in weber(ampere)$^{-1}$
Q) Write three factors on which the self-inductance of a coil depends
Q) Why the oscillations of a copper disc in a magnetic field are damped
Q) Prove that the charge induced is independent of time
Q) Write three factors on which the mutual inductance between a pair of coils depends
Q) The electric current in a wire in the direction from B to A is decreasing. What is the direction of induced current in the metallic loop kept above the wire as shown in figure



Q) How does the self inductance of a coil change when an iron rod is introduced in the coil
Q) A ring is fixed to the wall of a room. When south pole of a magnate is brought near the ring, what shall be the direction of induced current in the ring
Q) Does change in magnetic flux induce emf or current
Ans) The induced current will be produced only if the circuit is closed. However, the induced emf will be definitely produced
Q) When a magnet falls through a vertical coil, will its acceleration be different from the 'acceleration due to gravity'?
Q) A solenoid with an iron core and a bulb are connected to a d.c. source. How does the brightness of the bulb change, when the iron core is removed from the solenoid.
Q) Why resistance coils are usually double wound?
Ans) This is done to reduce a self-inductance
Q) Two identical loops, one of copper and another of constantan, are removed from a magnetic field within the same time interval. In which loop will the induced current be grater

Q) Two identical loops, one of copper and another of constantan, are removed from a magnetic field within the same time interval. In which loop will the induced current be greater

Q) Name the physical quantity whose SI unit is weber. Is it a scalar quantity?

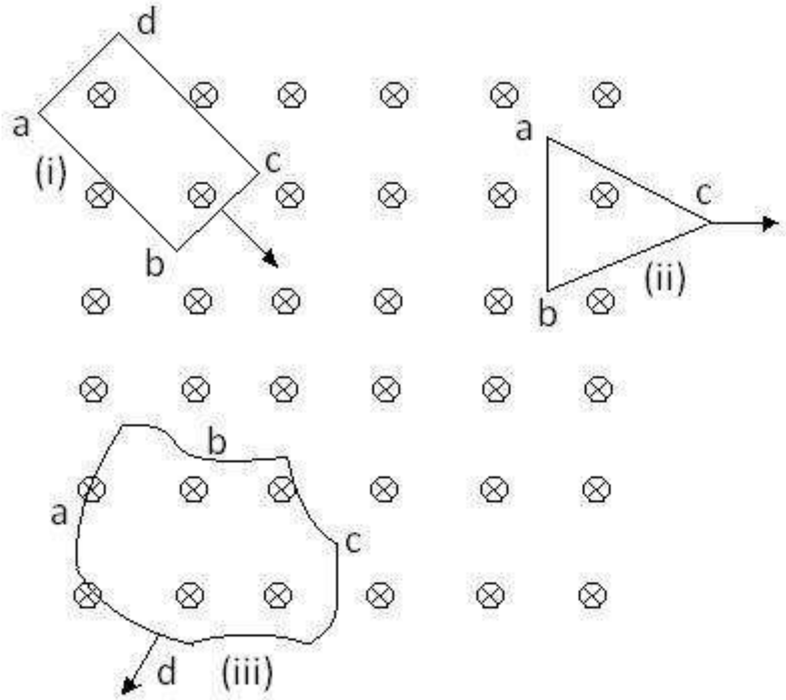
Q) An iron bar falling through the hollow region of a thick cylindrical shell made of copper experiences a retarding force. What can you conclude about the iron bar?

Q) A coil is wound on an iron core and looped back on itself so that the core has two sets of closely wounded wires in series carrying current in opposite senses. What do you expect about its self inductance? Will it be large or small

Ans) The self-inductance will be small due to the cancellation of induced emf effects. This is a special example of the situation when the winding I such that $L_{eq} = L_1 + L_2 - 2M$

Or $L_{eq} = L+L-2L = 0$

Q) Figure shows planar loops of different shapes moving out of or into a region of a magnetic field which is directed normal to the plane of the loop away from the reader. Determine the direction of induced current in each loop using Lenz's law



Ans) (i) The magnetic flux through the rectangular loop abcd increases, due to motion of the loop into region of magnetic field, The induced current must flow along the path bcdab so that it opposes the increasing flux

(ii) Due to the outward motion, magnetic flux through triangular loop abc decreases due to which the induced current flows along bacb, so as to oppose the change in flux

(iii) As the magnetic flux decreases due to motion of the irregular shaped loop abcd out of the region of magnetic field, the induced current flows along cdabc, so as to oppose change in flux

Q) Two similar circular co-axial loops carry equal currents in the same direction. If the loop be brought nearer what will happen to the current in them

Ans) When the loops are brought closer, there is an increase of magnetic flux. An induced emf has to oppose the change of magnetic flux. So the current in each loop will decrease.
Q) When a fan is switched off, a spark is produced in the switch. Why?
Ans) At the time of break of the circuit, a large emf is induced which opposes the decay of current in the circuit. This ionizes the air between the contact and of the switch. Consequently, spark is produced
Q) Two straight and parallel wires A and B are being brought towards each other, If current in A be I, what will be the direction of induced current in B? If A and B are taken away from each other, then?
Ans) In the first case, the induced current will be opposite to the current in A. This is because of repulsion. The induced current opposes the motion of B
In the second case, the direction of induced current will be the same as the direction of current i. This is because of “attraction”. The induced current opposes the motion of B
Q) How is the mutual inductance of pair of coils affected when
(i) separation between the coil is increased
(ii) the number of turns of each coil is increased
(iii) A thin iron sheet is placed between the two coils, other factors remaining the same? Explain your answer in each case
Ans) (i) When the separation between the two coils is increased, the flux linked with the secondary due to the current in the primary decreases. Hence the mutual inductance decreases
(ii) Mutual inductance increases when the number of turns in each coil increases because $M \propto N_1N_2$
(iii) When an iron sheet is placed between the two coils the mutual inductance increases, because M $\propto$ permeability (μ)
Q) A cylindrical bar magnet is kept along the axis of a circular coil. Will there be a current induced in the coil id the magnet is rotated about its axis
Ans) Since there is no change in magnetic flux therefore no current is induced
Q) Why birds fly off a high-tension wire when current is switched on
Ans) When current begins to increase from zero to maximum value, a current is induced in the body of the bird. This produces a repulsive force and the bird flies off
Q) Why the inductance per unit length for a solenoid near the centre is different from inductance per unit length near its end
Ans) This is because the magnetic field near the centre of the solenoid is $\mu_0 ni$. On the other hand, the magnetic field at the end is ($\mu_0 ni/2$)

----------------END--------------

LESSON 7

ALTERNATING CURRENT

Alternating current

As we have seen earlier a rotating coil in a magnetic field, induces an alternating emf and hence an alternating current. Since the emf induced in the coil varies in magnitude and direction periodically, it is called an alternating emf. The significance of an alternating emf is that it can be changed to lower or higher voltages conveniently and efficiently using a transformer. Also the frequency of the induced emf can be altered by changing the speed of the coil. This enables us to utilize the whole range of electromagnetic spectrum for one purpose or the other. For example domestic power in India is supplied at a frequency of 50 Hz. For transmission of audio and video signals, the required frequency range of radio waves is between 100 KHz and 100 MHz. Thus owing to its wide applicability most of the countries in the world use alternating current.

Measurement of AC

Since alternating current varies continuously with time, its average value over one complete cycle is zero. Hence its effect is measured by rms value of a.c.

RMS value of a.c.

The rms value of alternating current is defined as that value of the steady current, which when passed through a resistor for a given time, will generate the same amount of heat as generated by an alternating current when passed through the same resistor for the same time.

The rms value is also called effective value of an a.c. and is denoted by I_{rms} when an alternating current i=Io sin ωt flows through a resistor of resistance R, the amount of heat produced in the resistor in a small time dt is dH = i^2 R dt. We know that alternating current is given I = $i_0$sinωt

The total amount of heat produced in the resistance in one complete cycle is

$$H = \int_0^T i^2\, Rdt$$

$$H = \int_0^T i_0^2 sin^2\omega t\, Rdt$$

$$H = i_0^2 R \int_0^T \left(\frac{1 - cos2\omega t}{2}\right) dt$$

$$H = \frac{i_0^2 R}{2}\left[\int_0^T dt - \int_0^T cos2\omega tdt\right]$$

As $\int_0^T cos2\omega tdt = 0$

$$H = \frac{i_0^2 R}{2} T$$

But this heat is also equal to the heat produced by rms value of AC in the same resistor (R) and in the same time (T),

$$H = I_{rms}^2 RT$$

Thus

$$I_{rms}^2 RT = \frac{i_0^2 R}{2} T$$

$$\therefore I_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

We can calculate rms value as root mean square value :
The mean value or average value of ac over time T is given by

$$i_{rms}^2 = \frac{\int_0^T i^2 dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{\int_0^T i_0^2 sin^2(\omega t) dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{i_0^2 \int_0^T [1 - cos2\omega t] dt}{2T}$$

As $\int_0^T cos2\omega t dt = 0$

$$i_{rms}^2 = \frac{i_0^2 T}{2T} = \frac{i_0^2}{2}$$

$$\therefore i_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

Similarly

$$E_{rms} = \frac{E_0}{\sqrt{2}}$$

Solved numerical

Q) If the voltage in ac circuit is represented by the equation

$$V = 220\sqrt{2} sin(314t - \varphi)$$

Calculate (a) peak and rms value of the voltage

(b) average voltage

(c) frequency of ac

<u>Solution:</u>

(a) For ac voltage

$V = V_0 \sin(\omega t - \varphi)$

The peak value

$V_0 = 220\sqrt{2} = 311V$

The rms value of voltage

$$V_{rms} = \frac{V_0}{\sqrt{2}} = \frac{220\sqrt{2}}{\sqrt{2}} = 220V$$

(b) Average voltage in full cycle is zero, Average voltage in half cycle is

$$V_{ave} = \frac{2}{\pi} V_0 = \frac{2}{\pi} 311 = 198.71V$$

(c) As $\omega = 2\pi f$

$2\pi f = 314$

$f = 314/2\pi = 50\text{Hz}$

Q) Write the equation of a 25 cycle current sine wave having rms value of 30 A.

<u>Solution:</u>

Given: frequency f = 25 HZ and I_{rms} = 30A or $i_0 = 30\sqrt{2}$

$I = i_0 \sin(2\pi f)t$

$I = 30\sqrt{2} \sin(2\pi \times 25)t$

$I = 30\sqrt{2} \sin(50\pi)t$

Q) An electric current has both A.C. and D.C. components. The value of the D.C component is equal to 12A while the A.C. component is given as I = 9sinωt A. Determine the formula for the resultant current and also calculate the value of I_{rms}

Solution: Resultant current at any instant of time will be I = 12+9sinωt

Now $I_{rms} = \sqrt{\langle I^2 \rangle} = \sqrt{(12 + 9\sin\omega t)^2}$

$I_{rms} = \sqrt{\langle 144 + 216 sin\omega t + 81 sin^2 \omega t \rangle}$

Here, the average is taken over a time interval equal to the periodic time

Now $<144> = 144$

$216<\sin\omega t> = 0$

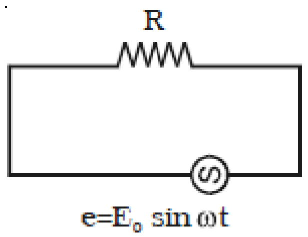
And $81<\sin^2 \omega t> = 81 \times (1/2) = 40.5$

$\therefore I_{rms} = \sqrt{144 + 40.5} = 13.58\ A$

Series AC Circuit

1) When only resistance is in an ac circuit

Consider a simple ac circuit consisting of resistor of resistance R and and ac generator, as shown in figure



According to Kirchhoff's loop law at any instant , the algebraic sum of the potential difference around a closed loop in a circuit must be zero

$e - V_R = 0$

$e - I_R R = 0$

$E_0 \sin\omega t - I_R R = 0$

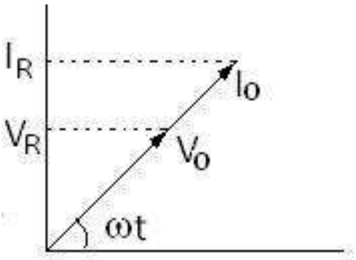
$I_R = E_0 \sin\omega t / R = I_0 \sin\omega t$ ---(1)

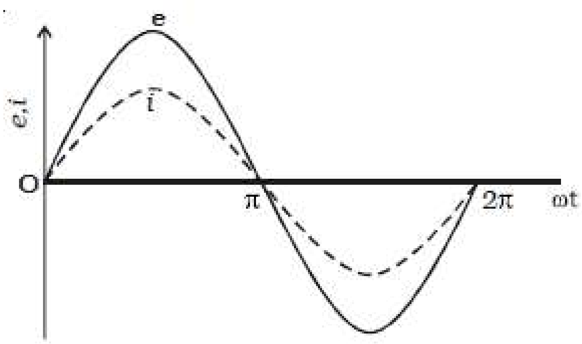
Where I_0 is the maximum current $I_0 = E_0/R$

From above equations, we see that the instantaneous voltage drop across the resistor is

$V_R = I_O R \sin\omega t$ ---(2)

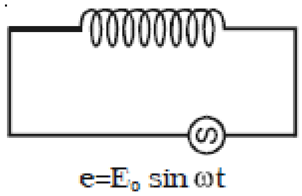
We see in equation (1) and (2) I_R and V_R both vary as $\sin\omega t$ and reach their maximum values at the same time as shown in graph they are said to be in phase.





2) When only Inductor is in an ac circuit

Consider an ac circuit consisting only of an indiuctor of inductance l connected to the termicals of ac generator, as shown in figure



The induced emf across the inductor is given by L(di/dt). On applying Kirchhoff's loop rule to the circuit

$$e - V_L = 0$$
$$e = L\frac{di}{dt}$$
$$E_0 sin\omega t = L\frac{di}{dt}$$

Integrating above expression as a function of time

$$i_L = \frac{E_0}{L}\int sin\omega t dt = -\frac{E_0}{\omega L}cos\omega t + C$$

For average value of current over one time period to be zero, C= 0

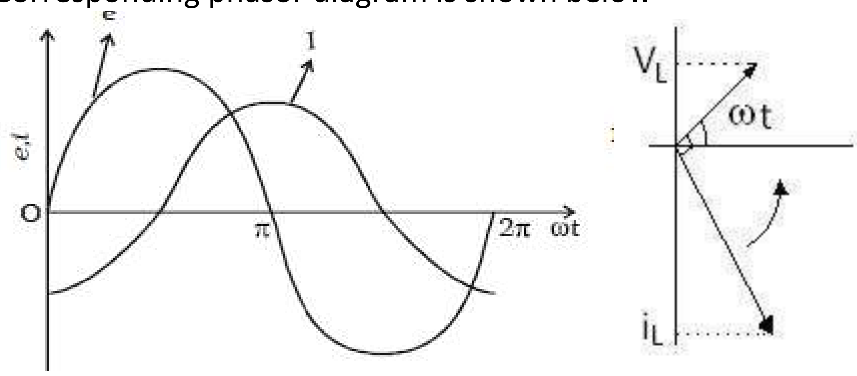
$$\therefore i_L = -\frac{E_0}{\omega L}cos\omega t$$

When we use the trigonometric identity $cos\omega t \;=\; -sin\left(\omega t - \frac{\pi}{2}\right)$

We can express equation as

$$i_L = \frac{E_0}{\omega L}sin\left(\omega t - \frac{\pi}{2}\right)$$

From above equation it is clear that current lags by π/2 to voltage. The voltage reaches maximum , one quarter of than oscillation period before current reaches maximum value. Corresponding phasor diagram is shown below



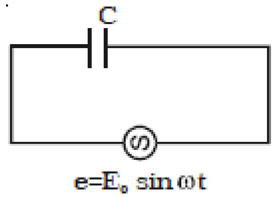
Secondly current is maximum when cosωt = 1

$$i_0 = \frac{E_0}{\omega L}$$

ωL is known as inductive reactance denoted by X_L

3) When only capacitor is in an ac circuit

Figure shows an ac circuit consisting of a capacitor of capacitance C connected across the terminals of an ac generator.



On applying Kirchhoff's rule to this circuit, we get

$$e - V_C = 0$$
$$V_C = e$$
$$V_C = E_0 sin\omega t$$

Where V_C is the instantaneous voltage drop across the capacitor. From the definition of capacitance V_C = Q/C, and this value of V_C substituted into equation gives

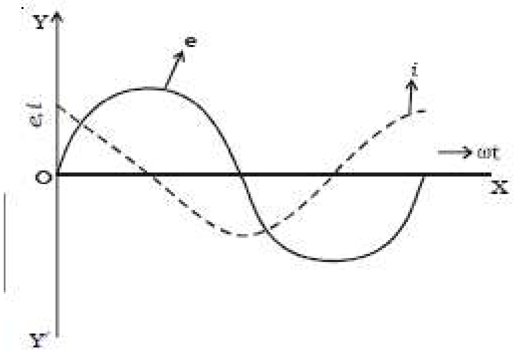
Q = C E_0sinωt

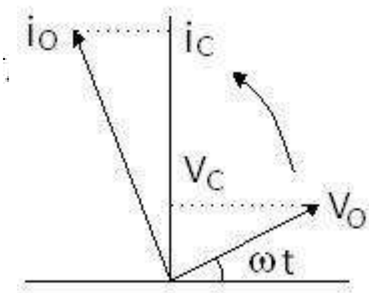
Since i = dQ/dt, on differentiating above equation gives the instantaneous current in the circuit

$$i_C = \frac{dQ}{dt} = \mathrm{C}E_0\omega\cos\omega t$$

From above equation it is clear that current leads the voltage by π/2

A plot of current and voltage versus times , shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches maximum value. The corresponding phasor diagram is shown





Secondly when cosωt = 1, in equation $\mathrm{i_C} = \mathrm{C}E_0\omega\cos\omega t$ the current in circuit is maximum

$$\mathrm{i_C} = \mathrm{C}E_0\omega = \frac{E_0}{X_C}$$

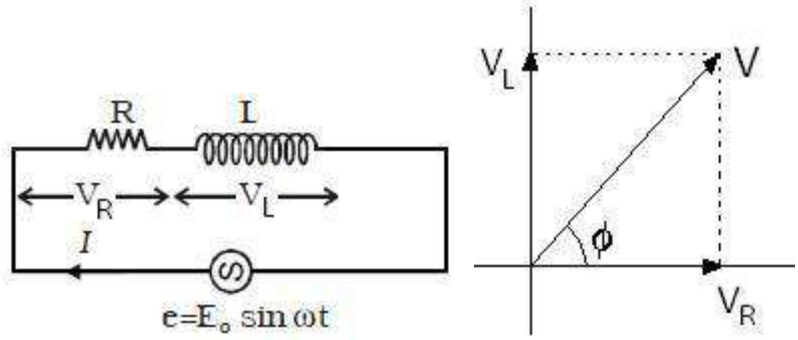
X_C is called the capacitive reactance

$$X_C = \frac{1}{\omega C}$$

For DC supply, ω = 0 therefore X_C will be infinite and current will not flow through capacitor once it is fully charged.

SERIES L-R Circuit

Now consider an ac circuit consisting of a resistor or resistance R and an inductor of inductance L in series with an ac source generator



Suppose in phasor diagram, current is taken along positive direction. The V_R is also along positive x-direction as there is no phase difference between i_R and V_R . While V_L will be along y direction as we know that current lags behind the voltage by 90^O

So we can write

$V = V_R + jV_L$

$V = i_RR + j(iX_L)$

$V= iZ$

Here $Z = R + jX_L = R + j(\omega L)$ is called as impedance of the circuit. Impedance plays the same role in ac circuit as the ohmic resistance does in DC circuit. The modulus of impedance is

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

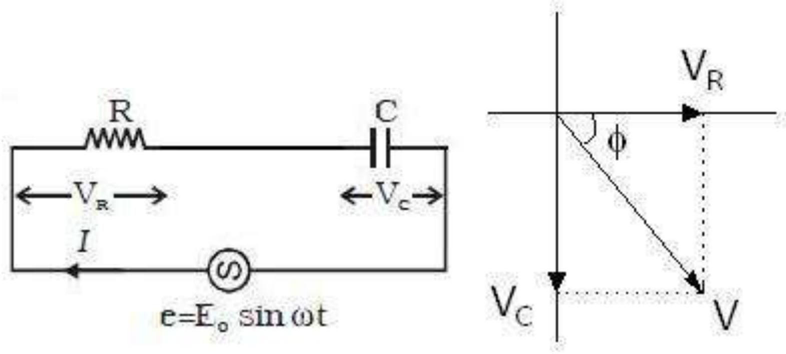
The potential difference leads the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_L}{V_R}\right| = tan^{-1}\left(\frac{X_L}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{\omega L}{R}\right)$$

SERIES R-C Circuit

Now consider an ac circuit consisting of resistance R and a capacitor of capacitance C in series with an ac source generator



Suppose in phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction but V_C is along negative y-direction as current leads the potential by phase 90^O so we can write

$V = V_R - jV_C$

$$V = iR - j\left(\frac{i}{\omega C}\right) = iZ$$

Here impedance

$$Z = R - j\left(\frac{1}{\omega C}\right)$$

And the potential difference lags the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_C}{V_R}\right| = tan^{-1}\left(\frac{X_C}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{1/\omega C}{R}\right) = tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Solved Numerical

Q) An alternating current voltage of 220 V r.m.s. at frequency of 40 cycles/ second is supplied to a circuit containing a pure inductance of 0.01H and a pure resistance of 6 ohm in series. Calculate (i) the current (ii) potential difference across the resistance (iii)potential difference across the inductance (iv) the time lag

Solution

The impedance of L-R circuit is given by

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

$$|Z| = \sqrt{R^2 + (2\pi f L)^2}$$

$$|Z| = \sqrt{(6)^2 + (2 \times 3.14 \times 40 \times 0.01)^2}$$

Z=6.504 ohms

(i) r.m.s value of current

$$i_{rms} = \frac{E_{rms}}{Z} = \frac{220}{6.504} = 33.83\ amp$$

(ii) The potential difference across the resistance is given by

$V_R = i_{rms} \times R = 33.83 \times 6 = 202.98$ Volt

(iii) Potential difference across inductance is given by

$V_L = i_{rms} \times (\omega L) = 33.83 \times 6 = 202.98$ volts

(iv) Phase angle

$$\varphi = tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\varphi = tan^{-1}\left(\frac{2\pi f L}{R}\right) = tan^{-1}\left(\frac{2 \times 3.14 \times 40 \times 0.01}{R}\right)$$

$$\varphi = tan^{-1}(0.4189) = 22^o 73'$$

Now time lag =

$$\frac{\varphi}{360} \times T = \frac{\varphi}{360} \times \frac{1}{f} = \frac{22^o 73'}{360 \times 40} = 0.001579\ s$$

Q) An ac source of an angular frequency ω is fed across a resistor R and a capacitor C in series. The current registered is i. If now the frequency of source is changed to ω/3 (but maintaining the same voltage), the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency

Solution:

At angular frequency ω , the current in R-C circuit is given by

$$i_{rms} = \frac{E_{rms}}{\sqrt{R^2 + \left(\frac{1}{\omega^2 C^2}\right)}} - - - eq(1)$$

When frequency changed to ω/3 , the current is halved. Thus

$$\frac{i_{rms}}{2} = \frac{E_{rms}}{\sqrt{\left\{R^2 + \left(\frac{1}{\frac{\omega^2 C^2}{3^2}}\right)\right\}}}$$

$$\frac{i_{rms}}{2} = \frac{E_{rms}}{\left\{R^2 + \frac{9}{\omega^2 C^2}\right\}} - - - eq(2)$$

From above equation (1) and (2) we have

$$\frac{E_{rms}}{\sqrt{R^2 + \left(\frac{1}{\omega^2 C^2}\right)}} = \frac{2E_{rms}}{\sqrt{\left\{R^2 + \left(\frac{9}{\omega^2 C^2}\right)\right\}}}$$

Solving the equation we get

$$3R^2 = \frac{5}{\omega^2 C^2}$$

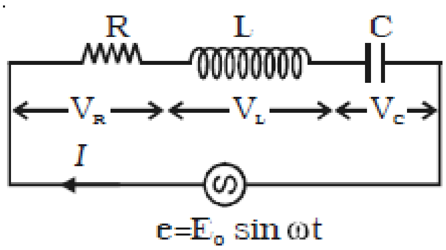
Hence ratio of reactance to resistance

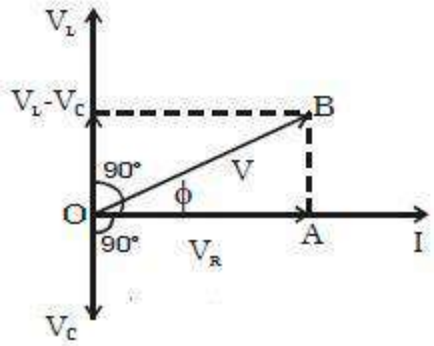
$$\frac{1/_{\omega C}}{R} = \sqrt{\frac{3}{5}}$$

SERIES L – C – R CIRCUIT

Consider an ac circuit consisting of resistance R, capacitor of capacitance C and an inductor of inductance L are in series with ac source generator

Suppose in a phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction, V_L along positive y-direction and V_C along negative y-direction, as potential difference across an inductor leads the current by 90^O in phase while that across a capacitor, lags by 90^O





$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

We can write V = V_R + jV_L – jV_C

V = iR +j(iX_L) – j(iX_C)

V = iR +j[i(X_L – X_C)] = iZ

Here impedance is

$$Z = R + j(X_L - X_C)$$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The potential difference leads the current by an angle

$$\varphi = tan^{-1}\left|\frac{V_L - V_C}{V_R}\right|$$

$$\varphi = tan^{-1}\left|\frac{X_L - X_C}{R}\right|$$

$$\varphi = tan^{-1}\left|\frac{\omega L - \frac{1}{\omega C}}{R}\right|$$

The steady current is given by

$$i = \frac{V_O}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} sin(\omega t + \varphi)$$

The peak current is

$$i_0 = \frac{V_O}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

It depends on angular frequency ω of ac source and it will be maximum when

$$\omega L - \frac{1}{\omega C} = 0$$

$$\omega = \sqrt{\frac{1}{LC}}$$

And corresponding frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This frequency is known as resonant frequency of the given circuit. At this frequency peak current will be $i_0 = \frac{V_0}{R}$

This resistance R in the LCR circuit is zero, the peak current at resonance is $i_0 = \frac{V_0}{R}$

It means, there can be a finite current in pure LC circuit even without any applied emf. When a charged capacitor is connected to pure inductor

This current in the circuit is at frequency $f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

Solved Numerical

Q) A resistor of resistance R, an inductor of inductance L and a capacitor of capacitance C all are connected in series with an a.c. supply. The resistance of R is 16 ohm. And for a given frequency, the inductive reactance of L is 24 ohms and capacitive reactance of C is 12 ohms. If the current in circuit is 5amp, find

(a) The potential difference across R, L and C

(b) the impedance of the circuit

(c) the voltage of ac supply

(d) Phase angle

Solution:

(a) Potential difference across resistance V_R = iR = 5×16 =80 volt

Potential difference across inductance

V_L = i × (ωL) = 5 × 24 = 120 volt

Potential across condenser

V_C = i × (1/ωC) = 5 × 12 = 60 volts

(b) Impedance

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$|Z| = \sqrt{16^2 + (24-12)^2} = 20\ ohm$$

(c) The voltahe of ac supply is given by

V = iZ = 5 ×20 = 100 volt

(c) Phase angle

$$\varphi = tan^{-1}\left|\frac{\omega L - \frac{1}{\omega C}}{R}\right|$$

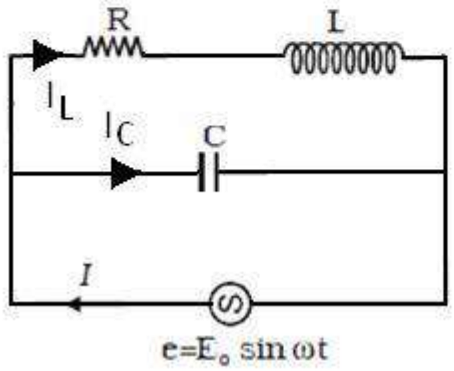
$$\varphi = tan^{-1}\left|\frac{24-12}{16}\right|$$

$$\varphi = tan^{-1}(0.75) = 36^{O}87'$$

PARALLEL AC CIRCUIT

Consider an alternating source connected across an inductor L in parallel with a capacitance C

The resistance in series with the inductance is R and with the capacitor as zero



Let the instantaneous value of emf applied be V and the corresponding current is I, I_L and I_C. Then

$I = I_L + I_C$

Or

$$\frac{V}{Z} = \frac{V}{R + j\omega L} - \frac{V}{\frac{j}{\omega C}}$$

$$\frac{V}{Z} = \frac{V}{R + j\omega L} + j\omega CV \ \ (as\ j^2 = -1)$$

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + j\omega C$$

$\frac{1}{Z}$ is called admittance Y

$$\frac{1}{Z} = Y = \frac{1}{R + j\omega L}\frac{R - j\omega L}{R - j\omega L} + j\omega C$$

$$Y = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$Y = \frac{R + j(\omega C R^2 + \omega^3 L^2 C - \omega L)}{R^2 + \omega^2 L^2}$$

Magnitude of admittance

$$|Y| = \frac{\sqrt{R^2 + (\omega C R^2 + \omega^3 L^2 C - \omega L)^2}}{R^2 + \omega^2 L^2}$$

The admittance will be minimum. When
$\omega CR^2 + \omega^3 L^2 C - \omega L = 0$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

It gives the condition of resonance and corresponding frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is known as resonance frequency. At resonance frequency admittance is minimum or impedance is maximum. Thus the parallel circuit does not allow this frequency from source to pass in the circuit. Due to this reason the circuit with such frequency is known as rejecter circuit

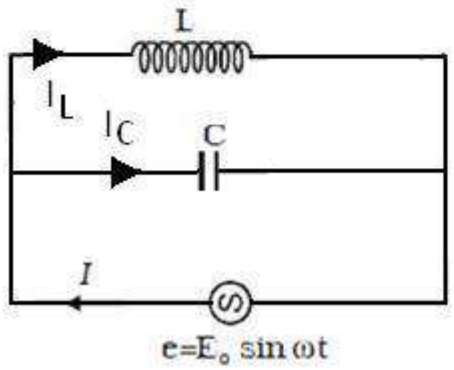
Note

If R = 0, resonance frequency

$f = \frac{1}{2\pi\sqrt{LC}}$ is same as resonance frequency in series circuit

Solved numerical

Q) For the circuit shown in figure. Current is inductance is 0.8A while in capacitance is 0.6A. What is the current drawn from the source



Solution:

In this circuit $E = E_0 \sin\omega t$ is applied across an inductance and capacitance in parallel, current in inductance will lag the applied voltage while across the capacitor will lead and so

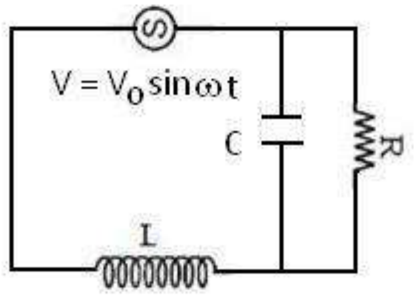
$$i_L = \frac{V}{X_L} sin\left(\omega t - \frac{\pi}{2}\right) = -0.8cos\omega t$$
$$i_C = \frac{V}{X_C} sin\left(\omega t + \frac{\pi}{2}\right) = +0.6cos\omega t$$

So current from the source

$i = i_L + i_C = -0.2\cos\omega t$

$|i_0| = 0.2A$

Q) An emf $V_0 \sin\omega t$ is applied to a circuit which consists of self-inductance L of negligible resistance in series with a variable capacitor C. the capacitor is shunted by a variable resistance R. Find the value of C for which the amplitude of the current is independent of R
Solution



Solution:
First we will calculate impedance of the circuit
The complex impedance of the circuit
Z = jωL +Z'
Here Z' is complex impedance of parallel combination of Capacitor and R

$$\frac{1}{Z'} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

$$Z' = \frac{R}{1 + j\omega CR} = \frac{R(1 - j\omega CR)}{(1 + j\omega CR)(1 - j\omega CR)} = \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$Z = \mathrm{j\omega L} + \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$Z = \mathrm{j\omega L} + \frac{R}{1 + \omega^2 C^2 R^2} - \frac{j\omega CR^2}{1 + \omega^2 C^2 R^2}$$

$$Z = j\left(\omega \mathrm{L} - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right) + \frac{R}{1 + \omega^2 C^2 R^2}$$

Magnitude of Z is given by

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 + \left(\omega \mathrm{L} - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right)^2$$

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} + \left(\frac{\omega CR^2}{1 + \omega^2 C^2 R^2}\right)^2$$

$$Z^2 = \left(\frac{R}{1 + \omega^2 C^2 R^2}\right)^2 (1 + \omega^2\ C^2\ R^2) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2}$$

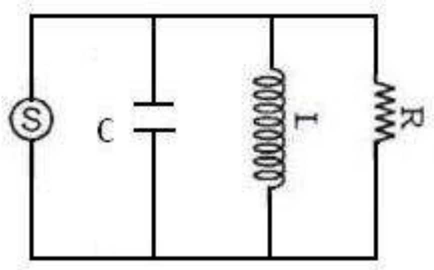
$$Z^2 = \left(\frac{R^2}{1 + \omega^2 C^2 R^2}\right) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2}$$

The value of current will be independent of R . It is possible when

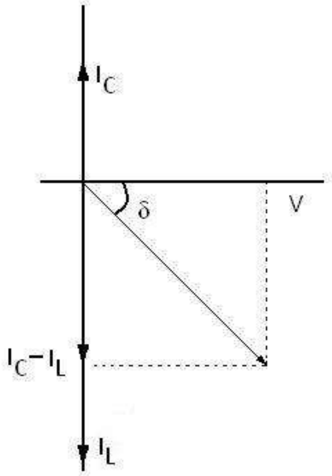
$$R^2 - 2\omega^2 LCR^2 = 0$$

$$C = \frac{1}{2}\omega^2 L$$

Q) Derive the expression for the total current flowing in the circuit using phaser diagram



Solution:
The phasor diagram of the voltage and current is as shown in figure. In order to obtain the total current, we shall have to consider the addition of the currents. From the diagram we have



$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

But

$$I_R = \frac{V}{R}\ ; I_L = \frac{V}{X_L}; I_C = \frac{V}{X_C}$$

$$I = V\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_L} - \frac{1}{X_C}\right)^2}$$

From figure

$$tan\delta = \frac{I_L - I_C}{I_R} = \frac{\frac{1}{X_L} - \frac{1}{X_C}}{\frac{1}{R}}$$

$$tan\delta = R\left(\frac{1}{X_L} - \frac{1}{X_C}\right)$$

Q-factor

The selectivity or sharpness of a resonant circuit is measured by the quality factor or Q factor. In other words it refers to the sharpness of tuning at resonance. The Q factor of a series resonant circuit is defined as the ratio of the voltage across a coil or capacitor to the applied voltage.

$$Q = \frac{voltage\ across\ L\ or\ C}{applied\ voltage} --(1)$$

Voltage across L = I ω_oL ...(2)

where ω_o is the angular frequency of the a.c. at resonance. The applied voltage at resonance is the potential drop across R, because the potential drop across L is equal to the drop across C and they are 180^o out of phase. Therefore they cancel out and only potential drop across R will exist.

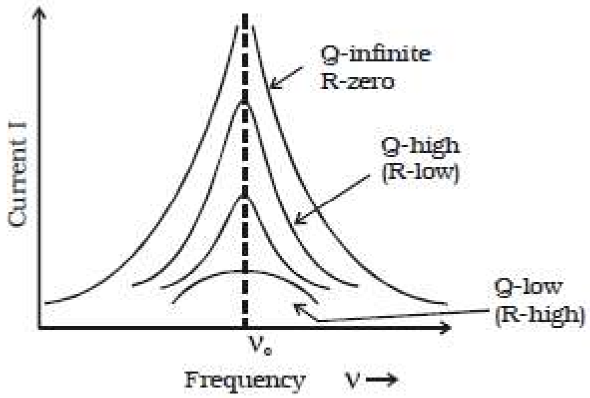
Applied Voltage = IR ...(3)

Substituting equations (2) and (3) in equation (1)

$$Q = \frac{\mathrm{I\ \omega o L}}{\mathrm{IR}} = \frac{\mathrm{\omega o L}}{\mathrm{R}}$$

$$Q = \frac{1}{\sqrt{RC}}\frac{L}{R}$$

Q is just a number having values between 10 to 100 for normal frequencies. Circuit with high Q values would respond to a very narrow frequency range and vice versa. Thus a circuit with a high Q value is sharply tuned while one with a low Q has a flat resonance. Q-factor can be increased by having a coil of large inductance but of small ohmic resistance. Current frequency curve is quite flat for large values of resistance and becomes more sharp as the value of resistance decreases. The curve shown in graph is also called the frequency response curve.



Sharpness of resonance

The amplitude of the current in the series *LCR* circuit is given by

$$i_{max} = \frac{v_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and is maximum when ω = ω_0 = 1/√(LC) The maximum value is $i_{max} = V_{max}/R$

For values of ω other than ω0, the amplitude of the current is less than the maximum value.

Suppose we choose a value of ω for which the current amplitude is 1/ √2 times its maximum value. At this value, the power dissipated by the circuit becomes half. From the curve in Fig. ,we see that there are two such values of ω, say, ω_1 and ω_2, one greater and the other smaller than ω_0 and symmetrical about ω_0. We may

write
$\omega_1 = \omega_0 + \Delta\omega$
$\omega_2 = \omega_0 - \Delta\omega$
The difference $\omega_1 - \omega_2 = 2\Delta\omega$ is often called the *bandwidth* of the circuit. The quantity $(\omega_0 / 2\Delta\omega)$ is regarded as a measure of the sharpness of resonance. The smaller the $\Delta\omega$, the sharper or narrower is the resonance.
We see from Fig. that if the resonance is less sharp, not only is the maximum current less, the circuit is close to resonance for a larger range $\Delta\omega$ of frequencies and the tuning of the circuit will not be good. So, less sharp the resonance, less is the selectivity of the circuit or vice versa.
Value of $\Delta\omega = \frac{R}{2L}$
we see that if quality factor is large, i.e., R is low or L is large, the circuit is more selective.

POWER IN AN AC CIRCUIT
In case of steady current the rate of doing work is given by,
P = VI
In an alternatin circuit, current and voltage both vary with time, so the work done by the source in time intrerval dt is given by
dw = Vidt
Suppose in an ac, the current is leading the voltage by an angle φ . Then we can write
$V = V_m \sin\omega t$ and
$I = i_m \sin(\omega t+\varphi)$
$dw = V_m\, i_m \sin\omega t \sin(\omega t+\varphi)dt$
$dw = V_m\, i_m (\sin^2 \omega t\cos\varphi + \sin\omega t\cos\omega t\sin\varphi)\, dt$
The total work done in a complete cycle is

$$W = V_m i_m cos\varphi \int_0^T sin^2\,\omega t dt + V_m i_m sin\varphi \int_0^T sin\omega t cos\omega t dt$$

$$W = \frac{1}{2} V_m i_m cos\varphi \int_0^T (1 - cos2\omega t)\, dt + \frac{1}{2} V_m i_m sin\varphi \int_0^T sin2\omega t dt$$

$$W = \frac{1}{2} V_m i_m T cos\varphi$$

The average power delivered by the source is, therefore
P=W/T

$$P = \frac{1}{2} V_m i_m cos\varphi$$

$$P = \frac{V_m}{\sqrt{2}} \frac{i_m}{\sqrt{2}} cos\varphi$$

$$P = V_{rms} i_{rms} cos\varphi$$

This can also be written as,
$P = I^2 Z\cos\phi$
Here, Z is impedance, the term $\cos\varphi$ is known as power factor

It is said to be leading if current leads voltage, lagging if current lags voltage . Thus , a power factor of 0.5 lagging means current lags voltage by 60^O (as $\cos^{-1}0.5= 60^O$). the product of V_{rms} and i_{rms} gives the apparent power. While the true power is obtained by multiplying the apparent power by the power factor cosφ.

(i) Resistive circuit: For φ=0^O, the current and voltage are in phase. The power is thus, maximum .

(ii) purely inductive or capacitive circuit: For φ=90^O, the power is zero. The current is then stated as wattless. Such a case will arise when resistance in the circuit is zero. The circuit is purely inductive or capacitive

(iii) *LCR series circuit*: In an *LCR* series circuit, power dissipated is given by P = $I^2Z\cos\phi$ where

$$\varphi = tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

So, ϕ may be non-zero in a *RL* or *RC* or *RCL* circuit. Even in such cases, power is dissipated only in the resistor.

(iv) *Power dissipated at resonance in LCR circuit*: At resonance $X_C - X_L = 0$, and ϕ = 0. Therefore, cosϕ = 1 and $P = I^2Z = I^2R$. That is, maximum power is dissipated in a circuit (through *R*) at resonance

Solved Numerical

Q) In an L-C-R A.C. series circuit L = 5H, ω = 100 rad s^{-1}, R = 100Ω and power factor is 0.5. Calculate the value of capacitance of the capacitor

Solution:

Power factor

$$cos\delta = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}}$$

Squaring on both side

$$cos^2\delta = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}$$

Cosδ = 0.5

$$\frac{1}{4} = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2}$$

$$R^2 + \left(\omega L - \frac{1}{\omega L}\right)^2 = 4R^2$$

$$\left(\omega L - \frac{1}{\omega L}\right)^2 = 3R^2$$

$$\omega L - \frac{1}{\omega C} = \sqrt{3}R$$

$$\omega L - \sqrt{3}R = \frac{1}{\omega C}$$

$$C = \frac{1}{\omega}\left(\frac{1}{\omega L - \sqrt{3}R}\right)$$

$$C = \frac{1}{100}\left(\frac{1}{100 \times 5 - \sqrt{3} \times 100}\right)$$

$$C = \frac{10^{-2}}{500 - 173.2} = \frac{10^{-2}}{326.8} = 30.6 \times 10^{-6}F$$

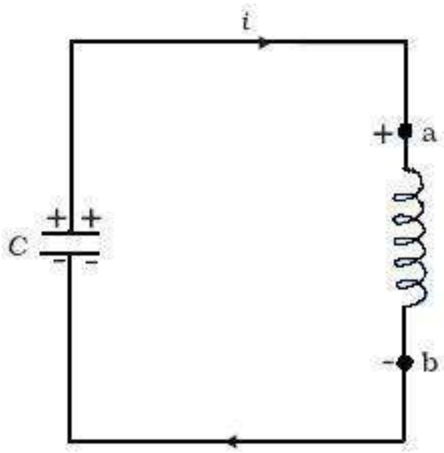
C=30.6μF

LC OSCILLATIONS

We know that a capacitor and an inductor can store electrical and magnetic energy, respectively.

When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electrical oscillations similar to oscillations in mechanical systems.

Let a capacitor be charged q_m (at t = 0) and connected to an inductor as shown in Fig..



The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let q and i be the charge and current in the circuit at time t.

Since di/dt is positive, the induced emf in L will have polarity as shown, i.e., $v_b < v_a$.

According to Kirchhoff's loop rule,

$$\frac{q}{C} - L\frac{di}{dt} = 0$$

i = – (dq/dt) in the present case (as q decreases, i increases).

Therefore, above equation becomes:

$$\frac{d^2q}{dt^2} - \frac{1}{LC}q = 0$$

Comparing above equation with standard equation for oscillation

$$\frac{d^2x}{dt^2} - \omega_0^2 x = 0$$

The charge, therefore, oscillates with a natural frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and varies sinusoidally with time as q = q_m cos(ω_0t +φ)
where *qm* is the maximum value of *q* and ϕ is a phase constant.
Since *q* = *qm* at *t* = 0, we have cos ϕ =1 or ϕ = 0. Therefore, in the present case,
q = q_m cos(ω_0t) current I = $i_m$sin(ω_0t) here i_m = q_m ω_0
Initially capacitor is fully charged , it stores energy in the form of electric field

$$U_E = \frac{1}{2} CV^2$$

At *t* = 0, the switch is closed and the capacitor starts to discharge As the current increases, it sets up a magnetic field in the inductor and thereby, some energy gets stored in the inductor in the form of magnetic energy:

$$U_B = \frac{1}{2} Li^2.$$

As the current reaches its maximum value i_m, (at *t* = *T*/4) all the energy is stored in the magnetic field:

$$U_B = \frac{1}{2} Li^2.$$

The capacitor now has no charge and hence no energy. The current now starts charging the capacitor, This process continues till the capacitor is fully charged (at *t* = *T*/2But it is charged with a polarity opposite to its initial state The whole process just described will now repeat itself till the system reverts to its original state. Thus, the energy in the system oscillates between the capacitor and the inductor.
Note that the above discussion of LC oscillations is not realistic for two reasons:
(i) Every inductor has some resistance. The effect of this resistance is to introduce a damping effect on the charge and current in the circuit and the oscillations finally die away.
(ii) Even if the resistance were zero, the total energy of the system would not remain constant. It is radiated away from the system in the form of electromagnetic waves (discussed in the next chapter). In fact, radio and TV transmitters depend on this radiation.

Solved Numerical

Q) A capacitor of capacitance 25μF is charged to 300V. It is then connected across a 10mH inductor. The resistance of the circuit is negligible
(a) Fins the frequaency of oscillation of the circuit
(b) Find the potential difference across capacior and magnitude of circuit cutrrent 1.2ms after the inductor and capacitor are connected
(c) Find the magnetic energy and electric energy at t=0 and t = 1.2 ms.

Solutions:

(a) The frequency of oscillation of the circuit is

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Substituting the given values we have

$$f = \frac{1}{2\pi\sqrt{(10 \times 10^{-3})(25 \times 10^{-6})}} = \frac{10^3}{\pi} Hz$$

(b) Charge across the capacitor at time t will be

$q = q_O \cos\omega_o t$ and $I = -q\omega_o \sin\omega_o t$

Here $q_o = CV_O = (25\times10^{-6})(300) = 7.5\times10^{-3}C$

Now, charge in the capacitor after t = 1.25×10^{-3} s is

$q = (7.5\times10^{-3})\cos(2\pi\times318.3)(1.2\times10^{-3})C = 5.53\times10^{-3}C$

∴ P.D across capacitor ,

$$V = \frac{|q|}{C} = \frac{5.53 \times 10^{-3}}{25 \times 10^{-6}} = 221.2\ volt$$

The magnitude of current in the circuit at t = 1.2 $\times10^{-3}$ s is

$|i| = q\omega_o \sin\omega_o t$

$|i| = (7.5\times10^{-3})(2\pi)(318.3)\sin(2\pi\times318.3)(1.2\times10^{-3})A = 10.13A$

(c) At t = 0, Current in the circuit is zero. Hence $U_L = 0$

Charge on the capacitor is maximum

Hence

$$U_C = \frac{1}{2}\frac{q_0^2}{C}$$

$$U_C = \frac{1}{2}\frac{(7.5 \times 10^{-3})^2}{25 \times 10^{-6}} = 1.125\ J$$

At t = 1.25ms, q =5.53×10^{-3}C

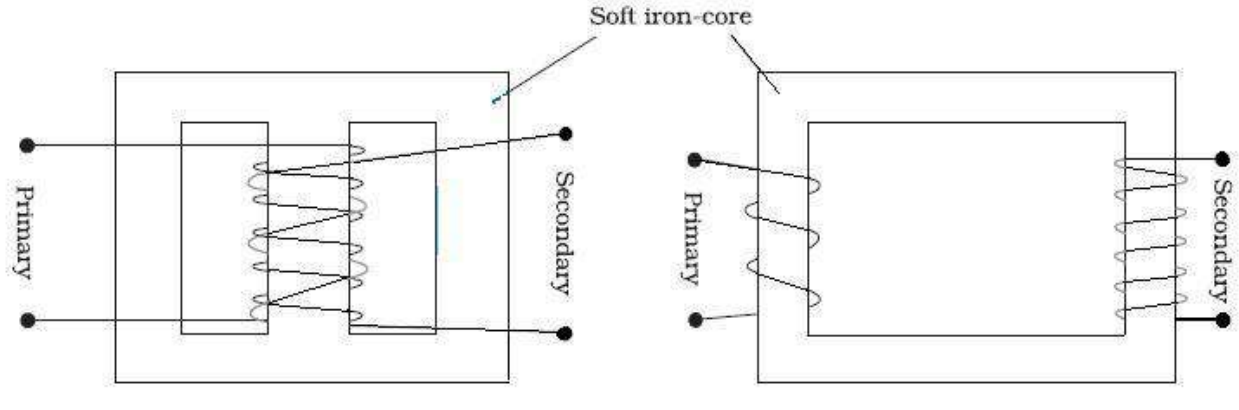
$$U_C = \frac{1}{2}\frac{q_0^2}{C}$$

$$U_C = \frac{1}{2}\frac{(5.53 \times 10^{-3})^2}{25 \times 10^{-6}} = 0.612\ J$$

TRANSFORMERS

For many purposes, it is necessary to change (or transform) an alternating voltage from one to another of greater or smaller value. This is done with a device called *transformer* using the principle of mutual induction. A transformer consists of two sets of coils, insulated from each other. They are wound on a soft-iron core, either one on top of the other as in Fig.a or on separate limbs of the core as in Fig. (b).

One of the coils called the *primary coil* has *Np* turns. The other coil is called the *secondary coil*; it has *Ns* turns. Often the primary coil is the input coil and the secondary coil is the output coil of the transformer



(a) (b)

When an alternating voltage is applied to the primary, the resulting current produces an alternating magnetic flux which links the secondary and induces an emf in it. The value of this emf depends on the number of turns in the secondary. We consider an ideal transformer in which the primary has negligible resistance and all the flux in the core links both primary and secondary windings. Let ϕ be the flux in each turn in the core at time *t* due to current in the primary when a voltage *vp* is applied to it.
Then the induced emf or voltage E*s*, in the secondary with *Ns* turns is

$$E_S = -N_S\frac{d\varphi}{dt}$$

The alternating flux φ also induces an emf, called back emf in the primary. This is

$$E_P = -N_P\frac{d\varphi}{dt}$$

But E*p* = *Vp.* If this were not so, the primary current would be infinite since the primary has zero resistance(as assumed). If the secondary is an open circuit or the current taken from it is small, then to a good approximation
E*s* = *Vs*
where *Vs* is the voltage across the secondary. Therefore, above equations can be written as

$$V_S = -N_S\frac{d\varphi}{dt}$$

$$V_P = -N_P\frac{d\varphi}{dt}$$

From above equations

$$\frac{V_S}{V_P} = \frac{N_S}{N_P}$$

Note that the above relation has been obtained using three assumptions:
(i) the primary resistance and current are small;
(ii) the same flux links both the primary and the secondary as very little flux escapes from the core, and
(iii) the secondary current is small.

If the transformer is assumed to be 100% efficient (no energy losses), the power input is equal to the power output, and since p = i V,

$$i_p V_p = i_s V_s$$

The equations obtained above apply to ideal transformers (without any energy losses). But in actual transformers, small energy losses do occur due to the following reasons:
(i) *Flux Leakage*: There is always some flux leakage; that is, not all of the flux due to primary passes through the secondary due to poor design of the core or the air gaps in the core. It can be reduced by winding the primary and secondary coils one over the other.
(ii) *Resistance of the windings*: The wire used for the windings has some resistance and so, energy is lost due to heat produced in the wire (*I* 2*R*). In high current, low voltage windings, these are minimized by using thick wire.
(iii) *Eddy currents*: The alternating magnetic flux induces eddy currents in the iron core and causes heating. The effect is reduced by having a laminated core.
(iv)Hysteresis: The magnetization of the core is repeatedly reversed by the alternating magnetic field. The resulting expenditure of energy in the core appears as heat and is kept to a minimum by using a magnetic material which has a low hysteresis loss.

The large scale transmission and distribution of electrical energy over long distances is done with the use of transformers. The voltage output of the generator is stepped-up (so that current is reduced and consequently, the I^2R loss is cut down). It is then transmitted over long distances to an area sub-station near the consumers. There the voltage is stepped down. It is further stepped down at distributing sub-stations and utility poles before a power supply of 240 V reaches our homes.

Solved Numerical

Q) In an ideal step-up transformer input voltage is 110V and current flowing in the secondary is 10A. If transformation ratio is 10, calculate output voltage, current in primary, input and out put power

Solution:
Transformer ratio

$$r = \frac{N_S}{N_P} = 10$$

(i)

$$\frac{E_S}{E_P} = \frac{N_S}{N_P}$$

$$E_S = E_P \frac{N_S}{N_P} = 110 \times 10 = 1100\ V$$

$$Ouput\ voltage\ E_S = 1100\ V$$

(ii)

$$E_P I_P = E_S I_S$$

$$I_P = \frac{E_S}{E_P} I_S = 10 \times 10 = 100\ A$$

(iii)
Input power = Output power for ideal transformer

$$E_S I_S = E_P I_p = (1100)(10) = 11000\ W$$

OSCILLATIONS

Periodic Motion and Oscillatory motion

If a body repeats its motion along a certain path, about a fixed point, at a definite interval of time, it is said to have a *periodic motion*

If a body moves to and fro, back and forth, or up and down about a fixed point in a definite interval of time, such motion is called an *oscillatory motion.* The body performing such motion is called an *oscillator*.

Simple Harmonic Motion

The periodic motion of a body about a fixed point, on a linear path, under the influence of the force acting towards the fixed point and proportional to displacement of the body from the fixed point is called a *simple harmonic motion* (SHM)

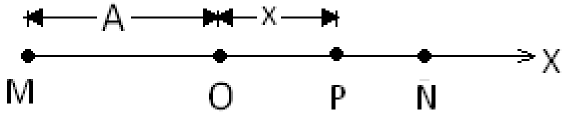
A body performing simple harmonic motion is known as *Simple Harmonic Oscillator (SHO)*

Simple harmonic motion is a special type of periodic motion in which

(i) The particle oscillates on a straight line.

(ii) The acceleration of the particle is always directed towards a fixed point on the straight line.

(iii) The magnitude of acceleration is proportional to the displacement of the particle from fixed point.



This fixed point is called the centre of the oscillation or *mean position*. Taking this point as origin "O".

The maximum displacement of oscillator on either side of the mean position is called *amplitude* denoted by A

The time required to complete one oscillation is known as *periodic time* (T) of oscillator

In other words, the least time interval of time after which the periodic motion of an oscillator repeat itself is called a periodic time of the oscillator. Distance travelled by oscillator is 4A in *periodic time*

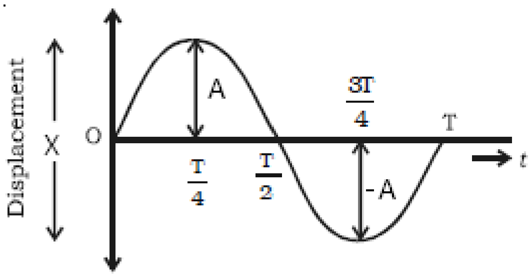
The number of oscillations completed by simple harmonic oscillator in one second is defined as *frequency*. SI unit is s^{-1} or hertz (H)

It is denoted by f and f = 1/T

2π times the frequency of an oscillator is called the *angular frequency* of the oscillator

It is denoted by ω. Its SI unit is rad s^{-1}, $\omega = 2\pi f$.

If we draw the graph of displacement of SHO against time as shown in figure, which is as shown in figure



Mathematical equation is the function of time

x(t) = Asin(ωt +ϕ) -----eq(1)

Here X(t) represents displacement at time t

A = Amplitude

(ωt + ϕ) = Phase

ϕ = Initial phase (epoch) for a given graph ϕ= 0.

If oscillation have started from negative x end (M) then initial phase is –π/2

If oscillation stated from positive end (N) then initial phase is π/2

ω = angular frequency

Velocity

Velocity of oscillator is

$$v(t) = \frac{dx(t)}{dt}$$

v(t) = ωAcos(ωt + φ)

$$v(t) = \pm\omega A\left(\sqrt{1 - sin^2(\omega t + \varphi)}\right)$$

$$v(t) = \pm\omega\left(\sqrt{A^2 - A^2 sin^2(\omega t + \varphi)}\right)$$

From eq(1)

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

Velocity is maximum at x = 0 or equilibrium position v = ±ωA

Velocity is minimum at x = A or extreme positions v= 0

Note that velocity is out of phase of displacement by π/2

Acceleration

$$a(t) = \frac{dv(t)}{dt}$$

a(t) = $-\omega^2$Asin(ωt + ϕ)

a(t) = $-\omega^2$ x(t)

At x = 0 ,acceleration is a = 0

And maximum at x = $\pm$A , a = $\mp\omega^2$A

Note velocity is out of phase of displacement by π

Solved Numerical

Q) A particle moving with S.H.M in straight line has a speed of 6m/s when 4m from the centre of oscillations and a speed of 8m/s when 3m from the centre. Find the amplitude of oscillation and the shortest time taken by the particle in moving from the extreme position to a point midway between the extreme position and the centre

Solution

From the formula for velocity

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

$$6 = \pm\,\omega\left(\sqrt{A^2 - 4^2}\right)$$

And

$$8 = \pm\,\omega\left(\sqrt{A^2 - 3^2}\right)$$

By taking ratio of above equations

$$\frac{6}{8} = \frac{\sqrt{A^2 - 4^2}}{\sqrt{A^2 - 3^2}}$$

On simplifying we get A = 5 m

On substituting value of A in equation

$$6 = \pm\,\omega\left(\sqrt{A^2 - 4^2}\right)$$

$$6 = \pm\,\omega\left(\sqrt{5^2 - 4^2}\right)$$

ω = 2 rad

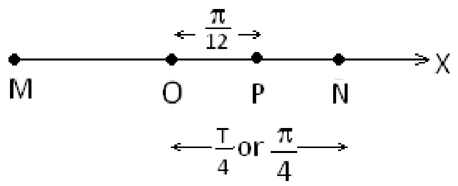
From the equation for oscillation

x(t) = Asin(ωt +ϕ)

$$\frac{A}{2} = A\sin 2t$$

0.5 = sin2t

⇒ 2t = π/6



t= π/12 this is the time taken by oscillator to move from centre to midway

Now Time taken by oscillator to move from centre to extreme position is T/4

As T = 2π /ω thus

Time taken to reach to extreme position from the centre =

$$\frac{T}{4} = \frac{2\pi}{4\omega} = \frac{2\pi}{4 \times 2} = \frac{\pi}{4}$$

Now time taken to reach oscillator to reach middle from extreme point is =

$$\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}\ sec$$

Q) A point moving in a straight line SHM has velocities v_1 and v_2 when its displacements from the mean position are x_1 and x_2 respectively. Show that the time period is

$$2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

Solution

From the formula for velocity

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

$$v_1^2 = \omega^2(A^2 - x_1^2)$$

$$A^2 = \frac{v_1^2}{\omega^2} + x_1^2 \quad ---eq(1)$$

$$A^2 = \frac{v_2^2}{\omega^2} + x_2^2 \quad ---eq(2)$$

From equation (1) and (2)

$$\frac{v_1^2}{\omega^2} + x_1^2 = \frac{v_2^2}{\omega^2} + x_2^2$$

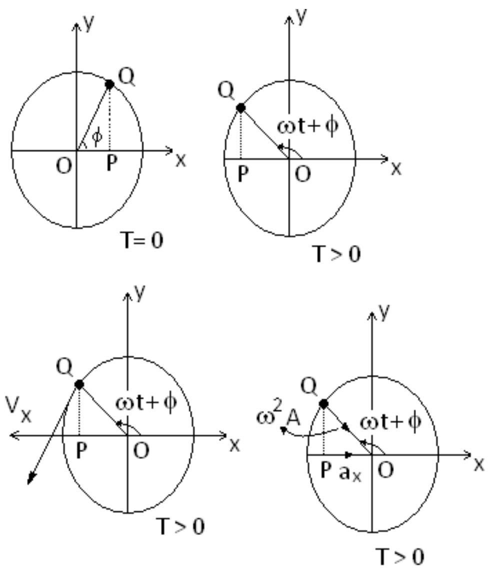
$$\frac{v_2^2}{\omega^2} - \frac{v_1^2}{\omega^2} = x_1^2 - x_2^2$$

$$\frac{1}{\omega^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$\frac{T^2}{(2\pi)^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

Relation between simple harmonic motion and uniform circular motion



In figure shown Q is a point moving on a circle of radius A with constant angular speed ω (in rad/sec). P is the perpendicular projection of Q on the horizontal diameter, along the x-axis.
Let us take Q as the reference point and the circle on which it moves the reference circle. As the reference point revolves, the projected point P moves back and forth along the horizontal diameter

Let the angle between the radius OQ and the x-axis at the time t=0 be called ϕ. At any time t later the angle between OQ and the x-axis is (ωt+ ϕ), the point Q moving with constant angular speed ω. The x-coordinate of Q at any time is, therefore

$X = A\cos(\omega t + \phi)$

i.e. P moves with simple harmonic motion

Thus, when a particle moves with uniform circular motion, its projection on a diameter moves with simple harmonic motion. The angular frequency ω of simple harmonic motion is the same as the angular speed of the reference point.

The velocity of Q is v = ωA. The component of v along the x-axis is

$V_x = -v\sin(\omega t + \phi)$

$V_x = -\omega A\sin(\omega t + \phi)$,

Which is also the velocity of p. The acceleration of Q is centripetal and has a magnitude,

$a = \omega^2 A$

The component of 'a' along the x-axis is

$A_x = -a\cos(\omega t + \phi)$,

$A_x = -\omega^2 A\cos(\omega t + \phi)$,

Which is the acceleration of P

The force law for simple harmonic motion

We know that

$F = ma$

As $a = -\omega^2 x(t)$

$F = -m\omega^2 x(t)$

This force is restoring force

According to Hook's law, the restoring force is given by

$F = -kx(t)$

With k as spring constant

Thus $k = m\,\omega^2$

∴ angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

And frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

In many cases, the simple harmonic motion can also occur even without spring. In that case k is called the force constant of SHM and it is restoring force per unit displacement (K = -F/x)

Solved numerical

Q) A spring balance has a scale that reads 50kg. The length of the scale is 20cm. A body suspended from this spring, when displaced and released, oscillates with period of 0.6s. Find the weight of the body

Solution:

Here m = 50kg

Maximum extension of spring x = 20- 0 = 20 cm = 0.2 m

Periodic time T = 0.6s

Maximum force F = mg

F = 50× 9.8 = 490 N

K = F/x = 490/0.2 = 2450 N m^{-1}

As

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$m = \frac{(0.6)^2 \times 2450}{4 \times (3.14)^2} = 22.36\ kg$$

∴ Weight of the body = mg = 22.36 ×9.8 = 219.1 N = 22.36 kgf

Examples of simple harmonic motion

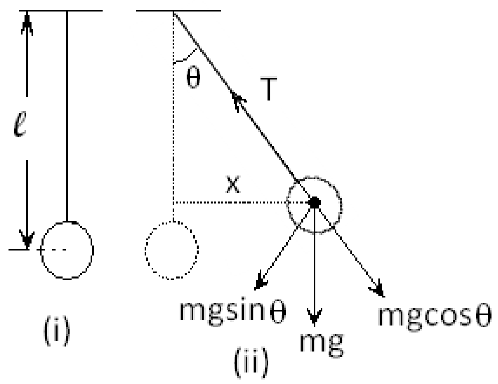
Simple pendulum

A simple pendulum consists of a heavy particle suspended from a fixed support through a light, inextensible and torsion less string

The time period of simple pendulum can be found by force or torque method and also by energy method

(a)Force method: the mean position or the equilibrium position of the simple pendulum is when θ = 0 as shown in figure(i). the length of the string is l, and mass of the bob is m

When the bob is displaced through distance 'x', the forces acting on it are shown in figure(ii)



The restoring force acting on the bob to bring it to the mean position is

F = - mgsinθ (-ve sign indicates that force is directed towards the mean position)

For small angular displacement

Sinθ ≈θ = x/l

$$\therefore F = -mg\frac{x}{l}$$

$$\therefore a = -g\frac{x}{l}$$

Comparing it with equation of simple harmonic motion a = $-\omega^2 x$

$$\omega^2 = \frac{g}{l}$$

$$\therefore \omega = \sqrt{\frac{g}{l}}$$

Time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(b)Torque method:

Now taking moment of force acting on the bob about O

$$\tau = -(mgsin\theta)l \ \ --- eq(1)$$

Also from Newton's second law

$$\tau = I\alpha \ \ --- eq(2)$$

From equation(1) and (2)

$$I\alpha = -(mgsin\theta)l$$

Since θ is small

$$I\alpha = -(mg\theta)l$$

But Moment of inertia I = ml^2

$$ml^2\alpha = -(mg\theta)l$$
$$\therefore \alpha = -\left(\frac{g}{l}\right)\theta$$

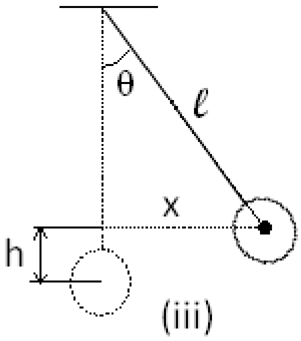
Comparing with simple harmonic motion equation ,$\alpha = -\omega^2\theta$

$$\omega = \sqrt{\frac{g}{l}}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Note: component of force mgcosθ cannot produce torque because it passes through fixed point

(c) Energy method:

Let the potential energy at the mean position be zero. Let the bob is displaced through an angle 'θ'. Let its velocity be 'v'



Then potential energy at the new position

U = mgl(1-cosθ)

Kinetic energy at this instant K = (1/2) mv^2

Total mechanical energy at this instant

E = U + K

E = mgl(1-cosθ) + (1/2) mv^2

We know, in simple harmonic motion E = constant

$$\frac{dE}{dt} = 0$$

$$\Rightarrow mgl\left[sin\theta\frac{d\theta}{dt}\right] + mv\frac{dv}{dt} = 0$$

Bur v = ωl

$$\therefore v = l\frac{d\theta}{dt}$$
$$\therefore mgl\left[sin\theta\frac{d\theta}{dt}\right] + ml\frac{d\theta}{dt}\frac{dv}{dt} = 0$$
$$\therefore g[sin\theta] + \frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = a = -g[sin\theta] \approx g\theta$$

But θ = x/l

$$\therefore a = -g\frac{x}{l}$$

But a=-ω^2x

$$\therefore \omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Periodic time

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(a)Simple pendulum in a lift:

If the simple pendulum is oscillating in a lift moving with acceleration a, then the effective g of the pendulum is

g_{eff} = g$\pm$a

+ sign is taken when lift is moving upward

- ve sign is taken when lift is moving downward

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{g \pm a}}$$

(b)Simple pendulum in the compartment of a train

If the simple pendulum is oscillating in a compartment of a train accelerating or retarding horizontally at the rate 'a' then the effective value of g is

$$g_{eff} = \sqrt{g^2 + a^2}$$

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}}$$

(c)Seconds pendulum

The pendulum having the time-period of two seconds, is called the second pendulum. It takes one second to go from one end to the other end during oscillation. It also crosses the mean position at every one second

Solved Numerical

Q) Length of a second's pendulum on the surface of earth is l_1 and l_2 at height 'h' from the surface of earth. Prove that the radius of the earth is given by

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Solution:
Period of second's pendulum is 2sec

$$2 = 2\pi\sqrt{\frac{l_1}{g}}$$

And g at surface of earth is given by

$$g = \frac{GM_e}{R_e^2}$$

$$\therefore 2 = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}} \quad ---eq(1)$$

g at height h from surface of earth is

$$g = \frac{GM_e}{(R_e + h)^2}$$

Thus at height h

$$2 = 2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} \quad ---eq(2)$$

From equation(1) and (2) we get

$$2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}}$$

$$l_2(R_e + h)^2 = l_1 R_e^2$$

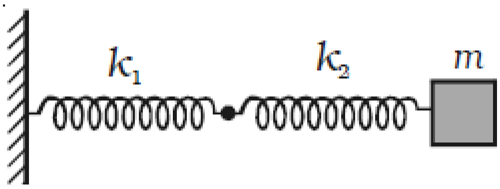
$$\sqrt{l_2}\,(R_e + h) = \sqrt{l_1}\,R_e$$

$$\sqrt{l_2}\,h = R_e\left(\sqrt{l_1} - \sqrt{l_2}\right)$$

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Combinations of springs

Series combination



As show in figure consider a series combination of two massless spring of spring constant k_1 and k_2

In this system when the combination of two springs is displaced to a distance y, it produces extension x_1 and x_2 in two springs of force constants k_1 and k_2.
$F = -k_1 x_1$; $F = -k_2 x_2$

where F is the restoring force.

$$x = x_1 + x_2 = -F\left[\frac{1}{k_1} + \frac{1}{k_2}\right]$$

We know that $F = -kx$
$x = -F/k$
From the above equations,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

Time period T

$$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

Frequency f

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

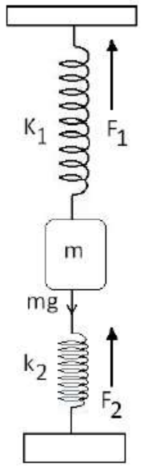
If both the springs have the same spring constant,
$k_1 = k_2 = k$. then equivalent spring constant k' = k/2

$$T = 2\pi\sqrt{\frac{m(2)}{k}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$$

Parallel combination

(i) Consider a situation as shown in figure where a body of mass m is attached in between the two massless springs of spring constant k_1 and k_2. let the body is left free for SHM in vertical plane after pulling mass m In this situation when a body is pulled lower through small displacement y. lower spring gets compressed by y, while upper spring elongate by y. hence restoring forces F_1 and F_2 set up in both these springs will act in the same direction.



Net restoring force will be

$F = F_1 + F_2$

$F = - k_1y \; -k_2y$

$F= - (k_1 + k_2)\, y$

If k' is the is equivalent spring constant then

$F = -k'y$ thus

$k' = k_1 + k_2$

Now periodic time T

$$T = 2\pi\sqrt{\frac{m}{k'}} = \sqrt{\frac{m}{k_1 + k_2}}$$

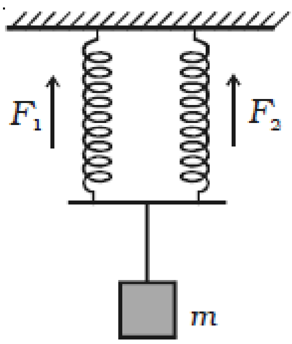
Frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}$$

If $k_1 = k_2 = k$ then

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

(ii) Two massless springs of equal lengths having force constants k_1 and k_2 respectively are suspended vertically from a rigid support as shown in figure. At their free ends, a block of mass m having non-uniform density distribution is suspended so that spring undergoes equal extension



In this situation two bodies are pulled down through a small distance y and the system is made to perform SHM in vertical plane

Here, the springs have different force constants. Moreover the increase in their length is same. Therefore, the load is distributed equally between the springs. Hence, the restoring force developed in each spring is different.

If F_1 and F_2 are the restoring forces set up due to extension of springs, then

$F_1 = -k_1\, y$ and $F_2 = k_2y$

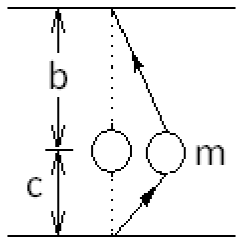
Also the total restoring force

$F = F_1 + F_2$

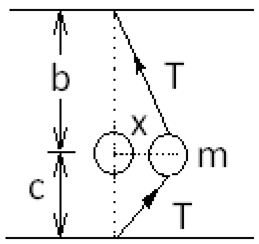
$F = F_1 + F_2$

$F = - k_1y - k_2y$
$F = - (k_1 + k_2) y$
If k' is the is equivalent spring constant then
$F = -k'y$ thus
$k' = k_1 + k_2$
Frequency and periodic time will be same as given in (i)

Q) A small mass m is fastened to a vertical wire, which is under tension T. What will be the natural frequency of vibration of the mass if it is displaced laterally a slight distance and then released?



Solution



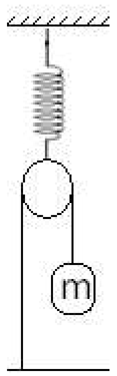
Sphere is displaced by a very small distance x thus angle formed is also small
Using trigonometry we can find Restoring force $F_b = T\sin\theta = T(x/b)$
Now restoring force per unit displacement $k_1 = F_b /x = T/b$
$F_c = T\sin\theta = T(x/c)$
Now restoring force per unit displacement $k_2 = F_c /x = T/c$
Since both restoring forces are parallel combination

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{T}{b}+\frac{T}{c}}{m}}$$

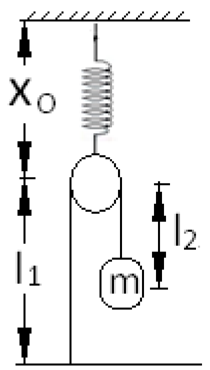
$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{1}{b}+\frac{1}{c}\right)}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{b+c}{bc}\right)}$$

Q) The spring has a force constant k. the pulley is light and smooth while the spring and the string are light. If the block of mass 'm' is slightly displaced vertically and released, find the period of vertical oscillation



Solution: When mass m is pulled by distance of x, increase in length of spring is x/2 as explained below



$x_0 + l_1$ = constant
dx_0 be the increase in length of spring
$dx_0 + dl_1 = 0$
and $dx_0 + dl_2 = x$
Thus $2dx_0 + dl_1 + dl_2 = x$
As string is inextensible $dl_1 + dl_2 = 0$
Thus $2dx_0 = x$ or $dx_0 = x/2$

Before pulling mass m spring was in equilibrium
$2T_0 = kx_0$
And $T_0 = mg$
Thus $2mg = kx_0$
When spring stretch by x/2 then tension in spring is T

$$F = k\left(x_0 + \frac{x}{2}\right)$$

$$\text{Or } 2T = k\left(x_0 + \frac{x}{2}\right)$$

$$2T = kx_0 + \frac{kx}{2}$$

$$2T = 2T_0 + \frac{kx}{2}$$

$$2T - 2T_0 = \frac{kx}{2}$$

$$T - T_0 = \frac{kx}{4}$$

Restoring force on mass m is $T-T_0$ which is proportional to displacement

Thus restoring force constant k' = k/4

Period of oscillation

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{\frac{k}{4}}} = 4\pi\sqrt{\frac{m}{k}}$$

Differential equation of simple harmonic motion

According to Newton's second law of motion

$$F = ma = m\frac{d^2y(t)}{dt^2}$$

Comparing this with F = -ky(t)

$$m\frac{d^2y(t)}{dt^2} = -ky(t)$$

$$\frac{d^2y(t)}{dt^2} = -\frac{k}{m}y(t)$$

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t)$$

$$\frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0$$

This is the second order differential equation of the simple harmonic motion. The solution of this equation is of the type

Y(t) = A sinωt or y(t)=Bcosωt

Or any linear combination of sine and cosine function

Y(t) = A sinωt + Bcosωt

Solved Numerical

Q) The SHM is represented by y = 3sin314t + 4cos314t. y in cm and in t in second. Find the amplitude, epoch the periodic time and the maximum velocity of SHO

Solution

Y = A sin(ωt + φ)

Y = A cosφsin ωt + Asinφcosωt

Comparing with 3sin314t + 4cos314t

3 = Acosφ and 4 = Asinφ

∴ $A^2 \cos^2\varphi + A^2\sin^2\varphi = 3^2 + 4^2$

$A^2 = 25$
A = 5 cm
The initial phase (epoch) is obtained as

$$tan\varphi = \frac{sin\varphi}{cos\varphi} = \frac{4}{3}$$

$$\varphi = tan^{-1}\left(\frac{4}{3}\right)$$

$$\varphi = 53^o 8'$$

Now

$$T = \frac{2\pi}{\omega\pi} = \frac{2}{314} = 0.02\ s$$

Maximum velocity
$V_{max} = \omega A = 314 \times 5 = 1570$ cm/s

Q) The vertical motion of a ship at sea is described by the equation $\frac{d^2x}{dt^2} = -4x$ where x in metre is the vertical height of the ship above its mean position. If it oscillates through a total distance of 1m in half oscillation, find the greatest vertical speed and the greatest vertical acceleration.
Solution
Comparing given with standard equation for oscillation we get
$\omega^2 = 4$ or $\omega = 2$
given amplitude A = 1/2
$V_{max} = \omega A = 2\times (1/2) = 1$m/s
$A_{max} = \omega^2 A = 2^2 \times (1/2) = 2$ m/s^2

Total Mechanical Energy in Simple Harmonic Oscillator

The total energy (E) of an oscillating particle is equal to the sum of its kinetic energy and potential energy if conservative force acts on it.

Kinetic energy

Kinetic energy of the particle of mass m is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) \quad ---eq(1)$$

Potential energy

From definition of SHM $F = -kx$ the work done by the force during the small displacement dx is $dW = -F.dx = -(-kx)\,dx = kx\,dx$
∴ Total work done for the displacement x is,

$$W = \int dw = \int_0^x kx\,dx$$

$k = \omega^2 m$

$$W = \int_0^x m\omega^2 x \, dx = \frac{1}{2} m\omega^2 x^2$$

This work done is stored in the body as potential energy

$$U = \frac{1}{2} m\omega^2 x^2 \quad --- (2)$$

Total energy *E* = *K* + *U*

$$E = \frac{1}{2} m\omega^2 (A^2 - x^2) + \frac{1}{2} m\omega^2 x^2$$

$$E = \frac{1}{2} m\omega^2 A^2$$

Special cases

(i) When the particle is at the mean position *x*= 0, from eqn (1) it is known that kinetic energy is maximum total energy is wholly kinetic $K_{max} = \frac{1}{2} m\omega^2 A^2$
and from eqn. (2) it is known that potential energy is zero.

(ii) When the particle is at the extreme position y = *+a*, from eqn. (1) it is known that kinetic energy is zero and from eqn. (2) it is known that Potential energy is maximum. Hence the total energy is wholly potential. $U_{max} = \frac{1}{2} m\omega^2 A^2$

(iii)When y = A/2

$$K = \frac{1}{2} m\omega^2 \left[A^2 - \left(\frac{A}{2} \right)^2 \right]$$

$$K = \frac{3}{4} \left(\frac{1}{2} m\omega^2 A^2 \right)$$

$$K = \frac{3}{4} (E)$$

$$U = \frac{1}{2} m\omega^2 \left(\frac{A}{2} \right)^2$$

$$U = \frac{1}{4} \left(\frac{1}{2} m\omega^2 A^2 \right)$$

If the displacement is half of the amplitude K and U are in the ratio 3 : 1,

Solved numerical

Q) If a particle of mass 0.2kg executes SHM of amplitude 2cm and period of 6 sec find(i0the total mechanical energy at any instant (ii) kinetic energy and potential energies when the displacement is 1cm

Solution:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6}$$

(i)Total mechanical energy at any instant is given by

$$E = \frac{1}{2}m\omega^2A^2$$

$$E = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (2 \times 10^{-2})^2$$

$$E = 0.1 \times \frac{4\pi^2}{36} \times 4 \times 10^{-4}$$

$$E = 4.39 \times 10^{-5}J$$

(ii)K.E. at the instant when displacement x is given by

$$K = \frac{1}{2}m\omega^2(A^2 - x^2)$$

$$K = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (4 \times 10^{-4} - 1 \times 10^{-4})\ J$$

$$K = 3.29 \times 10^{-5}J$$

P.E. energy at that instant = Total energy – K.E

= (4.39 -3.29) $\times 10^{-5}$

=1.1×10^{-5}J

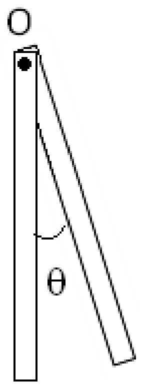
Angular Simple Harmonic Motion

A body to rotate about a given axis can make angular oscillations. For example, a wooden stick nailed to a wall can oscillate about its mean position in the vertical plane

The conditions for an angular oscillation to be angular harmonic motion are

(i)When a body is displaced through an angle from the mean position, the resultant torque is proportional to the angle displaced

(ii)This torque is restoring in nature and it tries to bring the body towards the mean position



If the angular displacement of the body at an instant is θ , then resultant torque on the body

τ = -kθ

if the momentum of inertia is I, the angular acceleration is

$$\alpha = \frac{\tau}{I} = -\frac{k}{I}\theta$$

Or

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta \quad ---eq(1)$$

Here

$$\omega = \sqrt{\frac{k}{I}}$$

Solution of equation (1) is

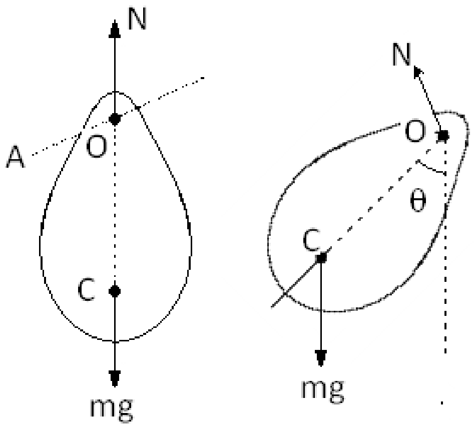
$$\theta = \theta_0 \sin(\omega t + \varphi)$$

Where θ_0 is the maximum angular displacement on either side. Angular velocity at time 't' is given by

$$\omega = \frac{d\theta}{dt} = \theta_0 \omega cos(\omega t + \varphi)$$

Physical pendulum

An rigid body suspended from a fixed support constitutes a physical pendulum.



As shown in figure is a physical pendulum. A rigid body is suspended through a hole at O. When the centre of mass C is vertically below O at a distance of 'l', the body may remain at rest.

The body is rotated through an angle θ about a horizontal axis OA passing through O and perpendicular to the plane of motion The torque of the forces acting on the body, about the axis OA is $\tau = mgl\sin\theta$, here $l =$ OC

If momentum of inertia of the body about OA is I, the angular acceleration becomes

$$\alpha = \frac{\tau}{I} = -\frac{mgl}{I} sin\theta$$

For small angular displacement $\sin\theta = \theta$

$$\alpha = -\left(\frac{mgl}{I}\right)\theta$$

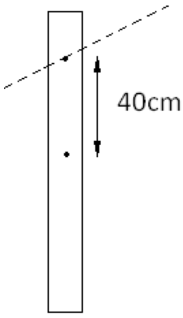
Comparing with $\alpha = -\omega^2\theta$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$$

Solved Numerical

Q) A uniform meter stick is suspended through a small-hole at the 10cm mark. Find the time period of small oscillations about the point of suspension

Solution:



Let the mass of stick be M. The moment of inertia of the stick about the axis of rotation through the point of suspension is

$$I = \frac{Ml^2}{12} + Md^2 = M\left(\frac{l^2}{12} + d^2\right)$$

Centre of mass of the stick is at 50 cm from top thus it is at distance 40 cm from the small hole thus d = 40 cm, given length of stick = 1m

Time period

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{M\left(\frac{l^2}{12} + d^2\right)}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{\left(\frac{l^2}{12} + d^2\right)}{gd}}$$

$$T = 2 \times 3.14\sqrt{\frac{\frac{1}{12} + (0.4)^2}{9.8 \times 0.4}}$$

T=1.56 sec

Torsional pendulum

In torsional pendulum, an extended body is suspended by a light thread or wire. The body is rotated through an angle about the wire as the axis of rotation.

The wire remains vertical during this motion but a twist ‘θ’ is produced in the wire. The twisted wire exerts a restoring torque on the body, which is proportional to the angle of the twist.

$\tau \propto -\theta$; $\tau = -k\theta$; k is proportionality constant and is called torsional constant of the wire. If l be the moment of inertia of the body about vertical axis, the angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-k}{I}\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{k}{I}}$$

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}}$

Q) The moment of inertia of the disc used in a torsional pendulum about the suspension wire is 0.2 kg-m^2. It oscillates with a period of 2s. Another disc is placed over the first one and the time period of the system becomes 2.5 s. Find the moment of inertia of the second disc about the wire

Solution

Let the torsional constant of the wire be k

$$T = 2\pi\sqrt{\frac{I}{k}}$$

$$2 = 2\pi\sqrt{\frac{0.2}{k}} \quad ---eq(1)$$

When a second disc having moment of inertia I_1 about the wire is added, the time period is

$$2.5 = 2\pi\sqrt{\frac{0.2+I_1}{k}} \quad ---eq(2)$$

From eq(1) and (2)

I_1 = 0.11 kg-m^2

Two Body System

In a two body oscillations, such as shown in the figure, a spring connects two objects, each of which is free to move. When the objects are displaced and released, they both oscillate. The relative separation x_1- x_2 gives the length of the spring at any time. Suppose its unscratched length is L; then x = (x_1 – x_2) – L is the change in length of the spring, and F = kx is the magnitude of the force exerted on each particle by the spring as shown in figure.

Applying Newton's second law separately to the two particles, taking force component along the x-axis, we get

$$m_1\frac{d^2x_1}{dt^2} = -kx \quad and \quad m_2\frac{d^2x_2}{dt^2} = +kx$$

Multiplying the first of these equations by m_2 and the second by m_1 and then subtracting,

$$m_1m_2\frac{d^2x_1}{dt^2} - m_1m_2\frac{d^2x_2}{dt^2} = -m_2kx - m_1kx$$

This can be written as

$$\frac{m_1m_2}{(m_1+m_2)}\frac{d^2}{dt^2}(x_1-x_2) = -kx$$

$$\mu\frac{d^2}{dt^2}(x_1-x_2) = -kx \quad ---eq(1)$$

Here μ is known as reduced mass and has dimension of mass

$$\mu = \frac{m_1m_2}{(m_1+m_2)}$$

Since L is constant

$$\frac{d^2}{dt^2}(x_1 - x_2) = \frac{d^2}{dt^2}(x + L) = \frac{dx}{dt}$$

Equation (i) becomes

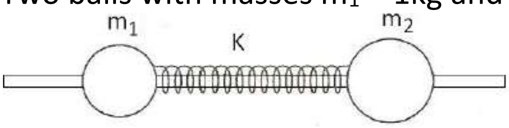
$$\mu\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x$$

Thus periodic time

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

Q) Two balls with masses m_1 = 1kg and m_2 = 2kgkg are slipped on a thin smooth horizontal rod. The balls are interconnected by a light spring of spring constant 24 N/m. the left hand ball is imparted the initial velocity v_1= 12 cm/s.



Find (a) the oscillation frequency of the system (b) the energy and amplitude of oscillation

Solution

Reduced mass

$$\mu = \frac{m_1 m_2}{(m_1 + m_2)} = \frac{1 \times 2}{1 + 2} = \frac{2}{3}\ kg$$

Frequency

f = 1/T

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

$$f = \frac{1}{2 \times 3.14}\sqrt{\frac{24 \times 3}{2}} = 0.955\ sec$$

Initial velocity given to mass m_1 is v_1

For undamped oscillation, this initial energy remains constant

Hence total energy of S.H.M. of two balls is given as

$$E = \frac{1}{2}\mu v_1^2$$

If amplitude of oscillation is A then

$$\frac{1}{2}\mu v_1^2 = \frac{1}{2}kA^2$$

$$A = \sqrt{\frac{\mu}{k}}v_1$$

$$A = \sqrt{\frac{2}{3\times 24}} \times 0.12 = 0.02\ m$$

Or A = 2 cm

Damped oscillation

Experimental studies showed that the resistive force acting on the oscillator in a fluid medium depends upon the velocity of the oscillator

Thus resistive force or damping force acting on the oscillator is

$F_d \propto v$

$\therefore F_d = -nv$

Here b is damping constant and has SI units kg/second. The negative sign indicates that the force F_d opposes the motion

Thus, a damped oscillator oscillate under the influence of the following forces

(i)Restoring force $F_x = -kx$ and

(ii) Resistive force $F_d = -bv$

Net force $F = F_x + F_d$

According to second law of motion

ma = -ky – bv

$$m\frac{d^2x}{dt^2} = -\mathrm{ky} - \mathrm{b}\frac{\mathrm{dx}}{\mathrm{dt}}$$

$$\frac{d^2x}{dt^2} + \frac{\mathrm{b}}{\mathrm{m}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = 0 \quad --- \mathrm{eq}(1)$$

Solution of this equation is

$$x(t) = A\ e^{-bt/2m}\ sin(\omega' t + \varphi)$$

Here $A\ e^{-bt/2m}$ is the amplitude of the damped oscillation and decreases exponentially with time

The angular frequency ω' of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Energy of oscillator

$$E = \frac{1}{2}kA^2e^{-bt/m}$$

Above equation is valid only if b << √(km)

Natural Oscillations

When a system capable of oscillating is given some initial displacement from its equilibrium position and left free (i.e. in absence of any external force) it begins to oscillate. Thus the oscillations performed by it in absence of any resistive forces are known as natural oscillations. The frequency of natural oscillations is known as natural frequency f_0 and corresponding angular frequency is denoted by ω^0

Forced Oscillation

Oscillations of the system under the influence of an external periodic force are forced oscillation

Consider an external periodic force F = F_0 sinωt acting on the system which is capable to oscillate

Equation for oscillation can be witten as

$$m\frac{d^2x}{dt^2} + \mathrm{b}\frac{\mathrm{dx}}{\mathrm{dt}} + \mathrm{ky} = \mathrm{F_0 sin\omega t}$$

$$\frac{d^2x}{dt^2} + \frac{\mathrm{b}}{\mathrm{m}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = \frac{\mathrm{F_0}}{\mathrm{m}}\mathrm{sin\omega t}$$

The solution of equation is given by

X = A sin(ωt + φ)

Here, A and φ are the constants of the solution they are found as,

$$A = \frac{F_0}{[m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2]^{1/2}}$$

$$\varphi = tan^{-1}\frac{\omega x}{v_0}$$

Here m is the mass of oscillator, v_0 is velocity of oscillator, x is the displacement of oscillator.

(i) for small damping factor

$$m(\omega_0^2 - \omega^2) \gg b\omega$$

Equation for amplitude becomes

$$A = \frac{F_0}{m^2(\omega_0^2 - \omega^2)^2}$$

(ii) For large damping factor

$$b\omega \gg m(\omega_0^2 - \omega^2)$$

$$A = \frac{F_0}{b\omega}$$

If when value of ω approaches $\omega_0$the amplitude becomes maximum. This phenomenon is known as resonance. The value of ω for which resonance occurs is known as the resonant frequency.

Solved numerical

Calculate the time during which the amplitude becomes $A/2^n$ in case of damped oscillations, where A = initial amplitude

Solution:

$$A(t) = Ae^{-bt/2m}$$

But $A(t)= A/2^n$

$$\therefore \frac{A}{2^n} = Ae^{-bt/2m}$$

Taking log to the base e on both sides

$$\frac{bt}{2m} = nln2$$

$$t = \frac{2mnln2}{bt} = \frac{2mn}{bt}(0.693)$$

----------------END--------------

SOUND WAVES

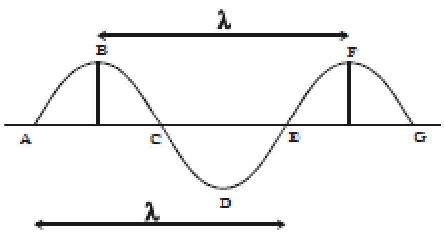
Wave motion is a mode of transmission of energy through a medium in the form of a disturbance. It is due to the repeated periodic motion of the particles of the medium about an equilibrium position transferring the energy from one particle to another.
The waves are of three types - mechanical, electromagnetic and matter waves. Mechanical waves can be produced only in media which possess elasticity and inertia. Water waves, sound waves and seismic waves are common examples of this type. Electromagnetic waves do not require any material medium for propagation. Radio waves, microwaves, infrared rays, visible light, the ultraviolet rays, X rays and γ rays are electromagnetic waves. The waves associated with particles like electrons, protons and fundamental particles in motion are matter waves.

Characteristics of wave motion

(i) Wave motion is a form of disturbance travelling in the medium due to the periodic motion of the particles about their mean position
(ii) It is necessary that the medium should possess elasticity and inertia.
(iii) All the particles of the medium do not receive the disturbance at the same instant (i.e) each particle begins to vibrate a little later than its predecessor.
(iv) The wave velocity is different from the particle velocity. The velocity of a wave is constant for a given medium, whereas the velocity of the particles goes on changing and it becomes maximum in their mean position and zero in their extreme positions.
(v) During the propagation of wave motion, there is transfer of energy from one particle to another without any actual transfer of the particles of the medium.
(vi) The waves undergo reflection, refraction, diffraction and interference

Mechanical wave motion

The two types of mechanical wave motion are (i) transverse wave motion and (ii) longitudinal wave motion



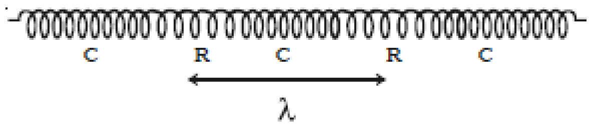
(i) Transverse wave motion

Transverse wave motion is that wave motion in which particles of the medium execute SHM about their mean positions in a direction perpendicular to the direction of propagation of the wave. Such waves are called transverse waves. Examples of transverse waves are waves produced by plucked strings of veena, sitar or violin and electromagnetic waves. Transverse waves travel in the form of crests and troughs. The maximum displacement of the particle in the positive direction i.e. above its mean position is called

crest and maximum displacement of the particle in the negative direction i.e below its mean position is called trough.
Thus if ABCDEFG is a transverse wave, the points B and F are crests while D is trough For the propagation of transverse waves, the medium must possess force of cohesion and volume elasticity.
Since gases and liquids do not have rigidity (cohesion), transverse waves cannot be produced in gases and liquids. *Transverse waves can be produced in solids and surfaces of liquids only*

(ii) Longitudinal wave motion

'Longitudinal wave motion is that wave motion in which each particle of the medium executes simple harmonic motion about its mean position along the direction of propagation of the wave.'
Sound waves in fluids (liquids and gases) are examples of longitudinal wave. When a longitudinal wave travels through a medium, it produces compressions and rarefactions.
In the case of a spiral spring, whose one end is tied to a hook of a wall and the other end is moved forward and backward, the coils of the spring vibrate about their original position along the length of the spring and longitudinal waves propagate through the spring



Compression and rarefaction in spring

The regions where the coils are closer are said to be in the state of compression, while the regions where the coils are farther are said to be in the state of rarefaction.
When sound waves passes through that region of air, the air molecules in certain region are pushed very close to each other during their oscillations. Hence, both density and pressure of air increase in such regions. In such region condensation is said to be formed.
In the regions between two consecutive condensations, the air molecules are found to be quit separated. In such region density and pressure of air decreases and rarefaction is said to be formed.
Compressive strain is produced during the propagation of waves, which is possible in solid, liquids and gases medium.

Important terms used in wave motion

(i) Wavelength (λ)

The distance travelled by a wave during which a particle of the medium completes one vibration is called wavelength. It is also defined as the distance between any two nearest particles on the wave having same phase.
Wavelength may also be defined as the distance between two successive crests or troughs in transverse waves, or the distance between two successive compressions or rarefactions in longitudinal waves.

(ii) Time period (T)
The time period of a wave is the time taken by the wave to travel a distance equal to its wavelength.
(iii) Frequency (n)
This is defined as the number of waves produced in one second. If T represents the time required by a particle to complete one vibration, then it makes 1/T*waves in one second.*
Therefore frequency is the reciprocal of the time period
(i.e) *F* = 1/T

Relationship between velocity, frequency and wavelength of a wave
The distance travelled by a wave in a medium in one second is called the velocity of propagation of the wave in that medium. If v represents the velocity of propagation of the wave, it is given by

$$Velocity\ = \frac{Distance}{time}$$

$$v = \frac{\lambda}{T} = \lambda f$$

The velocity of a wave (*v*) is given by the product of the frequency and wavelength.

Velocity of wave in different media
The velocity of mechanical wave depends on elasticity and inertia of the medium
Velocity of a transverse wave along a stretched string
If *m* is the mass per unit length of the string
If T is the tension in string. Then velocity of wave

$$v = \sqrt{\frac{T}{m}}$$

The speed of a wave along a stretched ideal string depends only on the tension and the linear mass density of the string and does not depend on the frequency of the wave

Velocity of longitudinal waves in an elastic medium
Velocity of longitudinal waves in an elastic medium is

$$v = \sqrt{\frac{E}{\rho}}$$

where E is the modulus of elasticity, ρ is the density of the medium.

(i) In the case of a solid rod

$$v = \sqrt{\frac{Y}{\rho}}$$

where *Y* is the Young's modulus of the material of the rod and ρ is the density of the rod.
(ii) In liquids,

$$v = \sqrt{\frac{B}{\rho}}$$

where *B* is the Bulk modulus and ρ is the density of the liquid

Newton's formula for the velocity of sound waves in air

Newton assumed that sound waves travel through air under isothermal conditions (i.e) temperature of the medium remains constant.
The change in pressure and volume obeys Boyle's law.
PV = constant
Differentiating we get
P dV + V dP = constant

$$\therefore P - V\ \frac{dP}{dV} = \frac{dP}{dV/_V} = Bulk\ Modulus\ B$$

Thus, isothermal bulk modulus B = Pressure P

$$\therefore v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{P}{\rho}}$$

Since solids and liquids are much less compressible than gases, speed of sound in gasses is higher.

Laplace's correction

The experimental value for the velocity of sound in air is 332 m s^{-1}. But the theoretical value of 280 m $^{s-1}$ is 15% less than the experimental value.
The above discrepancy between the observed and calculated values was explained by Laplace in 1816.
Sound travels in air as a longitudinal wave. The wave motion is therefore, accompanied by compressions and rarefactions. At compressions the temperature of air rises and at rarefactions, due to expansion, the temperature decreases. Air is a very poor conductor of heat. Hence at a compression, air cannot lose heat due to radiation and conduction. At a rarefaction it cannot gain heat, during the small interval of time. As a result, the temperature throughout the medium does not remain constant. Laplace suggested that sound waves travel in air under adiabatic condition and not under isothermal condition.
For an adiabatic change, the relation between pressure and volume is given by

$$PV^{\gamma} = constant$$

Differentiating the equation with respect to V

$$P\,\gamma V^{\gamma-1}+V^{\gamma}\frac{dP}{dV}=0$$
$$P\,\gamma+V\frac{dP}{dV}=0$$
$$\frac{-dP}{dV/V}=\gamma P$$
$$\therefore B=\gamma P$$

Thus, for an adiabatic process bulk modulus = γP
Using this value of B we get wave speed

$$v=\sqrt{\frac{B}{\rho}}=\sqrt{\frac{\gamma P}{\rho}}$$

Factors affecting velocity of sound in gases
(i) Effect of pressure
If pressure of the gas is changed keeping its temperature constant , P/ρ remains constant as the density of gas directly varies as the pressure. Therefore, the speed of sound in a gas does not depend on the pressure of the gas, at constant temperature and constant humidity
Density of water vapour is less than the density of dry air at same temperature. Hence the speed of sound increases with increase in humidity as equation

$$v=\sqrt{\frac{\gamma P}{\rho}}$$

(ii) Effect of temperature

For a gas, *PV = RT for one mole of gas*

$$P=\frac{RT}{V}$$

Substituting value of P in equation

$$v=\sqrt{\frac{\gamma P}{\rho}}$$

We get

$$v=\sqrt{\frac{\gamma}{\rho}\frac{RT}{V}}=\sqrt{\frac{\gamma RT}{m}}$$

Mass of gas is m = ρV
Speed of sound in gas is directly proportional to the square root of its absolute temperature (T)

$$v \propto \sqrt{T}$$

velocity of sound in air increases by 0.61 m s^{-1} per degree centigrade rise in temperature

(iii)Effect of wind

The velocity of sound in air is affected by wind. If the wind blows with the velocity *w* along the direction of sound, then the velocity of sound increases to *v + w*. If the wind blows in the opposite direction to the direction of sound, then the velocity of sound decreases to *v – w*. If the wind blows at an angle θ with the direction of sound, the effective velocity of sound will be (*v + w* cos θ).

Note: In a medium, sound waves of different frequencies or wavelengths travel with the same velocity. Hence there is no effect of frequency on the velocity of sound.

Solved Numerical

Q) The wavelength of a note emitted by a tuning fork of frequency 512Hz in air at 17^{O}C is 66.5 cm. If the density of air at S.T.P. is 1.293 g/lit, calculate γ of air

Solution:

Frequency of tuning fork = 512 Hz, T = 17+28-73 = 290K, λ = 0.665 m

Density of air = 1.293 g/litre = 1.293 kg/m^3 pressure P = 1 atm = 1.01×10^5 N/m^2

Velocity of sound v = fλ = 512 × 0.665 = 340.5 m/s

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

$$\gamma = \frac{v^2\rho}{P}$$

$$\gamma = \frac{(340.5)^2 \times 1.293}{1.01 \times 10^5} = 1.48$$

Q) The speed of transverse wave going on a wire having length 50cm and mass 5.0g is 80 m/s. The area of cross-section of the wire is 1.0 mm^2 and its Young's modulus is 16×10^{11}N/m^2. Find the extension of the wire over its natural length

Solution:

Length of wire L = 50 cm = 50×10^{-2} m

Mass of wire M = 5g = 5×10^{-3}kg

Cross sectional area of wire A = 1mm^2 = 10^{-6} m^2

Young's modulus of wire Y = 16×10^{11}N/m^2

Mass per unit length of wire m = M/L = 5×10^{-3}kg/50×10^{-2} m =10^{-2} kg/m

The wave speed in wire

$$v = \sqrt{\frac{T}{m}}$$

$$\therefore T = mv^2$$

$$\therefore T = 10^{-2} \times (80)^2 = 64N$$

Now Young's modulus

$$Y = \frac{F/_A}{\Delta L/_L}$$

Extension of wire ΔL

$$\Delta L = \frac{FL}{AY}$$

$$\Delta L = \frac{(64)(50 \times 10^{-2})}{(10^{-6})(16 \times 10^{11})} = 0.02\ mm$$

Progressive wave

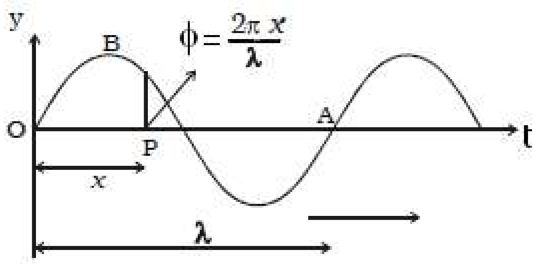
A progressive wave is defined as the onward transmission of the vibratory motion of a body in an elastic medium from one particle to the successive particle.

Equation of a plane progressive wave

An equation can be formed to represent generally the displacement of a vibrating particle in a medium through which a wave passes.

Thus each particle of a progressive wave executes simple harmonic motion of the same period and amplitude differing in phase from each other.

Let us assume that a progressive wave travels from the origin O along the positive direction of X axis, from left to right



The displacement of a particle at a given instant is y = A sinωt

where *a* is the amplitude of the vibration of the particle and ω = 2π*f*.

The displacement of the particle P at a distance *x* from O at a given instant is given by,

y = *a* sin (ωt - ϕ).

If the two particles are separated by a distance λ, they will differ by a phase of 2π. Therefore, the phase ϕ of the particle P at a distance x is

$$\varphi = \frac{2\pi x}{\lambda}$$

$$y = Asin\left(\omega t - \frac{2\pi x}{\lambda}\right)$$

$$But\ k = \frac{2\pi}{\lambda}\ wave\ vector$$

$$y = Asin(\omega t - kx)\ ---(1)$$

$$But\ \omega = \frac{2\pi}{T}$$
$$y = Asin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$
$$y = Asin2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \ ----(2)$$

If the wave travels in opposite direction, the equation becomes

$$y = Asin\ 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

(i)Variation of phase with time

The phase changes continuously with time at a constant distance. At a given distance *x* from O let ϕ_1 and ϕ_2 be the phase of a particle at time t_1 and t_2 respectively.

$$\varphi_1 = 2\pi\left(\frac{t_1}{T} - \frac{x}{\lambda}\right)$$

$$\varphi_2 = 2\pi\left(\frac{t_2}{T} - \frac{x}{\lambda}\right)$$

$$\varphi_2 - \varphi_1 = \frac{2\pi}{T}(t_2 - t_1)$$
$$\Delta\varphi = \frac{2\pi}{T}\Delta t$$

This is the phase change $\Delta\phi$ of a particle in time interval Δt. If $\Delta t = T$, $\Delta\phi = 2\pi$. This shows that after a time period T, the phase of a particle becomes the same

Thus $d\varphi/dt$ = constant

$$\therefore \frac{d}{dt}(\omega t - kx) = 0$$
$$\omega - k\frac{dx}{dt} = 0$$
$$\frac{dx}{dt} = \frac{\omega}{k} = v$$

Here v is the phase speed of wave. Which is same as speed of wave

(ii) Variation of phase with distance

At a given time t phase changes periodically with distance *x*. Let ϕ_1 and ϕ_2 be the phase of two particles at distance x_1 and x_2 respectively from the origin at a time t.

$$\varphi_1 = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

$$\varphi_2 = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$$

$$\varphi_2 - \varphi_1 = -\frac{2\pi}{T}(x_2 - x_1)$$
$$\Delta\varphi = -\frac{2\pi}{T}\Delta x$$

The negative sign indicates that the forward points lag in phase when the wave travels from left to right. When $\Delta x = \lambda$, $\Delta\phi = 2\pi$, the phase difference between two particles having a path difference λ is 2π

Characteristics of progressive wave

1. Each particle of the medium executes vibration about its mean position. The disturbance progresses onward from one particle to another.
2. The particles of the medium vibrate with same amplitude about their mean positions.
3. Each successive particle of the medium performs a motion similar to that of its predecessor along the propagation of the wave, but later in time.
4. The phase of every particle changes from 0 to 2π.
5. No particle remains permanently at rest. Twice during each vibration, the particles are momentarily at rest at extreme positions, different particles attain the position at different time.
6. Transverse progressive waves are characterised by crests and troughs. Longitudinal waves are characterised by compressions and rarefactions.
7. There is a transfer of energy across the medium in the direction of propagation of progressive wave.
8. All the particles have the same maximum velocity when they pass through the mean position.
9. The displacement, velocity and acceleration of the particle separated by mλ are the same, where m is an integer.

Intensity and sound level

The loudness of a sound depends on intensity of sound wave and sensitivity of the ear. The intensity is defined as the amount of energy crossing per unit area per unit time perpendicular to the direction of propagation of the wave.
Intensity is measured in W m^{-2}.
The intensity of sound depends on (i) Amplitude of the source ($I \propto A^2$),
(ii) Surface area of the source ($I \propto S$),
(iii) Density of the medium ($I \propto \rho$),
(iv) Frequency of the source ($I \propto f^2$) and
(v) Distance of the observer from the source ($I \propto 1/r^2$

Solved Numerical

Q) A simple harmonic wave has the equation y = 0.3sin(314t – 1.57x), where t, x and y are in seconds, meteres and cm respectively. Find the frequency and the wavelength of the wave. Another wave has the equation y' = 0.10 sin (314t – 1.57x + 1.57). Deduce the phase difference and the ratio of intensities of wave

Solution:

Since y is in cm thus given equation in meters unit is

$$y = \frac{0.3}{100} sin(314t - 1.57x)$$

Comparing with standard equation y= a sin(ωt –kx)

We get

$$\omega = 2\pi f = 314$$

Frequency:

$$f = \frac{314}{2 \times 3.14} = 50Hz$$

Wavelength

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k} = \frac{2 \times 3.14}{1.57} = 4m$$

Phase difference between two wave is (314t – 1.57x + 1.57) – (314t – 1.57x) = 1.57 radian

Or

$$\frac{1.57 \times 180}{\pi} = 90^o$$

The ratio of intensity

$$\frac{I_1}{I_2} = \frac{a_1^2}{a_2^2} = \frac{9}{1}$$

Q) Given the equation for a wave in a string y = 0.03sin(3x – 2t)

Where y and x are in metres and t is in seconds

(a) When t = 0, what is the displacement at x = 0.1 m?
(b) When x = 0.1m, what is the displacement at t=0 and t = 0.2s?
(c) What is the equation for the velocity of oscillation of particle of string and what is the maximum velocity
(d) What is the velocity of propagation of waves

Solution:

Given equation y = 0.03sin(3x – 2t)

(a) At t=0 , x = 01m , y = 0.03sin(0.3) = 8.86×10^{-3}m
(b) X = 0.1 m and t = 0 y = 8.86×10^{-3}m

   At x= 0.1 at t = 0.2s

   Y = 0.03sin(0.3-0.4) = -0.03sin(0.1) m = -2.997×10^{-3}m
(c) Particle velocity

$$V_P = \frac{dy}{dt} = -0.06\cos(3x - 2t)$$

$|V_P|_{max} = 6\times10^{-2}$ m/s

(d) Wave velocity = ω/k = 2/3 = 0.67 m/s

Superposition principle

When two waves travel in a medium simultaneously in such a way that each wave represents its separate motion, then the resultant displacement at any point at any time is equal to the vector sum of the individual displacements of the waves.

Let us consider two simple harmonic waves of same frequency travelling in the same direction. If a_1 and a_2 are the amplitudes of the waves and ϕ is the phase difference between them, then their instantaneous displacements are

$$y_1 = Asin\omega t$$
$$y_1 = Asin(\omega t + \varphi)$$

According to the principle of superposition, the resultant displacement is represented by

$y = y_1 + y_2$

$= A_1 \sin \omega t + A_2 \sin (\omega t + \phi)$

$= A_1 \sin \omega t + A_2 (\sin \omega t. \cos \phi + \cos \omega t.\sin \phi)$

$= (A_1 + A_2 \cos \phi) \sin \omega t + a2 \sin \phi \cos \omega t$ ...(3)

Put $A_1 + A_2 \cos \phi = A \cos \theta$ ...(4)

$A_2 \sin \phi = A \sin \theta$ ...(5)

Where A and θ are constants, then

$y = A \sin \omega t. \cos \theta + A \cos \omega t. \sin \theta$

or $y = A \sin (\omega t + \theta)$ ...(6)

This equation gives the resultant displacement with amplitude A.

From eqn. (4) and (5)

$A^2 cos^2 \theta + A^2 sin^2 \theta = (A_1 + A_2 cos \phi)^2 + (A_2 sin \phi)^2$

$\therefore A^2 = A_1^2 + A_2^2 + 2A_1A_2 cos \phi$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos\varphi}$$

And

$$tan\theta = \frac{A_2sin\varphi}{A_1 + A_2cos\varphi}$$

We know that intensity is directly proportional to the square of the amplitude

$I \propto A^2$

$\therefore I\ \alpha\ (A_1^2 + A_2^2 + 2A_1A2\ cos\ \phi)$

Special cases

The resultant amplitude A is maximum, when *cos* $\phi = 1$ *or* $\phi = 2m\pi$ where *m* is an integer (i.e) I_{max} α $(A_1 + A_2)^2$
The resultant amplitude A is minimum when
$\cos \phi = -1$ or $\phi = (2m + 1)\pi$
I_{min} α $(A_1 - A_2)^2$
The points at which interfering waves meet in the same phase
$\phi = 2m\pi$ i.e $0, 2\pi, 4\pi, ...$ are points of maximum intensity, where constructive interference takes place.

Reflection of wave

Reflection of wave from a rigid support

Suppose a wave propagates in the decreasing value of x, represented by equation
$y = A \sin(\omega t + kx)$ reaches a point x = 0
When the wave arrives at rigid support, support exerts equal and opposite force on the sting. This reaction force generates a wave at the support which travels back (along increasing values of x) along the string . This wave is known as reflected wave
At the support x= 0. Resultant displacement due to incident and reflected wave is zero
Thus if incident wave is
$y_i = A\sin(\omega t)$ since x = 0
then according to super position principle reflected wave equation at x= 0 is
$y_r = -A\sin(\omega t)$
or $y_r = A\sin(\omega t + \pi)$
Above equation shows that during reflection of wave phase increase by π
The reflected wave is travelling in positive x direction so the equation for reflected wave may be written as
$y_r = A \sin(\omega t + \pi - kx)$
$y_r = -A \sin(\omega t - kx)$

If incident wave equation is $y_i = A\sin(\omega t - kx)$ the equation for reflected wave is
$y_r = -A\sin(\omega t + kx)$
(b)Reflection of waves from a free end:
Suppose one end of the string is tied to a very light ring which can move or slide on the vertical rod without friction. Such end is called free end
Suppose crest reaches such free end then ring moves in upward direct as it is not fixed. As a result phase of reflected wave is same of incident wave. Or phase of reflected and incident wave is same.
If $y_i = A\sin(\omega t + kx)$ represent incident wave then
$Y_r = - A\sin(\omega t - kx)$ represent reflected wave

Beats

When two waves of nearly equal frequencies travelling in a medium along the same direction superimpose upon each other, beats are produced. The amptitude of the resultant sound at a point rises and falls regularly.
The intensity of the resultant sound at a point rises and falls regularly with time. When the intensity rises to maximum we call it as waxing of sound, when it falls to minimum we call it as waning of sound.
The phenomenon of waxing and waning of sound due to interference of two sound waves of nearly equal frequencies are called beats.
The number of beats produced per second is called beat frequency, which is equal to the difference in frequencies of two waves.

Analytical method
Let us consider two waves of slightly different frequencies f_1 and f_2 ($f_1 > f_2$) having equal amplitude travelling in a medium in the same direction. At time t = 0, both waves travel in same phase. The equations of the two waves are
$y_1 = a \sin 2\pi f_1 t$ and $y_1 = a \sin 2\pi f_2 t$
When the two waves superimpose, the resultant displacement is given by
$y = y_1 + y_2$
$y = a \sin (2\pi f_1) t + a \sin (2\pi f_2) t$

$$y = 2a sin 2\pi \left(\frac{f_1 + f_2}{2}\right) t cos 2\pi \left(\frac{f_1 - f_2}{2}\right) t$$

Substituting

$$A = 2a cos 2\pi \left(\frac{f_1 - f_2}{2}\right) \quad and\ f = \frac{f_1 + f_2}{2}$$

$$y = A \sin 2\pi f t$$

This represents a simple harmonic wave of frequency $f = \frac{f_1+f_2}{2}$
and amplitude A which changes with time.

(i) The resultant amplitude is maximum (i.e) ± 2a, if

$$cos 2\pi \left(\frac{f_1 - f_2}{2}\right) = \pm 1$$

$$2\pi \left(\frac{f_1 - f_2}{2}\right) = n\pi$$

(where n = 0, 1, 2 ...) or $(f_1 - f_2)\, t = n$
The first maximum is obtained at $t_1 = 0$
The second maximum is obtained at

$$t_2 = \frac{1}{f_1 - f_2}$$

The third maximum at

$$t_3 = \frac{2}{f_1 - f_2}$$

and so on.

The time interval between two successive maxima is

$t_2 - t_1 = t_3 - t_2 = 1 / (f_1 - f_2)$

Hence the number of beats produced per second is equal to the reciprocal of the time interval between two successive maxima.

(ii) The resultant amplitude is minimum (i.e) equal to zero, if

$$cos2\pi\left(\frac{f_1 - f_2}{2}\right) = 0$$

$$2\pi\left(\frac{f_1 - f_2}{2}\right) = \frac{\pi}{2} + n\pi$$

$$(f_1 - f_2)t = \frac{2n + 1}{2}$$

where n = 0, 1, 2 ...

The first minimum is obtained at n = 0

$$t'_1 = \frac{1}{2\left(f_1 - f_2\right)}$$

The second minimum is obtained at n = 1

$$t'_2 = \frac{2}{2\left(f_1 - f_2\right)}$$

The third minimum is obtained at

$$t'_3 = \frac{5}{2\left(f_1 - f_2\right)}$$

Time interval between two successive minima is

$$t'_2 - t'_1 = t'_3 - t'_2 = \frac{1}{f_1 - f_2}$$

Hence, the number of beats produced per second is equal to the reciprocal of time interval between two successive minima.

Uses of beats

(i) The phenomenon of beats is useful in tuning two vibrating bodies in unison. For example, a sonometer wire can be tuned in unison with a tuning fork by observing the beats. When an excited tuning fork is kept on the sonometer and if the sonometer wire is also excited, beats are heard, when the frequencies are nearly equal. If the length of the wire is adjusted carefully so that the number of beats gradually decreases to zero, then the two are said to be in unison. Most of the musical instruments are made to be in unison based on this method.

(ii) The frequency of a tuning fork can be found using beats. A standard tuning fork of frequency N is excited along with the experimental fork. If the number of beats per second is n, then the frequency of experimental tuning fork is N+n. The experimental tuning fork is then loaded with a little bees' wax, thereby decreasing its frequency.
Now the observations are repeated. If the number of beats increases, then the frequency of the experimental tuning fork is N-n, and if the number of beats decreases its frequency is N + n.

Solved Numerical

Q) Two wires are fixed on a sonometer. Their tensions are in ratio 8:1, the lengths in the ratio 36:35, the diameters in the ratio 4:1, the densities in the ratio 1:2. Find the frequency of beats if the note of higher pitch has a frequency of 360Hz.

Solution:

$$v = \sqrt{\frac{T}{m}}$$

T is tension and m is mass of string per unit length

$$v = \sqrt{\frac{T}{\left(\frac{\pi d^2}{4}\right)\rho}}$$

$$\frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}\frac{d_2^2}{d_1^2}\frac{\rho_2}{\rho_1}} = \sqrt{\left(\frac{8}{1}\right)\left(\frac{1}{16}\right)\left(\frac{2}{1}\right)} = 1$$

Thus $v_1 = v_2 = v$(say)

$$f_1 = \frac{v}{2L_1} = 360 - x$$

$$f_2 = \frac{v}{2L_2} = 360$$

$$\frac{360 - x}{360} = \frac{L_2}{L_1} = \frac{35}{36}$$

$$360 - x = 350$$

X = 10 beats

Q) A wire of a sonometer 1m long weight 5g and is stretched by a force of 10kg wt. When the length of the vibrating portion of wire is 28 cm three beats per second are heard, if the wire and unkown frequency are sound together. The wire is slightly shorted and 4 beats per second are then heard. What is the frequency of the fork?

Solution:

The frequency of the sonometer wire

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

L = 0.28m, T = 10×9.8 = 98 N, m = 5gm = 5×10^{-3} kg/m

$$f_1 = \frac{1}{2\times 0.28}\sqrt{\frac{98}{5\times 10^{-3}}} = 250Hz$$

The sonometer wire has a frequency which differs from the tuning fork by 3Hz.
If the wire is shorted, the frequency of the wire increases. The number of beats also increases to 4.
His means that the frequency of the wire is greater than the frequency of the tuning fork
∴ the frequency of the tuning fork = 250 -3 =247 Hz

Stationary waves

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.

Analytical method

Let us consider a progressive wave of amplitude *a* and wavelength λ travelling in the direction of X axis.

$$y_1 = asin2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

This wave is reflected from a ***free end*** and it travels in the negative direction of X axis, then

$$y_1 = asin2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

According to principle of superposition, the resultant displacement is $y = y_1 + y_2$

$$y = a\left[sin2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + sin2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right]$$

$$y = a\left[2sin\frac{2\pi t}{T}cos\frac{2\pi x}{\lambda}\right]$$

$$y = 2acos\frac{2\pi x}{\lambda}sin\frac{2\pi t}{T}$$

This is the equation of a stationary wave.

(i) At points where *x* = 0, λ/2, λ, 3λ/2 the values of

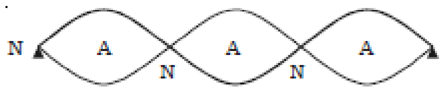
$$cos\frac{2\pi x}{\lambda} = \pm 1$$

∴ A = + 2a. At these points the resultant amplitude is maximum. They are called *antinodes* (Fig.).

(ii) At points where *x* = λ/4, 3λ/4 , 5λ/4 the values of

$$cos\frac{2\pi x}{\lambda} = 0$$

$\therefore A$ = 0. At these points the resultant amplitude is minimum. They are called *nodes* (Fig.).



The distance between any two successive antinodes or nodes is equal to λ/2
and the distance between an antinode and a node is λ/4
(iii) When t = 0, T/2, T, 3T/2, …

$$sin\frac{2\pi t}{T} = 0$$

Displacement is zero
(iv) When t = T/4, 3T/4, 5T/4, …..

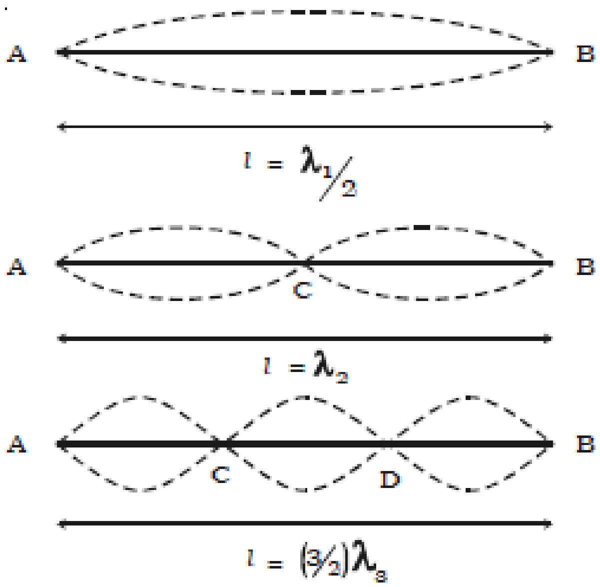
$$sin\frac{2\pi t}{T} = \pm 1$$

Displacement is maximum
Characteristics of stationary waves
1. The waveform remains stationary.
2. Nodes and antinodes are formed alternately.
3. The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.
4. Pressure changes are maximum at nodes and minimum at antinodes.
5. All the particles except those at the nodes, execute simple harmonic motions of same period.
6. Amplitude of each particle is not the same, it is maximum at antinodes decreases gradually and is zero at the nodes.
7. The velocity of the particles at the nodes is zero. It increases gradually and is maximum at the antinodes.
8. Distance between any two consecutive nodes or antinodes is equal to λ / 2
whereas the distance between a node and its adjacent antinode is equal to λ/4
.
9. There is no transfer of energy. All the particles of the medium pass through their mean position simultaneously twice during each vibration
10. Particles in the same segment vibrate in the same phase and between the neighboring segments, the particles vibrate in opposite phase.

Modes of vibration of stretched string

(i) Fundamental frequency

If a wire is stretched between two points, a transverse wave travels along the wire and is reflected at the fixed end. A transverse stationary wave is thus formed as shown in Fig. When a wire AB of length L is made to vibrate in one segment then



$$L = \frac{\lambda_1}{2}$$

$\lambda_1 = 2L$. This gives the lowest frequency called fundamental frequency

$$f_1 = \frac{v}{\lambda_1}$$

We know that velocity of wave in stretched sting is given by $v = \sqrt{\frac{T}{m}}$

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

(ii) Overtones in stretched string

If the wire AB is made to vibrate in two segments then $L = \lambda_2$

But

$f_2 = \frac{v}{\lambda_2}$

$$f_2 = \frac{1}{L}\sqrt{\frac{T}{m}}$$

f_2 is the frequency of the first overtone.

Since the frequency is equal to twice the fundamental, it is also known as second harmonic.

Similarly, higher overtones are produced, if the wire vibrates with more segments.

If there are *n* segments, the length of each segment is

$$\lambda_n = \frac{2L}{n}$$

Frequency f_n

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{m}} = nf_1$$

(i.e) n^{th} harmonic corresponds to $(n-1)^{th}$ overtone

Laws of transverse vibrations of stretched strings

The laws of transverse vibrations of stretched strings are

(i) the law of length

For a given wire (*m* is constant), when *T* is constant, the fundamental frequency of vibration is inversely proportional to the vibrating length (i.e)

$$f \propto \frac{1}{L} \quad or\ fL = constant$$

(ii) law of tension

For constant *L* and *m,* the fundamental frequency is directly proportional to the square root of the tension (i.e) $n \alpha \sqrt{T}$.

(iii) the law of mass.

For constant *L* and *T*, the fundamental frequency varies inversely as the square root of the mass per unit length of the wire (i.e) $n \alpha 1/\sqrt{m}$

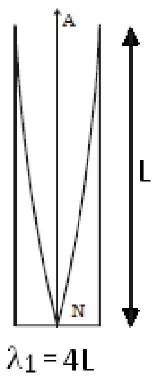
Vibrations of air column in pipes

Musical wind instruments like flute, clarinet etc. are based on the principle of vibrations of air columns. Due to the superposition of the incident wave and the reflected wave, longitudinal stationary waves are formed in the pipe.

Organ pipes

Organ pipes are musical instruments which are used to produce musical sound by blowing air into the pipe. Organ pipes are two types (i) closed organ pipes, closed at one end (ii) open organ pipe, open at both ends.

(i) Closed organ pipe : If the air is blown lightly at the open end of the closed organ pipe, then the air column vibrates (Fig.a) in the fundamental mode. There is a node at the closed end and an antinode at the open end. If *L* is the length of the tube, $\lambda_1 = 4L$



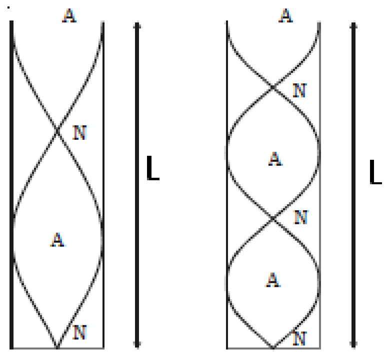
(a)

If f_1 is the fundamental frequency of the vibrations and *v* is the velocity of sound in air, then

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$$

If air is blown strongly at the open end, frequencies higher than fundamental frequency can be produced. They are called overtones.

b & c shows the mode of vibration with two or more nodes and antinodes



$\lambda_3 = 4L/3$
(b)

$\lambda_5 = 4L/5$
(c)

L =3λ₃/4 or λ₃ = 4L/3

$$\therefore f_3 = \frac{v}{\lambda_3} = \frac{3v}{4L} = 3f_1$$

This is the first overtone or third harmonics
Similarly

$$f_5 = \frac{5v}{4L} = 5f_1$$

This is the second overtone or fifth harmonic

Solved Numerical

Q) A disc contains 30 small holes evenly distributed along the rim and is rotated at the uniform rate of 540 revolutions per minute. A jet of air is blown through the hole in to a pipe whose other end is closed by movable piston. The length of the pipe can be varied between 90cm to 120 cm. What should be the exact length of the pipe so that the sound produced at the frequency on interruption of air jet is loudest? (velocity of sound in air =330 m/s)

Solution:

The frequency of notes produced by the disc = number of holes × frequency of rotating disc

$$f = 60 \times \frac{540}{60} = 270\ Hz$$

This note is produced at the mouth of the closed pipe
Let us first assume that closed pipe is vibrates in its fundamental mode
Then fundamental frequency

$$f_1 = \frac{v}{4L}$$

$$L = \frac{v}{4f_1}$$

$$L = \frac{330}{4 \times 270} = 0.306m = 30.6\ cm$$

This length is much shorter than the length given
The same fundamental frequency can also occur at a length
From the formula for first over tone

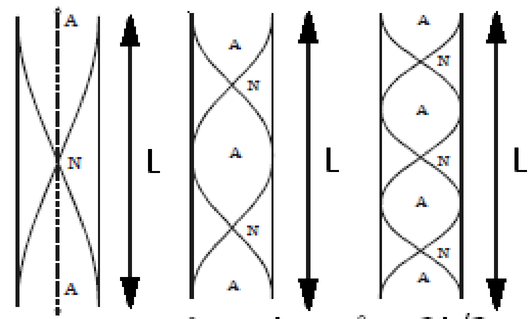
$$f_3 = \frac{3v}{4L}$$

if length is changed from L to 3L then we get

$$f = \frac{3v}{4(3L)} = \frac{v}{4L} = f_1$$

Thus the same fundamental frequency can also occur at a length 3 ×30.6 = 91.8 cm. This is greater than the minimum length 90cm of tube. The tube should be adjusted to this length to get the loudest note as desired
Note: If in place of air if any other gas is used then calculate velocity of sound in that gas and used in place of v to determine length

(ii) Open organ pipe –



$\lambda_1 = 2L$ (a) $\lambda_2 =$ L (b) $\lambda_3 = 2L/3$ (c)

When air is blown into the open organ pipe, the air column vibrates in the fundamental mode Fig.a. Antinodes are formed at the ends and a node is formed in the middle of the pipe. If L is the length of the pipe, then λ_1=2L
$v = f_1\lambda_1 = f_1 2L$
The fundamental frequency

$$f_1 = \frac{v}{2L}$$

In the next mode of vibration additional nodes and antinodes are formed as shown in fig(b) and fig(c)
$L = \lambda_2$ or $v = f_2\lambda_2 = f_2 L$

$$f_2 = \frac{v}{L} = 2f_1$$

Similarly,

$$f_2 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

This is the second overtone or third harmonic.

Therefore the frequency of nth overtone is *(n + 1)* f_1 where f_1 is the fundamental frequency. The frequencies of harmonics are in the ratio of 1 : 2 : 3 ....

Solved Numerical

Q) An open pipe filled with air has a fundamental frequency of 500Hz. The first harmonic of another organ pipe closed at one end and filled with carbon dioxide has the same frequency as that of the first harmonic of the open organ pipe. Calculate the length of each pipe. Assume that the velocity of sound in air and in carbondioxide to be 330 m/s and 264 m/s respectively

Solution:

The fundamental frequency is the first harmonics. In case of open pipe containing air at 30^{O}C. Let L_O be the length of the pipe. Then

$$f_1 = \frac{v}{2L_O}$$

$$L_O = \frac{v}{2f}$$

$$L_O = \frac{330}{2 \times 500} = 33\ cm$$

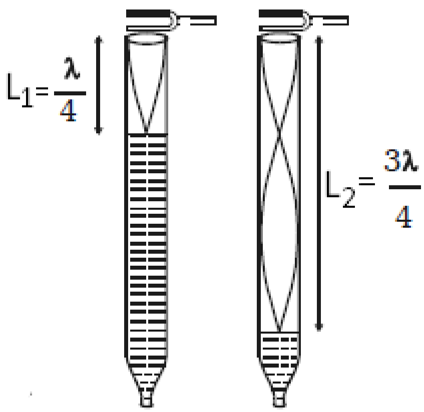
Let L_C be the length of the closed pipe. For the fundamental frequency of the ipipe

$$f_1 = \frac{v_{CO_2}}{4L_C}$$

$$L_C = \frac{264}{4 \times 500} = 13.2\ cm$$

Resonance air column apparatus

The glass tube is mounted on a vertical stand with a scale attached to it. The glass tube is partly filled with water. The level of water in the tube can be adjusted by raising or lowering the reservoir.



A vibrating tuning fork of frequency *f* is held near the open end of the tube.

The length of the air column is adjusted by changing the water level. The air column of the tube acts like a closed organ pipe.

When this air column resonates with the frequency of the fork the intensity of sound is maximum.

Here longitudinal stationary wave is formed with node at the water surface and an antinode near the open end. If L_1 is the length of the resonating air column

$$\frac{\lambda}{4} = L_1 + e \quad ---eq(1)$$

where e is the end correction.

The length of air column is increased until it resonates again with the tuning fork. If L_2 is the length of the air column

$$\frac{3\lambda}{4} = L_2 + e \quad ---eq(2)$$

From equations (1) and (2)

$$\frac{\lambda}{2} = L_2 - L_1$$

The velocity of sound in air at room temperature

$$v = f\lambda = 2f\,(L_2 - L_1)$$

End correction

The antinode is not exactly formed at the open end, but at a small distance above the open end. This is called the end correction

It is found that e = 0.61r*,* where *r* is the radius of the glass tube.

Doppler Effect

When a sound source and an observer are in relative motion with respect to the medium in which the wave propagate, the frequency of wave observed is different from the frequency of sound emitted by the source. This phenomenon is called Doppler effect. This is due to the wave nature of sound propagation and is therefore applicable to light waves also.

Calculation of apparent frequency

Suppose V is the velocity of sound in air, V_o is the velocity of observer (O) and f is the frequency of the source

(i)Source moves towards stationary observer

If the source S were stationary, the f waves sent out in one second towards the observer O would occupy a distance V and the wave length would bt v/f

If S is moving with velocity V_s towards stationary observer, the f waves emitted in one second occupy a distance (V-V_s) because S has moves a distance V_s towards O in 1 sec..

So apparent frequency would be

$$\lambda' = \left(\frac{V - V_s}{f}\right)$$

∴ apparent frequency

$$f' = \frac{velocity\ of\ sound\ relative\ to\ O}{wavelength\ of\ wave\ reaching\ O}$$

$$f' = \frac{V}{\lambda'} = f\left(\frac{V}{V - V_s}\right)$$

(ii)Source moves away from stationary observer:

Apparent wave length

$$\lambda' = \left(\frac{V + V_s}{f}\right)$$

$$f' = \frac{V}{\lambda'} = f\left(\frac{V}{V+V_s}\right)$$

(iii) Observer moving towards stationary source

$$f' = \frac{velocity\ of\ sound\ relative\ to\ O}{wavelength\ of\ wave\ reaching\ O}$$

Velocity of sound relative to O = V +V_o
And wavelength of waves reaching O = V/f

$$f' = \frac{V+V_0}{V/f} = f\left(\frac{V+V_0}{V}\right)$$

(iv) Observer moves away from the stationary source:

Velocity of sound relative to O = V –V_o
And wavelength of waves reaching O = V/f

$$f' = \frac{V-V_0}{V/f} = f\left(\frac{V-V_0}{V}\right)$$

(v) Source and observer both moves towards each other

Velocity of sound relative to O = V +V_o
And wavelength of waves reaching O = (V-V_s)/f

$$f' = \frac{V+V_0}{\frac{V-V_s}{f}} = f\left(\frac{V+V_0}{V-V_s}\right)$$

(vi) Source and observer both are moving away from each other

Velocity of sound relative to O = V - V_o
And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V-V_0}{\frac{V+V_s}{f}} = f\left(\frac{V-V_0}{V+V_s}\right)$$

(vii) Source moves towards observer but observer moves away from source

Velocity of sound relative to O = V - V_o
And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V-V_0}{\frac{V+V_s}{f}} = f\left(\frac{V-V_0}{V+V_s}\right)$$

(viii)Source moves away from observer but observer moves towards source

Velocity of sound relative to O = V + V_o
And wavelength of waves reaching O = (V+V_s)/f

$$f' = \frac{V+V_0}{\frac{V+V_s}{f}} = f\left(\frac{V+V_0}{V+V_s}\right)$$

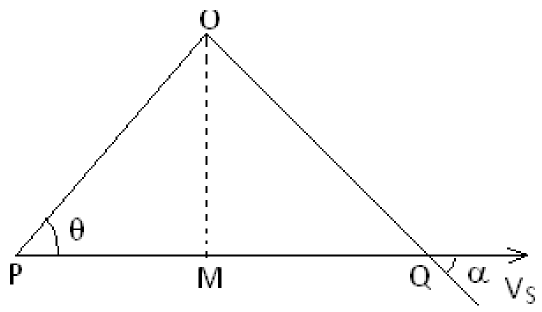
General equation can be written as

$$\boldsymbol{f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)}$$

If **observer** moving **towards** source V_o is +Ve
If **observer** I moving **away** from source V_0 is –Ve

If **source** is moving **towards** or approaching observer V_s is -Ve
If **source** moving **away** from observer V_s is +Ve
Effect of wind velocity
If wind velocity(w) is in the direction of sound(v) then we can add wind velocity
If wind velocity (w) is opposite in the direction of sound we can subtract wind velocity for final velocity of sound

Doppler effect when the source is moving at an angle to the observer



Let O be a stationary observer and let a source of sound of frequency f be moving along the line PQ with constant speed $_s$
When the source is at O, the line PO makes angle θ with PQ, which is the direction of V_s
The component of velocity V_s along PO is $V_s\cos\theta$ and it is towards the observer

The apparent frequency in this case

$$f_a = f\left(\frac{V}{V - V_s cos\theta}\right)$$

As the source moves along PQ, θ increases cosθ decreases and the apparent frequency continuously diminishes. At M, $\theta=90^O$ and hence $f_a = f$
When the source is at Q, the component of velocity V_s is $V_s\cos\alpha$ which is directed away from the observer. Hence the apparent frequency

$$f_a = f\left(\frac{V}{V + V_s cos\alpha}\right)$$

Solved Numerical

Q) A train travelling at a speed of 20 m/s and blowing a whistle with frequency of 240 Hz is approaching a train B which is at rest. Assuming the speed of sound to be 340 m/s calculate the following
(a) Wavelength in air(i) in front and (ii) behind the train A

(b) Frequencies measured by a listener in train B while train A is (i)approaching and (ii) receding from train B

(c)If train B starts moving with speed of 10 m/s, what will be the frequencies heard bya passenger in train B if both were (i)approaching and (ii) receding

Solution:

(a)(i) The train A is approaching the train B which is at rest

Thus $V_O = 0$, since source is approaching thus V_S is -Ve

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$f' = f\left(\frac{V}{V - V_s}\right)$$

Now f = V/λ

$$\therefore \frac{V}{\lambda'} = \frac{V}{\lambda}\left(\frac{V}{V - V_s}\right)$$

$$\therefore \lambda' = \lambda\left(\frac{V - V_s}{V}\right) = \frac{\lambda}{V}(V - V_s)$$

$$\therefore \lambda' = \frac{1}{f}(V - V_s)$$

$$\therefore \lambda' = \frac{1}{240}(340 - 20) = 1.33\ m$$

(ii)For observer behind the train A since source is moving away V_S is positive Thus

$$\therefore \lambda' = \frac{1}{f}(V + V_s)$$

$$\therefore \lambda' = \frac{1}{240}(340 + 20) = 1.5\ m$$

(b)(i)Frequency as measured by listener un train B when A is approaching B

Source is approaching V_S is –Ve, Listener is stationary $V_O = 0$

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$\therefore f' = f\left(\frac{V}{V - V_s}\right)$$

$$\therefore f' = 240\left(\frac{340}{340 - 20}\right) = 255\ Hz$$

(ii) Frequency as measured by listener in B as A recedes from hin

Source moving away V_S is +Ve, Listener is stationary $V_O = 0$

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$\therefore f' = f\left(\frac{V}{V+V_s}\right)$$

$$\therefore f' = 240\left(\frac{340}{340+20}\right) = 227\ Hz$$

(C)(i) Now both the trains are approaching each other
observer moving towards source V_o is +Ve and
Source is approaching V_S is –Ve

$$f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = 240\left(\frac{340+10}{340-20}\right) = 263\ Hz$$

(ii) Now both trains are moving away
observer moving away source V_o is -Ve and
Source is moving away V_S is +Ve

$$f' = f\left(\frac{V+V_o}{V-V_s}\right)$$

$$\therefore f' = f\left(\frac{V-V_o}{V+V_s}\right)$$

$$\therefore f' = 240\left(\frac{340-10}{340+20}\right) = 220\ Hz$$

Q) A spectroscope examination of light from a certain star shows that the apparent wavelength of certain spectral line from a certain star is 5001Å. Whereas the observed wavelength of the same line produced by terrestrial source is 5000. In what direction and what speed do these figure suggest that the star is moving relative to the earth.

Solution:

Actual wavelength of the spectral line = 5000 Å
Apparent wave length of the same line = 5001 Å
Since wavelength increases, the star is moving away from earth (red shif) V_s is positive
Observer is stationary and velocity of light V = c

$$f' = f\left(\frac{V \pm V_o}{V \pm V_s}\right)$$

$$f' = f\left(\frac{c}{c+V_s}\right)$$

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\left(\frac{c}{c+V_s}\right)$$

$$\lambda' = \lambda\left(\frac{c+V_s}{c}\right)$$

$$V_S = \frac{\lambda' c}{\lambda} - c$$

$$V_S = c\left(\frac{\lambda' - \lambda}{\lambda}\right)$$

$$V_S = 3\times10^8\left(\frac{10^{-10}}{5000\times10^{-10}}\right) = 6\times10^4\ m/s$$

Q) A whistle of frequency 1000 Hz is blown continuously in front of a board made of plaster of paries. If the board is made to move away from the whistle with a velocity of 1.375 m/s, calculate the number of beats heard per second by a stationary observer situated infront of the board in line with the whistle (velocity of sound in air = 330 m/s)

Solution

The frequency of whistle = 1000Hz

The reflecting board is moving away from the whistle with velocity of 1.375 m/s

The reflected source of sound for the observer is the image of the whistle behind the board, the image is moving away from the board with velocity 2×1.375 = 2.750 m/s

Therefore the frequency of sound heard by the observer due to reflection from the board is

$$f' = f\left[\frac{V}{V+V_S}\right]$$

$$f' = 1000\left[\frac{330}{330+2.75}\right] = 992\ (approx)$$

∴ number of beats heard by the observer = 1000 - 992 = 8 per second

----------------END--------------

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TRANSFER OF HEAT

Transfer of heat

There are three ways in which heat energy may get transferred from one place to another. These are conduction, convection and radiation.

Conduction

Heat is transmitted through the solids by the process of conduction only. When one end of the solid is heated, the atoms or molecules of the solid at the hotter end becomes more strongly agitated and start vibrating with greater amplitude. The disturbance is transferred to the neighboring molecules.

Coefficient of thermal conductivity

Let us consider a metallic bar of uniform cross section A whose one end is heated. After sometime each section of the bar attains constant temperature but it is different at different sections. This is called steady state.

In this state there is no further absorption of heat.

If Δx is the distance between the two sections with a difference in temperature of ΔT and ΔQ is the amount of heat conducted in a time Δt, then it is found that the rate of conduction of heat is

(i) directly proportional to the area of cross section (A)

(ii) directly proportional to the temperature difference between the two sections (ΔT)

(iii) inversely proportional to the distance between the two sections (Δx).

$$\frac{\Delta Q}{\Delta T} \propto -A\frac{\Delta T}{\Delta x}$$

Negative sign indicates as x increases temperature decreases

$$\frac{\Delta Q}{\Delta T} = KA\frac{\Delta T}{\Delta x} \quad ----eq(1)$$

where K is a constant of proportionality called co-efficient of thermal conductivity of the metal

$\frac{\Delta T}{\Delta x}$ is called as temperature gradient

Coefficient of thermal conductivity of the material of a solid is equal to the rate of flow of heat per unit area per unit temperature gradient across the solid. Its unit is W m^{-1} K^{-1}.

Or Cal $s^{-1}m^{-1}K^{-1}$

Thermal steady state

When a rod is heated , after sufficiently long time the temperature of all parts of rod become steady. These steady temperatures decreases along the length of the rod from hot end to its cold end. In this situation amount of heat energy received by the hot end in some interval is equal to the amount of heat lost by the cold end in the same time interval. Hence, any cross section of the rod, along its entire length, has the same value of heat current dQ/dt. Further along the entire length of the rod the value of the temperature gradient dT/dx is also the same along the length.

Now both dQ/dt and dT/dx remains constant with time. This condition of the rod is called 'thermal steady state" of the rod.
In thermal steady state the temperature of two ends of the rod T_1 and T_2 with $T_1>T_2$. As dT/dx is same all along the length of the rod

$$\frac{dT}{dx} = -\left[\frac{T_1 - T_2}{L}\right]$$

From eq(1)

$$\frac{dQ}{dt} = KA\left[\frac{T_1 - T_2}{L}\right] \quad ----eq(2)$$

As dQ/dt is constant

$$\frac{Q}{t} = KA\left[\frac{T_1 - T_2}{L}\right]$$

$$Q = KA\left[\frac{T_1 - T_2}{L}\right]t$$

Above equation gives the amount of heat flowing through the rod in a steady thermal state in time t.

If we represent dQ/dt as heat current (H) , which caused due to temperature difference then from equation (2)

$$H = \left[\frac{T_1 - T_2}{\frac{L}{KA}}\right]$$

Comparing the above equation with I = V/R we get thermal resistance R_H

$$R_H = \frac{L}{KA}$$

Unit of thermal resistance is Kelvin/watt and its dimensional formula is $M^{-1}L^{-2}T^3K$

Formula for effective thermal resistance when thermal conductors are connected in series

$$(R_H)_S = (R_H)_1 + (R_H)_2$$

For parallel connection

$$\frac{1}{(R_H)_P} = \frac{1}{(R_H)_1} + \frac{1}{(R_H)_2}$$

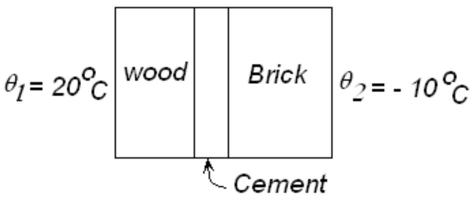
Solved Numerical

Q) An electric heater is sued in a room of total wall area 137 m^2 to maintain a temperature of $20^{O}C$ inside it, when the outside temperature is $-10^{O}C$. the walls have three different layers of materials. The innermost layer is of wood of thickness 2.5cm, the middle layer is of thickness 1.0cm and the outermost layer is of brick of thickness 25.0cm. Find the power of the electric heater. Assume that there is no heat loss through the floor and the ceiling.

The thermal conductivity of wood , cement and brick are 0.125 W/m ^{O}C, 1.5 W/m ^{O}C and 1.0 W/m ^{O}C respectively

Solution

Situation is as show in figure



The thermal resistance of wood, the cement and the brick layers are

$$R_W = \frac{L}{KA}$$

$$R_W = \frac{2.5 \times 10^{-2}}{0.125 \times 137} = \frac{0.2}{137}$$

$$R_C = \frac{1.0 \times 10^{-2}}{1.5 \times 137} = \frac{0.0067}{137}$$

$$R_B = \frac{25.0 \times 10^{-2}}{1.0 \times 137} = \frac{0.25}{137}$$

As the layers are connected in series, the equivalent

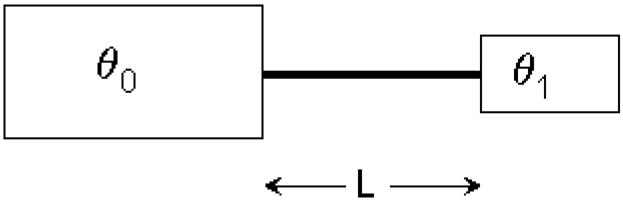
$$R = R_W + R_C + R_B$$

$$R = \frac{0.2 + 0.0067 + 0.25}{137} = 3.33 \times 10^{-3}$$

The heat current

$$i = \frac{\theta_1 - \theta_2}{R} = \frac{20 - (-10)}{3.33 \times 10^{-3}} = 9000\, W$$

Q) The figure shows a large tank of water at a constant temperature θ_0 and a small vessel containing a mass m of water at an initial temperature θ_1 ($<\theta_0$). A metal rod of length L, area of cross-section A and thermal conductivity K connects the two vessels. Find the time taken for the temperature of water in the smaller vessel to become θ_2 ($\theta_1 < \theta_2 < \theta_0$). Specific heat capacity of water is s and all other heat capacities are negligible.



Solution

Suppose, the temperature of the water in the smaller vessel is θ at time t. In the next time interval dt, a heat ΔQ is transferred from the big vessel

$$\Delta Q = \frac{KL}{A}(\theta_0 - \theta)dt \qquad ----eq(1)$$

This heat increases the temperature of the water in small tank to θ+dθ where

$$\Delta Q = ms\, d\theta \qquad ---------eq(2)$$

From equation (1) and (2)

$$ms\, d\theta = \frac{KL}{A}(\theta_0 - \theta)dt$$

$$dt = \frac{Lms}{KA}\frac{d\theta}{\theta_0 - \theta}$$

Or

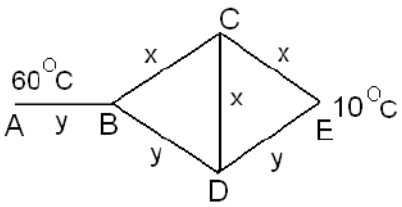
$$\int_0^T dt = \frac{Lms}{KA}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\theta_0-\theta}$$

Where T is the time required for the temperature of the water to become θ_2

Thus

$$T = \frac{Lms}{KA} ln\left(\frac{d\theta}{\theta_0-\theta}\right)$$

Q) three rods of material x and three rods of material y are connected as shown. All the rods are of identical length and cross sectional area. If the end A is maintained at 60^{O}c and the junction E is at 10^{O}C, calculate the temperature of the junctions B,C and D. The thermal conductivity of x is 0.92c.g.s unit and that of y is 0.46c.g.s units



Solution:

Since end A or rod AB is maintained at temperature higher than the end b heat is conducted from A to B

Now the total heat entering junction B is equal to the total heat leaving it (all by conduction alone)

Let the temperature of junction B, C, D be T_1, T_2 and T_3 respectively

Let the cross-sectional area of each rod be A and the length of rod be L. Then heat entering joint B per second =

$$= \frac{K_x A(60-T_1)}{L}$$

Heat leaving B per second = heat passing through BC + heat passing through BD

$$\frac{K_x A(T_1-T_2)}{L} + \frac{K_y A(T_1-T_3)}{L}$$

Thus

$$\frac{K_x A(60-T_1)}{L} = \frac{K_x A(T_1-T_2)}{L} + \frac{K_y A(T_1-T_3)}{L}$$

Given $K_X = 2K_Y$

$60 – T_1 = 2 (T_1 – T_2) + (T_1 – T_2)$

Or $4T_1 – 2T_2 – T_3 = 60$ ------eq(1)

Similarly for junction c

Heat received per second = Heat passing through CD + Heat passing through CE

$$\frac{K_x A(T_1-T_2)}{L} = \frac{K_x A(T_2-10)}{L} + \frac{K_x A(T_2-T_3)}{L}$$

OR $T_1 – T_2 = T_2 – 10 + T_2 - T_3$

Or $T_1 -3T_2 + T_3 = -10$ ----------eq(2)

For Junction D

$$\frac{K_y A(T_1-T_2)}{L} = \frac{K_x A(T_3-T_2)}{L} + \frac{K_y A(T_3-10)}{L}$$

$T_1 - T_2 = 2(T3 - T_2) + T_3 - 10$

$T_1 + 2T_2 - 4T_3 = -10$ ----eq(3) Solving equation (1) (2) and (3)

we get $T_1 = 30^OC$, $T_2 = 20^O$, $T_3 = 20^OC$

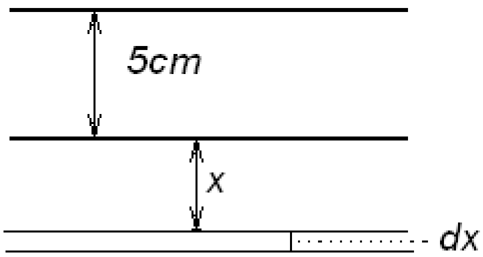
Temperature of junction B = 30^OC

Temperature of junction C = 20^OC

Temperature of junction D = 20^OC

Q) The thickness of ice layer on the surface of lake is 5cm. Temperature of environment is -10^OC. Find the time required for the thickness of the ice layer to become double. Thermal conductivity of ice is 0.004 cal/cms ^OC, density of ice is 0.92 g/cm^3 and latent heat of fusion is 80cal/gm

Solution



Consider a layer of thickness dx and surface area A. Heat required to b taken out from such layer is

dQ = Adx ρ L

Here ρ is density of ice and L is latent heat of melting

If time required for passage of heat through a thickness of 5x is dt Then

$$dQ = KA\frac{\Delta T}{5+x}dt$$

Thus

$$KA\frac{\Delta T}{5+x}dt = \text{Adx ρ L}$$

$$K(-10-0)dt = \text{A}(5+\text{x})\text{dx ρ L}$$

Integrating

$$-10K\int_0^t dt = \text{ρ L}\int_0^5 (5+\text{x})\text{dx}$$

$$-10Kt = \rho L\left\{[5x]_0^5 + \left[\frac{x^2}{2}\right]_0^5\right\}$$

$$-10Kt = \rho L(25+12.5)$$

$$-10 \times 0.004 \times t = 0.92 \times 80 \times (37.5)$$

t= 69,000 seconds

t = 19.16 hours

Thermal expansion

The increase in dimension of a substance due to absorption of heat is called thermal expansion and decrease in dimensions of the substance by releasing the heat is called thermal contraction

Linear expansion

The increase in the length of a body with increase in temperature is called linear expansion

For small change in temperature, the increase in length $\Delta\ell$ is directly proportional to original length ℓ and increase in temperature ΔT

$\Delta\ell \propto \ell$ and $\Delta\ell \propto \Delta T$

$\Delta\ell = \alpha\,\ell$

Here α is a constant of proportionality called coefficient of linear expansion of material of the body. The value of α depends on the type of material of body and temperature. If temperature interval is very large , then α do not depends on the temperature

The unit of α is $(^{O}C)^{-1}$ or K^{-1}

$\ell' = \ell\,(1+ \alpha\,\Delta T)$

Some substances exhibits uniform thermal expansion in all directions. Such substances are called isotropic substances. For such substance

Increase in area $\Delta A = 2\alpha A\Delta T$

Increase in volume $\Delta V= \gamma\Delta T$

$V' = V(1+ \gamma\alpha)$

For density

$P'=\rho(1+ \gamma\alpha)$

Thermal expansion is more in liquid than solid and it is maximum in gases

Solved Numerical

Q) The design of some physical instrument requires that there be a constant difference in length of 10cm between ion rod and copper rod laid side at all temperatures. Find their length ($\alpha_{Fe} = 11\times10^{-6}\ ^{O}C^{-1}$, $\alpha_{Cu} = 17\times10^{-6}\ ^{O}C^{-1}$)

Solution:

Since $\alpha_{Cu} > \alpha_{Fe}$ so length of iron rod should be greater than the length of cooper rod

Let initial length of iron rod be l_1 and copper rod be l_2 then

$l_1 – l_2 = 10cm$ ----------eq(1)

Also since the difference has to be constant at all the temperatures, so

$\Delta l = l_1\,\alpha_{Fe}\Delta T = l_2\,\alpha_{Cu}\,\Delta T$

$$\frac{l_1}{l_2} = \frac{\alpha_{Cu}}{\alpha_{Fe}} \quad ---eq(2)$$

Solving equation (1) and (2), we get

$l_1 = 28.3$ cm and $l_2 = 18.3$ cm

Q) A sphere of diameter 7.0cm and mass 266.5 g floats in a bath of liquid . As a temperature is raised, the sphere begins to sink at a temperature of $35^{O}C$. If the density of the liquid is 1.527 g/cm^2 at $0^{O}C$, find the coefficient of cubical expansion of the liquid. Neglect the expansion of the sphere

Solution

It is given that the expansion of the sphere is negligible as compared to the expansion of liquid. At $0^{O}C$, the density of the liquid is $\rho_0 = 1.527$ g/cm^3. At $35^{O}C$, the density of the liquid equals the density of the sphere. Thus

$$\rho_{35} = \frac{266.5}{\frac{4}{3}\pi(3.5)^2} = 1.484\ g/cm^3$$

We have density ρ ∝ (1/V) thus

$$\frac{\rho_\theta}{\rho_0} = \frac{V_0}{V_\theta} = \frac{1}{(1+\gamma\theta)}$$

Or

$$\rho_\theta = \frac{\rho_0}{(1+\gamma\theta)}$$

$$\gamma = \frac{\rho_0 - \rho_{35}}{\rho_{35}(35)} = \frac{(1.527 - 1.484)}{1.484 \times 35}$$

$$\gamma = 8.28\times10^{-4}\ /\ ^{O}C$$

Convection

It is a phenomenon of transfer of heat in a fluid with the actual movement of the particles of the fluid. When a fluid is heated, the hot part expands and becomes less dense. It rises and upper colder part replaces it. This again gets heated, rises up replaced by the colder part of the fluid. This process goes on.

This mode of heat transfer is different from conduction where energy transfer takes place without the actual movement of the molecules.

Application

It plays an important role in ventilation and in heating and cooling system of the houses.

Radiation

It is the phenomenon of transfer of heat without any material medium. Such a process of heat transfer in which no material medium takes part is known as radiation.

Thermal radiation

The energy emitted by a body in the form of radiation on account of its temperature is called thermal radiation. It depends on,

(i) temperature of the body,

(ii) nature of the radiating body

The wavelength of thermal radiation ranges from 8 × 10-7 *m* to 4 × 10-4 *m*. They belong to infra-red region of the electromagnetic spectrum.

Properties of thermal radiations

1. Thermal radiations can travel through vacuum.
2. They travel along straight lines with the speed of light.
3. They can be reflected and refracted. They exhibit the phenomenon of interference and diffraction.
4. They do not heat the intervening medium through which they pass.
5. They obey inverse square law.

Absorptive and Emissive power

Absorptive power

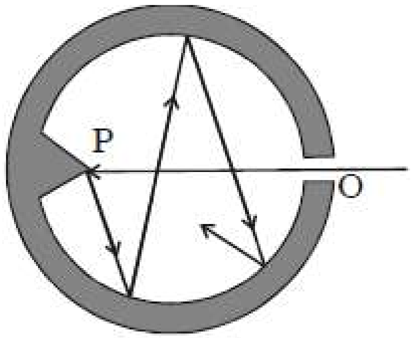
Absorptive power of a body for a given wavelength and temperature is defined as the ratio of the radiant energy absorbed per unit area per unit time to the total energy incident on it per unit area per unit time. It is denoted by a_λ.

Emissive power

Emissive power of a body at a given temperature is the amount of energy emitted per unit time per unit area of the surface for a given wavelength. It is denoted by e_λ.
Its unit is W m^{-2}.

Perfect black body

A perfect black body is the one which absorbs completely heat radiations of all wavelengths which fall on it and emits heat radiations of all wavelengths when heated. Since a perfect black body neither reflects nor transmits any radiation, the absorptive power of a perfectly black body is unity.



Fery's black body

Fery's black body consists of a double walled hollow sphere having a small opening O on one side and a conical projection P just opposite to it ref.figure Its inner surface is coated with lamp black. Any radiation entering the body through the opening O suffers multiple reflections at its inner wall and about 97% of it is absorbed by lamp black at each reflection. Therefore, after a few reflections almost entire radiation is absorbed. The projection helps in avoiding any direct reflections which even otherwise is not possible because of the small opening O. When this body is placed in a bath at fixed temperature, the heat radiations come out of the hole. The opening O thus acts as a black body radiator.

Kirchoff's Law

According to this law, the ratio of emissive power to the absorptive power corresponding to a particular wavelength and at a given temperature is always a constant for all bodies. This constant is equal to the emissive power of a perfectly black body at the same temperature and the same wavelength. Thus, if e_λ is the emissive power of a body corresponding to a wavelength λ at any given temperature, a_λ is the absorptive power of the body corresponding to the same wavelength at the same temperature and E_λ is the emissive power of a perfectly black body corresponding to the same wavelength and the same temperature, then according to Kirchoff's law

$$\frac{a_\lambda}{e_\lambda} = constant = E_\lambda$$

From the above equation it is evident that if a_λ is large, then e_λ will also be large (i.e) if a body absorbs radiation of certain wavelength strongly then it will also strongly emit the radiation of same wavelength. In other words, good absorbers of heat are good emitters also.

Applications of Kirchoff's law

(i) The silvered surface of a thermos flask is a bad absorber as well as a bad radiator. Hence, ice inside the flask does not melt quickly and hot liquids inside the flask do not cool quickly.

(ii) Sodium vapours on heating, emit two bright yellow lines. These are called D_1 and D_2 lines of sodium. When continuous white light from carbon arc passes through sodium vapour at low temperature, the continuous spectrum is absorbed at two places corresponding to the wavelengths of D_1 and D_2 lines and appear as dark lines. This is in accordance with Kirchoff's law.

Wien's displacement law

Wien's displacement law states that the wavelength of the radiation corresponding to the maximum energy (λ_m) decreases as the temperature T of the body increases.

(i.e) $\lambda_m T = b$ where b is called Wien's constant. Its value is 2.898×10^{-3} m K

Stefan's law

Stefan's law states that the total amount of heat energy radiated per second per unit area of a perfect black body is directly proportional to the fourth power of its absolute temperature.

(i.e) $E \propto T^4$ or $E = \sigma e_\lambda T^4$

where σ is called the Stefan's constant. Its value is 5.67×10^{-8} W m^{-2} K^{-4}.

It is also called Stefan - Boltzmann law, as Boltzmann gave a theoretical proof of the result given by Stefan.

If the body of temperature T is kept in an environment with temperature T_s ($T > T_S$), then rate at which the body loses heat is given by

$$\frac{dQ}{dt} = e_\lambda \sigma A(T^4 - T_S^4)$$

A : is area of surface , $e_\lambda = 1$ for perfectly black body

Solved numerical

Q) From 1 m^2 area of surface of Sun 6.3×10^7 J energy is emitted per second $\sigma = 5.669\times10^{-8}$ Wm^{-2}K^{-4}. Find the temperature of the surface of the sun

Solution.

A = 1 m^2 , dQ/dt = 6.3×10^7 J/s, $e_\lambda = 1$

$$\frac{dQ}{dt} = e_\lambda \sigma A T^4$$

$6.3\times10^7 = 5.669\times10^{-8}\times1\times1\times T^4$

T = 5841 K

Q) How many times faster the temperature of a cup of tea will decrease by 1^{O}C at 373K, then at 303K? Consider tea as a black body. Take room temperature as 293K

($\sigma = 5.7\times10^{-8}$ W m^{-2} K^{-4})

Solution

$$\frac{dQ_1}{dt} = e_\lambda \sigma A(373^4 - 293^4)$$
$$\frac{dQ_2}{dt} = e_\lambda \sigma A(303^4 - 293^4)$$

Take the ratio of above equations we get cup at 373K will decrease its temperature 11.32 times faster than cup at 303K temperature

Newton's law of cooling

Newton's law of cooling states that *the rate of cooling of a body is directly proportional to the temperature difference between the body and the surroundings.*

The law holds good only for a small difference of temperature. Loss of heat by radiation depends on the nature of the surface and the area of the exposed surface.

We know that the amount of heat required to change the temperature of a bosy of mass m and specific heat c, by ΔT is

ΔQ = mcΔT

Therefore, the rate of loss of heat

$$\frac{dQ}{dt} = -mc\frac{dT}{dt}$$

According to Newton's law, the rate of loss of heat by a body depends on the difference of temperature ($T - T_S$) between body and its surrounding

$$\therefore \frac{dQ}{dt} = -mc\frac{dT}{dt} \propto (T - T_S)$$
$$\therefore \frac{dQ}{dt} = -k'(T - T_S)$$

Here dT/dt is the rate of decrease in temperature of a body at temperature T.

The constant k' depends on the mass and the specific heat of the cooling body. Negative sign indicates temperature of body decreases with time

Note that Newton's law of cooling is true only for small interval of diierence of temperature between the body and its surrounding

If the amount of heat lost by the body due to radiation is very small, this law hold true for large interval of temperature also.

For natural convection, the law of cooling given by Langmuir – Lorenz is as under

$$-\frac{dT}{dt} \propto (T - T_S)^{\frac{5}{4}}$$

Solved Numerical

Q) A body at 80^{O}C cools down to 64^{O}C in 5 minutes and in 10 minutes it cools down to 52^{O}C. What will be its temperature after 20 minutes? What is the temperature of the environment

Solution

For the first 5 minute

$\Delta T = T_2 - T_1 = 64 - 80 = -16$ and $\Delta t = 5$

$$\therefore \frac{+16}{5} = k'\left(\frac{80+64}{2} - T_s\right) \qquad ----eq(1)$$

Here we have taken the average of initial and final temperatures as the temperature of the body

Similarly for next 5 minutes

$\Delta T = 52-64 = -12$

$$\therefore \frac{+12}{5} = k'\left(\frac{52+64}{2} - T_s\right) \qquad ----eq(2)$$

Dividing eq(1) by eq(2) we get

$$\frac{16}{5} \times \frac{5}{12} = \frac{72 - T_S}{58 - T_S}$$

$232 - 4T_S = 216 - 3T_S$

$232 - 216 = T_S$

$T_S = 16^{O}C$

Substituting value of T_S in equation(1)

$$\frac{16}{5} = k'(72 - 16)$$

$K' = 2/35$

Now for third stage $\Delta t = 10$ minutes

$\Delta T = T - 52$, where T is final temperature

$$\frac{52 - T}{10} = \frac{2}{35}\left(\frac{52 + T}{2} - 16\right)$$

On solving we get

$T = 36^{O}C$

ATOM

Thomson atom model

From the study of discharge of electricity through gases, it became clear that an atom consists of positive and negative charges. J.J. Thomson tried to explain the arrangement of positive charge and the electrons inside the atom.

According to him, an atom is a sphere of positive charge having a radius of the order of 10^{-10}m. The positive charge is uniformly distributed over the entire sphere and the electrons are embedded in the sphere of positive charge .

The total positive charge inside the atom is equal to the total negative charge carried by the electrons, so that every atom is electrically neutral.

According to Thomson, if there is a single electron in the atom (like a hydrogen atom), the electron must be situated at the centre of the positive sphere.

For an atom with two electrons (helium atom), the electrons should be situated symmetrically with respect to the centre of the sphere i.e., opposite sides of the centre at a distance of r/2 where *r* is the radius of the positive sphere. In a three electron system of the atom, the electrons should be at the corners of a symmetrically placed equilateral triangle, the side of which was equal to the radius of the sphere. In general, the electrons of an atom are located in a symmetrical pattern with respect to the centre of the sphere.

It was suggested that spectral radiations are due to the simple harmonic motion of these electrons on both sides of their mean positions. Moreover, the stability of the atom was very well explained on the basis of this model

Drawbacks

(i) According to electromagnetic theory, the vibrating electron should radiate energy and the frequency of the emitted spectral line should be the same as the electron. In the case of hydrogen atom,

Thomson's model gives only one spectral line of about 1300 Å. But the experimental observations reveal that hydrogen spectrum consists of five different series with several lines in each series.

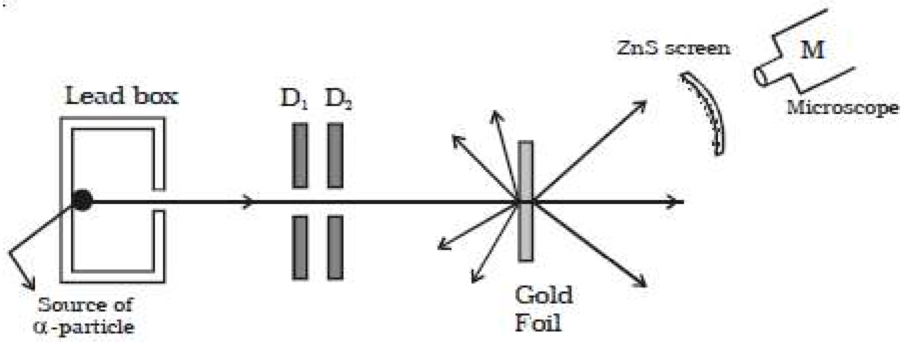
(ii) It could not account for the scattering of α-particles through large angles.

Rutherford's α - particle scattering experiment

Rutherford and his associates studied the scattering of the α - particles by a thin gold foil in order to investigate the structure of the atom. An α-particle is a positively charged particle having a mass equal to that of helium atom and positive charge in magnitude equal to twice the charge of an electron. They are emitted by many radioactive elements. The scattering of α-particles provide useful information about the structure of the atom.

Experimental arrangement

A fine stream of α−particles was obtained from a radioactive material Bi_{83}^{214} by placing it in a lead box with narrow opening as shown in Fig



The α-particles of energy 5.5MeV emitted from the source in all possible directions, but only a narrow beam emerges from the lead box. The remaining α-particles are absorbed in the lead box itself. After passing through the diaphragms D1 and D2, a narrow beam of α-particles incident on a thin gold foil of thickness 2.1×10^{-7} m, are scattered through different angles.
The scattered α-particles strike a fluorescent screen coated with zinc sulphide. When the α-particles strike the screen, tiny flashes of light are produced.
The observations can be made with the help of a low power microscope.

Observations and conclusions

(i) Most of the α particles either passed straight through the gold foil or were scattered by only small angles of the order of a few degrees.
This observation led to the conclusion that an atom has a lot of empty space.
(ii) A few α particles were scattered in the backward direction, which led Rutherford to conclude that the whole of the positive charge was concentrated in a tiny space of about 10^{-14} m. This region of the atom was named as nucleus. Only a small number of particles approaches the nucleus of the atom and they were deflected at large angles.

Distance of closest approach

An α particle directed towards the centre of the nucleus will move close up to a distance r_o as shown in Fig, where its kinetic energy will appear as electrostatic potential energy. After this, the α particle begins to retrace its path. This distance r_o is known as the distance of the closest approach.
Let *m* and *v* be the mass and velocity of the α particle directed towards the centre of the nucleus. Then, the kinetic energy of the particle,

$$E_k = \frac{1}{2}mv^2$$

Since, charge of an α-particle is 2e and that of the nucleus of the atom is Ze, the electrostatic potential energy of the α particle, when at a distance r_o from the centre of the nucleus is given by,

$$E_P = \frac{1}{4\pi\varepsilon_0}\frac{(2e)(Ze)}{r_0}$$

where Z is the atomic number of the atom and ε_o, the permittivity of free space.
On reaching the distance of the closest approach r_o, the kinetic energy of the α particle appears as its potential energy.

$$E_p = E_k$$

$$\frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{(2e)(Ze)}{r_0}$$

$$r_0 = \frac{1}{4\pi\varepsilon_0}\frac{4Ze^2}{mv^2}$$

Drawbacks
Rutherford atom model offered serious difficulties as regards the stability of the atom. Following are the two drawbacks of Rutherford's model:
(i)The electron in the circular orbit experiences a centripetal acceleration. According to electromagnetic theory, an accelerated electric charge must radiate energy in the form of electromagnetic waves.
Therefore, if the accelerated electron loses energy by radiation, the energy of the electron continuously decreases and it must spiral down into the nucleus. Thus, the atom cannot be stable. But, it is well known that most of the atoms are stable.
(ii) According to classical electromagnetic theory, the accelerating electron must radiate energy at a frequency proportional to the angular velocity of the electron. Therefore, as the electron spiral towards the nucleus, the angular velocity tends to become infinity and hence the frequency of the emitted energy will tend to infinity.
This will result in a continuous spectrum with all possible wavelengths. But experiments reveal only line spectra of fixed wavelength from atoms.

Bohr atom model

Neils Bohr in 1913, modified Rutherford's atom model in order to explain the stability of the atom and the emission of sharp spectral lines. He proposed the following postulates:

(i) An electron cannot revolve round the nucleus in all possible orbits. The electrons can revolve round the nucleus only in those allowed or permissible orbits for which the angular momentum of the electron is an integral multiple of h/2π
(where *h* is Planck's constant = 6.626×10^{-34} Js).
These orbits are called stationary orbits or non-radiating orbits and an electron revolving in these orbits does not radiate any energy.
If *m* and *v* are the mass and velocity of the electron in a permitted orbit of radius *r* then angular momentum of electron = $mvr = nh/2\pi$,

where n is called principal quantum number and has the integral values 1,2,3 … This is called Bohr's quantization condition.
(ii) An atom radiates energy, only when an electron jumps from a stationary orbit of higher energy to an orbit of lower energy. If the electron jumps from an orbit of energy E_2 to an orbit of energy E_1, a photon of energy $h\nu = E_2 - E_1$ is emitted. This condition is called Bohr's frequency condition.

Orbital velocity of electrons

Consider an atom whose nucleus has a positive charge Ze, where Z is the atomic number that gives the number of protons in the nucleus and e, the charge of the electron which is numerically equal to that of proton. Let an electron revolve around the nucleus in the nth orbit of radius r_n.
By Coulomb's law, the electrostatic force of attraction between the nucleus and the electron

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r_n^2}$$

Since, the electron revolves in a circular orbit, it experiences a centripetal force

$$\frac{mv_n^2}{r_n}$$

The necessary centripetal force is provided by the electrostatic force of attraction
For equilibrium, from equations

$$\frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r_n^2} = \frac{mv_n^2}{r_n}$$

$$v_n^2 = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r_n m}$$

$$v_n = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n m}} ---eq(1)$$

Radius of the nth orbit (r_n)

The angular momentum of an electron in nth orbit is,

$$L = mv_n r_n ---eq(2)$$

By Bohr's first postulate, the angular momentum of the electron
$L = nh/2\pi$ -----eq(2)
where n is an integer and is called as the principal quantum number.
From equations. (2) and (2),

$$mv_n r_n = \frac{nh}{2\pi} ----eq(3)$$

Substituting value of velocity from eq(1)

$$m\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n m}}r_n = \frac{nh}{2\pi}$$

Squaring on both sides

$$m^2\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n m}r_n^2 = \frac{n^2h^2}{4\pi^2}$$

$$r_n = \frac{n^2h^2\varepsilon_0}{\pi m Ze^2} \; - - - \; eq(4)$$

From equation (4), it is seen that the radius of the nth orbit is proportional to the square of the principal quantum number.

Therefore, the radii of the orbits are in the ratio 1 : 4 : 9....

For hydrogen atom, Z = 1 and n = 1 we get r_1 = 0.53Å

Energy of an electron in the nth orbit (En)

The total energy of the electron is the sum of its potential energy and kinetic energy in its orbit. The potential energy of the electron in the nth orbit is given by,

$$E_p = \frac{1}{4\pi\varepsilon_0}\frac{-Ze^2}{r_n}$$

The kinetic energy of the electron in the n^{th} orbit is,

$$E_k = \frac{1}{2}mv_n^2$$

Substituting value of velocity from equation (1)

$$E_k = \frac{1}{2}m\frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r_n m}$$

$$E_k = \frac{1}{8\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

Total energy = E_P + E_K

$$E = \frac{1}{4\pi\varepsilon_0}\frac{-Ze^2}{r_n} + \frac{1}{8\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

$$E = \frac{-1}{8\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

Substituting value of r_n from equation(4)

$$E = \frac{-Z^2me^4}{8\varepsilon_0 n^2h^2} \; - - - - eq(5)$$

From equation (5), it is seen that the radius of the n^{th} orbit is inversely proportional to the square of the principal quantum number

For hydrogen atom, Z = 1, and substituting other values in equation (5) we get

$$E_n = \frac{-13.6}{n^2} eV \quad ----eq(6)$$

As there is a negative sign in equation (6), it is seen that the energy of the electron in its orbit increases as n increases.

Frequency of spectral line

According to Bohr's second postulate, when an electron jumps from an outer orbit of quantum number n to an inner orbit of quantum number m, the frequency of the photon emitted is given by,

$$\nu = \frac{E_m - E_n}{h}$$

Using equation for energy, equation(6)

$$\nu = \frac{Z^2 m e^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \quad ----eq(7)$$

The wave number $\bar{\nu}$ of a radiation is defined as number of waves per unit length. It is equal to reciprocal of the wavelength

$$\bar{\nu} = \frac{1}{\lambda} = \frac{\nu}{c} \quad [\because c = \nu\lambda]$$

From equation(7)

$$\bar{\nu} = \frac{Z^2 m e^4}{8\varepsilon_0^2 c h^3}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) ----eq(8)$$

For Hydrogen Z = 1

$$\bar{\nu} = \frac{m e^4}{8\varepsilon_0^2 c h^3}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

$$\bar{\nu} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

where R is a constant called Rydberg's constant
Substituting the known values, we get $R = 1.094 \times 10^7 m^{-1}$

Spectral series of hydrogen atom

Whenever an electron in a hydrogen atom jumps from higher energy level to the lower energy level, the difference in energies of the two levels is emitted as a radiation of particular wavelength.
It is called a spectral line. As the wavelength of the spectral line depends upon the two orbits (energy levels) between which the transition of electron takes place, various spectral lines are obtained.
The different wavelengths constitute spectral series which are the characteristic of the atoms emitting them. The following are the spectral series of hydrogen atom.

(i) Lyman series

When the electron jumps from any of the outer orbits to the first orbit, the spectral lines emitted are in the ultraviolet region of the spectrum and they are said to form a series called Lyman series

$$\bar{\nu} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

Here n = 1 and m = 2, 3, 4, ….

$$\bar{\nu} = R\left(1 - \frac{1}{m^2}\right)$$

(ii) Balmer series

When the electron jumps from any of the outer orbits to the second orbit, we get a spectral series called the Balmer series. All the lines of this series in hydrogen have their wavelength in the visible region. Here n=2, m = 3,4,5 … The wave number of the Balmer series is,

$$\bar{\nu} = R\left(\frac{1}{2^2} - \frac{1}{m^2}\right)$$

The first line in this series (m = 3), is called the H_α-line, the second (m=4), the H_β-line and so on.

(iii) Paschen series

This series consists of all wavelengths which are emitted when the electron jumps from outer most orbits to the third orbit. Here m = 4,5,6 … and n = 3. This series is in the infrared region with the wave number given by

$$\bar{\nu} = R\left(\frac{1}{3^2} - \frac{1}{m^2}\right)$$

(iv) Brackett series

The series obtained by the transition of the electron from m = 5, 6...and n = 4 is called Brackett series. The wavelengths of these lines are in the infrared region. The wave number is

$$\bar{\nu} = R\left(\frac{1}{4^2} - \frac{1}{m^2}\right)$$

(v) Pfund series

The lines of the series are obtained when the electron jumps from any state m = 6, 7... and n=5. This series also lies in the infrared region. The wave number is,

$$\bar{\nu} = R\left(\frac{1}{5^2} - \frac{1}{m^2}\right)$$

Shortcomings of Bohr's theory

Bohr's theory was able to explain successfully a number of experimental observations and has correctly predicted the spectral lines of hydrogen atom. However, the theory fails in the following aspects.

(i) The theory could not account for the spectra of atoms more complex than hydrogen.

(ii) The theory does not give any information regarding the distribution and arrangement of electrons in an atom.

(iii) It does not explain the experimentally observed variations in intensity of the spectral lines of the element.

(iv) When the spectral line of hydrogen atom is examined by spectrometers having high resolving power, it is found that a single line is composed of two or more close components. This is known as the fine structure of spectral lines. Bohr's theory could not account for the fine structure of spectral lines.

(v) It is found that when electric or magnetic field is applied to the atom, each of the spectral line split into several lines. The former effect is called as Stark effect, while the latter is known as Zeeman effect.

Excitation and Ionization Potential

Suppose electron revolving in a stationary orbit of an atom absorbs specific energy, and jumps to an orbit of higher energy. This process is called excitation and the atom is said to be in the excited state. The energy absorbed to move electron from one orbit to the other is called excitation potential,

If n = 1 is called ground state, n = 2 is called first excited state, n =3 is called second excited state

If the energy supplied is large enough to remove an electron from the atom, then the atom is said to be ionized. The minimum energy needed to ionize an atom is called ionization energy. Note that energy for $n = \infty$ is taken as zero.

The minimum amount of energy to release an electron from ground state is called the first or principal ionization energy. The ionization of electrons from higher energy state is termed depending on the their quantum number . For n = 2, second ionization energy, for n = 3, third ionization energy etc.

De Broglie's explanation of Bohr's second postulate of quantization

Since the electron do not radiate energy in stationary orbits, the De broglie wave associated with it must be stationary wave. The electron-wave selects only those orbits for which the circumference of the orbit is equal to an integral number (n) of the associated de Broglie wavelength. If it is not then, each wave travel around the orbit will not be in phase and would interfere in such a way that their average intensity would be zero. In this case, an electron cannot be found in such an orbit.

Thus, necessary condition for permitted orbits is
2πr = nλ
Where n = number of waves (integer) and r = radius of orbit
And $\lambda = \text{wavelength of de Broglie wave} = \text{h/mv}$

$$\therefore 2\pi r = \frac{nh}{mv}$$

Now angular momentum

$$l = mvr = \frac{nh}{2\pi}$$

Solved Numerical

Q) The wavelength of the K_α line emitted by a hydrogen atom like element is 0.32Å. Determine the wavelength of K_β line emitted by the same element

Solution
From formula for hydrogen like element

$$\bar{\nu} = \frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

For K_α n = 1 and m = 2

$$\frac{1}{0.32 \times 10^{-10}} = R\left(1 - \frac{1}{2^2}\right)$$

$$\frac{1}{0.32 \times 10^{-10}} = R\left(\frac{3}{4}\right) \quad ---eq(1)$$

For K_β n = 1 and m = 3

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{3^2}\right)$$

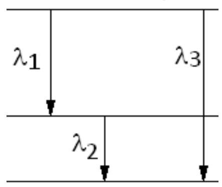
$$\frac{1}{\lambda} = R\left(\frac{8}{9}\right) \quad ---eq(2)$$

Taking ratio of equation (1) and (2) we get

$$\lambda = 0.32 \times 10^{-10}\left(\frac{3}{4}\right)\left(\frac{9}{8}\right) = 0.27 \times 10^{-10}$$

λ = 0.27 Å

Q) Find the relation between three wavelengths λ_1, λ_2, λ_3 from the energy level diagram shown in figure



Solution

From figure $E_3 = E_1 + E_2$ thus

$$\frac{hc}{\lambda_3} = \frac{hc}{\lambda_1} + \frac{hc}{\lambda_2}$$
$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

$$\lambda_3 = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$

Q) The wavelength of first line of Lyman series of hydrogen is identical to that of the second line of Balmer series for some hydrogen like ion X. Calculate energies of the first four levels of X. Also find its ionization potential (Given: Ground state binding energy of hydrogen atom 13.6 eV)

Solution:

From the formula

$$\frac{1}{\lambda} = Z^2 R \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

For first line of Lyman series of hydrogen Z = 1, n = 1 and m = 2

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{2^2}\right) = R\left(\frac{3}{4}\right) \quad ---eq(1)$$

For second line of Balmer series of X, n = 2 and m = 4

$$\frac{1}{\lambda} = Z^2 R \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

$$\frac{1}{\lambda} = ZR\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = Z^2 R\left(\frac{3}{16}\right) ----eq(2)$$

From equation (1) and (2)

$$R\left(\frac{3}{4}\right) = Z^2 R\left(\frac{5}{16}\right)$$

Z = 2

From the formula for energy of electron in nth orbit

$$E = \frac{-Z^2 m e^4}{8\varepsilon_0 n^2 h^2}$$

For ionization potential n = 1

Ionization potential for H ; Z =1

$$E_H = \frac{-me^4}{8\varepsilon_0 h^2}$$

Ionization potential for X ; Z =2

$$E_X = \frac{-2^2 m e^4}{8\varepsilon_0 h^2} = 4E_H$$

$$E_x = 4 \times (-13.6) = 54.4 \ eV$$

Ionization potential of x = 54.4 ev

Energy of ground state or first energy level n=1

$$E_2 = \frac{-54.4}{1^2} = -54.4 \, eV$$

Energy of first excited state or second energy level n = 2

$$E_2 = \frac{-54.4}{2^2} = -13.6 \, eV$$

Energy of first excited state or third energy level n = 3

$$E_3 = \frac{-54.4}{3^2} = -6.04 \, eV$$

Energy of first excited state n = 4 or fourth energy level n = 4

$$E_3 = \frac{-54.4}{4^2} = -3.4 \, eV$$

Q) A body of mass m is attached at one end of spring of force constant K. It is given a motion on a circular path of radius r. Assuming that there are integer number of waves representing this particle on the circumference of the circle and using Bohr's quantum condition prove that the quantized energy is given by $E_n = nh\omega/2\pi$

given angular frequency $\omega = \sqrt{\frac{k}{m}}$

Solution:

Total energy = Kinetic energy + spring potential energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \ ----eq(1)$$

Here k is restoring force constant of spring and k = $m\omega^2$ and $\omega=v/r$

$$k = \frac{mv^2}{r^2}$$

On substituting value of k in equation (1) we get

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2 \ ----eq(2)$$

According to Bohr's postulate angular momentum

$$mvr = \frac{nh}{2\pi}$$

$$\therefore mv^2 = \frac{nhv}{2\pi r} \ --- eq(3)$$

From equation(2) and (3) we get

$$E = \frac{nhv}{2\pi r}$$

But $\omega=v/r$

$$\therefore E = \frac{nh\omega}{2\pi}$$

Q) Suppose the potential energy between electron and proton at a distance r is given by $-ke^2/3r^3$. Use Bohr's theory to obtain energy levels of such a hypothetical atom
Solution:
We know that for conservative force field
F = -dU/dr

$$\therefore F = -\frac{d}{dr}\left(\frac{ke^2}{3r^3}\right) = \frac{ke^2(3)}{3r^4} = \frac{ke^2}{r^4}$$

Now electrostatic force = centripetal force at equilibrium

$$\frac{ke^2}{r^4} = \frac{mv^2}{r}$$

$$\therefore v^2 = \frac{ke^2}{r^3}$$

Now according to Bohr's postulate

$$mvr = \frac{nh}{2\pi}$$

Squaring on both sides

$$m^2v^2r^2 = \frac{n^2h^2}{(2\pi)^2}$$

Substituting value of v

$$m^2\frac{ke^2}{r^3}r^2 = \frac{n^2h^2}{(2\pi)^2}$$

$$r = \frac{m^2ke^2(2\pi)^2}{n^2h^2}$$

Now we know that total energy of electron = (1/2) potential energy

$$\therefore E = \frac{ke^2}{6r^3}$$

By substituting value of r from above

$$E = \frac{ke^2}{6}\frac{(n^2h^2)^3}{(m^2ke^2(2\pi)^2)^3}$$

$$\therefore E = \frac{(nh)^6}{6(ke^2)^2m^3(2\pi)^6}$$

Q) The energy of an electron in an excited hydrogen atom is -3.4eV. Determine the angular momentum using Bohr's theory. Given: Ground state energy of hydrogen atom = -13.6 eV
Solution:

$$-3.4 = -\frac{13.6}{n^2}$$

n=2
Angular momentum

$$L = \frac{nh}{2\pi} = \frac{2 \times 6.625 \times 10^{-34}}{2 \times 3.14} = 2.1 \times 10^{-34} Js$$

Q) At what temperature will the average molecular kinetic energy in gaseous hydrogen be equal to the binding energy of a hydrogen atom? Boltzmann constant $k_B = 1.3\times10^{-23}$ JK^{-1}
Solution
Kinetic energy of molecule of gas is given by $\frac{3}{2}k_BT$

$$\therefore \frac{3}{2}k_BT = 13.6eV$$

$$T = \frac{13.6 \times 1.6 \times 10^{-19} \times 2}{3 \times k_B}$$

$$T = \frac{13.6 \times 1.6 \times 10^{-19} \times 2}{3 \times 1.3 \times 10^{-23}} = 11.16 \times 10^4$$

$$T = 1.116 \times 10^5 K$$

----------------END--------------

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NUCLEUS

Nucleus

The nucleus consists of the elementary particles, protons and neutrons which are known as nucleons. A proton has positive charge of the same magnitude as that of electron and its rest mass is about 1836 times the mass of an electron. A neutron is electrically neutral, whose mass is almost equal to the mass of the proton. The nucleons inside the nucleus are held together by strong attractive forces called nuclear forces.

A nucleus of an element is represented as $_{Z}X^{A}$,

Where, X = Chemical symbol of the element.

Z = Atomic number which is equal to the number of protons

A = Mass number which is equal to the total number of protons and neutrons.

The number of neutrons is represented as N which is equal to A−Z.

For example: The chlorine nucleus is represented as $_{17}Cl^{35}$. It contains 17 protons and 18 neutrons.

Atomic mass is expressed in atomic mass unit (u), defined as (1/12)th of the mass of the carbon (C^{12}) atom. According to this definition.

$$1u = \frac{1.992647 \times 10^{-26}}{12} kg$$

$$1u = 1.660539 \times 10^{-27} \text{ kg}$$

Classification of nuclei

(i) Isotopes

Isotopes are atoms of the same element having the same atomic number Z but different mass number A. The nuclei $_{1}H^{1}$, $_{1}H^{2}$ and $_{1}H^{3}$ are the isotopes of hydrogen. As the atoms of isotopes have identical electronic structure, they have identical chemical properties and placed in the same location in the periodic table.

The relative abundance of different isotopes differs from element to element. Chlorine, for example, has two isotopes having masses 34.98 u and 36.98 u, which are nearly integral multiples of the mass of a hydrogen atom. The relative abundances of these isotopes are 75.4 and 24.6 per cent, respectively. Thus, the average mass of a chlorine atom is obtained by the weighted average of the masses of the two isotopes, which works out to be

$$= \frac{(75.4 \times 34.98) + (24.6 \times 36.98)}{100} = 35.47\ u$$

(ii) Isobars

Isobars are atoms of different elements having the same mass number A, but different atomic number Z. The nuclei $_8O^{16}$ and $_7N^{16}$ represent two isobars. Since isobars are atoms of different elements, they have different physical and chemical properties.

(iii) Isotones

Isotones are atoms of different elements having the same number of neutrons. $_6C^{14}$ and $_8O^{16}$ are some examples of isotones.

(iv) Isomers

For some nuclei Z values are same and A values are also same but their radioactive properties are different. They are called isomers of each other. $_{35}B^{80}$ has one pair of isomers

Discovery of Neutron

James Chadwick who observed emission of neutral radiation when beryllium nuclei were bombarded with alpha-particles. (α-particles are helium nuclei).
It was found that this neutral radiation could knock out protons from light nuclei such as those of helium, carbon and nitrogen.
Application of the principles of conservation of energy and momentum showed that if the neutral radiation consisted of photons, the energy of photons would have to be much higher than is available from the bombardment of beryllium nuclei with α-particles.
The clue to this puzzle, which Chadwick satisfactorily solved, was to assume that the neutral radiation consists of a new type of neutral particles called *neutrons*.
From conservation of energy and momentum, he was able to determine the mass of new particle 'as very nearly the same as mass of proton'. Mass of neutron m_n = 1.00866 u
Or 1.6749×10^{-27} kg

General properties of nucleus

Nuclear size
According to Rutherford's α−particle scattering experiment, the distance of the closest approach of α − particle to the nucleus was taken as a measure of nuclear radius, which is approximately 10^{-15}m.
If the nucleus is assumed to be spherical, an empirical relation is found to hold good between the radius of the nucleus R and its mass number A. It is given by

$$R = R_0 A^{\frac{1}{3}}$$

Where, $R_0 = 1.2 \times 10^{-15}$ m. or is equal to 1.2 F (1 Fermi, F = 10^{-15} m)
This means the volume of the nucleus, which is proportional to R^3 is proportional to A.
Thus the density of nucleus is a constant, independent of A.

Nuclear density

The nuclear density ρ_N can be calculated from the mass and size of the nucleus

$$\rho_N = \frac{nuclear\ mass}{nuclear\ volume}$$

Nuclear mass = Am_N

where, A = mass number

m_N = mass of one nucleon and is approximately equal to 1.67×10^{-27} kg

Nuclear volume V_N

$$V_N = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(R_0 A^{\frac{1}{3}}\right)^3$$

$$\rho_N = \frac{Am_N}{\frac{4}{3}\pi\left(R_0 A^{\frac{1}{3}}\right)^3} = \frac{m_N}{\frac{4}{3}\pi R_0^3}$$

Substituting the known values, the nuclear density is calculated as 1.816×10^{17} kg m^{-3} which is almost a constant for all the nuclei irrespective of its size. The high value of the nuclear density shows that the nuclear matter is in an extremely compressed state.

Nuclear mass

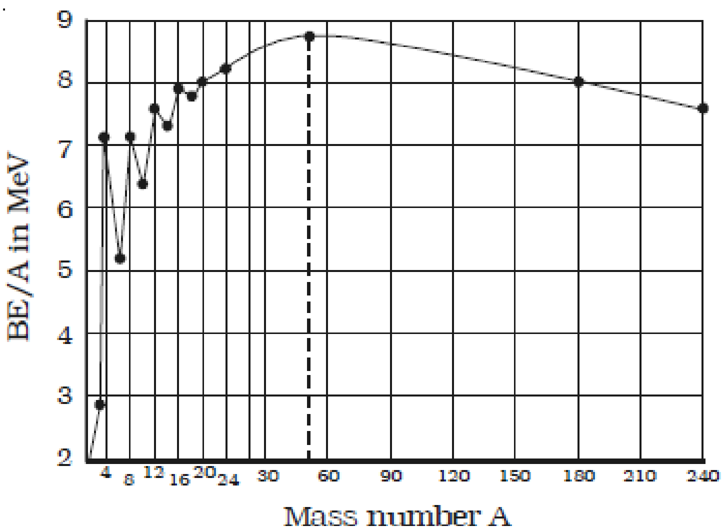
As the nucleus contains protons and neutrons, the mass of the nucleus is assumed to be the mass of its constituents.
Assumed nuclear mass = $Zm_P + Nm_N$,
Where, m_p and m_N are the mass of a proton and a neutron respectively
Z = number of protons
N = number of neutrons
However, from the measurement of mass by mass spectrometers, it is found that the mass of a stable nucleus (m) is less than the total mass of the nucleons.
i.e mass of a nucleus, $m < (Zm_p + Nm_N)$
$Zm_p + Nm_N - m = \Delta m$
where Δm is the mass defect
Thus, the difference in the total mass of the nucleons and the actual mass of the nucleus is known as the mass defect.

Note : In any mass spectrometer, it is possible to determine only the mass of the atom, which includes the mass of Z electrons.
If M represents the mass of the atom, then the mass defect can be written as

$$\Delta m = Zm_P + Nm_N + Zm_e - M$$

energy equivalent of 1 amu = 931 MeV

Binding energy



When the protons and neutrons combine to form a nucleus, the mass that disappears (mass defect, Δm) is converted into an equivalent amount of energy (Δmc^2). This energy is called the binding energy of the nucleus.
∴ Binding energy = $[Zm_P + Nm_N - m]$ c^2
Binding energy = $\Delta m\, c^2$
The binding energy of a nucleus determines its stability against disintegration. In other words, if the binding energy is large, the nucleus is stable and vice versa.
The binding energy per nucleon is

$$\frac{BA}{A} = \frac{Binding\ energy\ of\ nucleus}{Total\ number\ of\ nucleons}$$

It is found that the binding energy per nucleon varies from element to element. A graph is plotted with the mass number A of the nucleus along the X−axis and binding energy per nucleon along the Y-axis.

Explanation of binding energy curve

(i) The binding energy per nucleon increases sharply with mass number A upto 20. It increases slowly after A = 20.
For A<20, there exists recurrence of peaks corresponding to those nuclei, whose mass numbers are multiples of four and they contain not only equal but also even number of protons and neutrons. Example: ${}_2He^4$, ${}_4Be^8$, ${}_6C^{12}$, ${}_8O^{16}$, and ${}_{10}Ne^{20}$.
The curve becomes almost flat for mass number between 30 and 170. Beyond 170, it decreases slowly as A increases.

(ii) The binding energy per nucleon reaches a maximum of 8.8 MeV at A=56, corresponding to the iron nucleus (${}_{26}Fe^{56}$). Hence, iron nucleus is the most stable.

(iii) The average binding energy per nucleon is about 8.5 MeV for nuclei having mass number ranging between 30 and 170. These elements are comparatively more stable and non radioactive.

(iv) For higher mass numbers the curve drops slowly and the BE/A is about 7.6 MeV for uranium. Hence, they are unstable and radioactive.

(v) The lesser amount of binding energy for lighter and heavier nuclei explains nuclear fusion and fission respectively. A large amount of energy will be liberated if lighter nuclei are fused to form heavier one (fusion) or if heavier nuclei are split into lighter ones (fission).

Nuclear force

The nucleus of an atom consists of positively charged protons and uncharged neutrons. According to Coulomb's law, protons must repel each other with a very large force, because they are close to each other and hence the nucleus must be broken into pieces. But this does not happen. It means that, there is some other force in the nucleus which overcomes the electrostatic repulsion between positively charged protons and binds the protons and neutrons inside the nucleus. This force is called nuclear force.

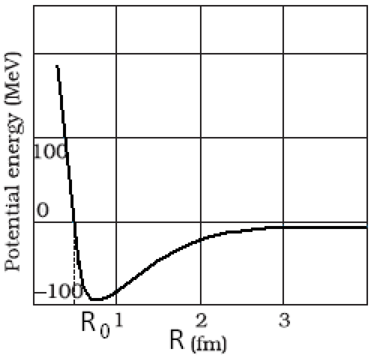
(i) Nuclear force is charge independent. It is the same for all the three types of pairs of nucleons (n−n), (p−p) and (n−p). This shows that nuclear force is not electrostatic in nature

(ii) Nuclear force is the strongest known force in nature. Nuclear force is about 1040 times stronger than the gravitational force.

(iii) Nuclear force is a short range force. It is very strong between two nucleons which are less than 10^{-15} m apart and is almost negligible at a distance greater than this. On the other hand electrostatic, magnetic and gravitational forces are long range forces that can be felt easily.

The nuclear force between two nucleons falls rapidly to zero as their distance is more than a few femtometres. This leads to *saturation of forces* in a medium or a large-sized nucleus, which is the reason for the constancy of the binding energy per nucleon.

A rough plot of the potential energy between two nucleons as a function of distance is shown in the Fig.



The potential energy is a minimum at a distance R_0 of about 0.8 fm. This means that the force is attractive for distances larger than 0.8 fm and repulsive if they are separated by distances less than 0.8 fm.

However, the present view is that the nuclear force that binds the protons and neutrons is not a fundamental force of nature but it is secondary.

Radioactivity

The phenomenon of spontaneous emission of highly penetrating radiations such as α, β and γ rays by heavy elements having atomic number greater than 82 is called radioactivity and the substances which emit these radiations are called radioactive elements. The radioactive phenomenon is spontaneous and is unaffected by any external agent like temperature, pressure, electric and magnetic fields etc.
Experiments performed showed that radioactivity was a nuclear phenomenon in which an unstable nucleus undergoes a decay. This is referred to as *radioactive decay*. Three types of radioactive decay occur in nature :

(i) α-decay in which a helium nucleus $_2He^4$ is emitted.

(ii) β-decay in which electrons or positrons (particles with the same mass as electrons, but with a charge exactly opposite to that of electron) are emitted;

(iii) γ-decay in which high energy (hundreds of keV or more) photons are emitted.
Each of these decay will be considered in subsequent sub-sections

Law of radioactive decay

In any radioactive sample, which undergoes α, β or γ-decay, it is found that the number of nuclei undergoing the decay per unit time is proportional to the total number of nuclei in the sample. If *N* is the number of nuclei in the sample and Δ*N* undergo decay in time Δ*t* then

$$\frac{\Delta N}{\Delta t} \propto N$$

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

where λ is called the radioactive *decay constant* or *disintegration constant*.
The change in the number of nuclei in the sample is dN = – ΔN in time Δt. Thus the rate of change of N is (in the limit Δt → 0)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N \quad ---(1)$$
$$\frac{\mathrm{d}N}{N} = -\lambda \mathrm{d}t$$

Now, integrating both sides of the above equation, we get,

$$\int_{N_0}^{N} \frac{\mathrm{d}N}{N} = -\lambda \int_{t_0}^{t} dt$$

$$\ln N - \ln N_0 = -\lambda\,(t - t_0)$$

Here N_0 is the number of radioactive nuclei in the sample at some arbitrary time t0 and N is the number of radioactive nuclei at any subsequent time t. Setting t_0 = 0 and rearranging Equation gives us

$$ln\frac{N}{N_0} = -\lambda t$$
$$N = N_0 e^{-\lambda t} \quad ---(2)$$

Above equation represents law of radioactive decay
Differentiating equation (2) we get

$$\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t}$$
$$-\frac{dN}{dt} = \lambda N_0 e^{-\lambda t}$$

Term -dN/dt is called the rate of disintegration or activity I of element at time t
From equation (1) we get I = λN
Thus

$$I = I_0 e^{-\lambda t}$$

Is alternative form of the law of *law of radioactive decay*
The SI unit for activity is becquerel, named after the discoverer of radioactivity, Henry Becquerel. It is defined as

1 becquerel = 1Bq = 1 decay per second

An older unit, the curie, is still in common use:

1 curie = 1 Ci = 3.7 × 10^{10} Bq (decays per second

Half life period

The half life period of a radioactive element is defined as the time taken for one half of the radioactive element to undergo disintegration.
From the law of disintegration

$$N = N_0 e^{-\lambda t}$$

Let $T_{½}$ be the half life period. Then, at t = $T_{½}$, N =N_0/2

$$\frac{N_0}{2} = N_0 e^{\lambda T_{1/2}}$$

$$\ln 2 = \lambda T_{½}$$

$$T_{1/2} = \frac{ln2}{\lambda}$$

$$T_{1/2} = \frac{0.693}{\lambda}$$

Fraction of radioactive substance left undecayed is,

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^n$$

where n is the number of half lives.

$$n = \frac{total\ time}{half\ life}$$

The half life period is inversely proportional to its decay constant. For a radioactive substance, at the end of T½, 50% of the material remain unchanged. After another T½ i.e., at the end of 2 T½, 25% remain unchanged. At the end of 3 T½, 12.5% remain unchanged and so on.

Solved Numerical

1. The half life of radon is 3.8 days. After how many days 19/20 of the sample will decay

Solution

If we take 20 parts as N_0 then N=1
From formula

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^n$$

$$\frac{1}{20} = \left(\frac{1}{2}\right)^n$$

$$20 = 2^n$$

using log we get log20=nlog2
1.3010 = n× 0.3010 thus n = 4.322

From formula

$$n = \frac{total\ time}{half\ life}$$

$$4.322 = \frac{t}{3.8}$$

t=16.42 days

Q) An archaeologist analysis of the wood in a prehistoric structure revels that the ratio of ^{14}C (half life = 5700 years) to ordinary carbon is only one fourth in the cells of living plants. What is the age of the wood?

Solution:

If we take $N_0 = 1$ then N = ¼

From formula

$$N = N_0 e^{-\lambda t}$$

$$\frac{N}{N_0} = e^{-\lambda t}$$

$$\frac{1}{4} = e^{-\lambda t}$$

$$4 = e^{\lambda t}$$

Taking log to the base e on both sides

ln4 = λt

Converting to log to base 10

2.303 log4 = λt

From formula for half life

$$\lambda = \frac{0.693}{T_{1/2}}$$

$$2.303\ log4\ = \frac{0.693}{T_{1/2}} t$$

$$t = \frac{2.303\ log4\ \times T_{1/2}}{0.693}$$

$$t = \frac{2.303\ \times 0.6021\ \times 5700}{0.693}$$

t =11400 years

Q) A radioactive nucleus X decays to nucleus Y with a decay constant $\lambda_X = 0.1\ s^{-1}$.

Y further decays to a stable nucleus Z with decay constant $\lambda_Y = \frac{1}{30} s^{-1}$

Initially there are only X nuclei and their number is $N_0 = 10^{20}$.

Set up the rate equation for the population of X, Y, and Z. The population of the Y nucleus as function of time is given by

$$N_Y = \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}\left(e^{-\lambda_y t} - e^{-\lambda_x t}\right)$$

Find the time at which N_Y is maximum and determine the population X and Z at that instant

Solution

Rate equation for X from Law of radioactive decay

$$\frac{dN_X}{dt} = -\lambda_X N_X \ \ ---eq(1)$$

Rate of decay of Y depends on generation of Y due to decay of X and population of Y at that instant thus

$$\frac{dN_Y}{dt} = \lambda_X N_X - \lambda_y N_y \ ---eq(2)$$

Rate of disintegration of Z depends only on rate of generation of Y thus

$$\frac{dN_Z}{dt} = \lambda_Y N_Y \ ---eq(3)$$

For N_Y to be maximum eq(2) should become zero

$$\lambda_X N_X - \lambda_y N_y = 0$$

$$\lambda_X N_X = \lambda_y N_y ---eq(4)$$

We know that

$$N_X = N_0 e^{-\lambda_x t} ---eq(5)$$

Given

$$N_Y = \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}\left(e^{-\lambda_y t} - e^{-\lambda_x t}\right) \ ---eq(6)$$

Substituting values of N_X and N_Y from equation (5) and (6) in equation (4) we get

$$\lambda_X N_0 e^{-\lambda_x t} = \lambda_y \frac{N_0\lambda_X}{\lambda_X - \lambda_Y}\left(e^{-\lambda_y t} - e^{-\lambda_x t}\right)$$

$$e^{-\lambda_x t} = \lambda_y \frac{1}{\lambda_X - \lambda_Y}\left(e^{-\lambda_y t} - e^{-\lambda_x t}\right)$$

$$\frac{\lambda_X - \lambda_Y}{\lambda_y} = \frac{\left(e^{-\lambda_y t} - e^{-\lambda_x t}\right)}{e^{-\lambda_x t}}$$

$$\frac{\lambda_X}{\lambda_y} - 1 = \frac{e^{-\lambda_y t}}{e^{-\lambda_x t}} - 1$$

$$\frac{\lambda_X}{\lambda_y} = e^{(\lambda_x t - \lambda_y t)}$$

Taking log on both side

$$(\lambda_x - \lambda_y)t = ln\left(\frac{\lambda_X}{\lambda_y}\right)$$

$$t = \frac{1}{(\lambda_x - \lambda_y)} ln\left(\frac{\lambda_X}{\lambda_y}\right)$$

$$t = \frac{1}{0.1 - \frac{1}{30}} ln\left(\frac{0.1}{1/30}\right)$$

$$t = 15ln(3)$$
$$t = 2.303 \times 15 \times log(3)$$

t=16.48s is time when population of Y is maximum

To find population of X and Z at t = 16.48s

We will use equation

$$N_X = N_0 e^{-\lambda_x t}$$

$$N_X = 10^{20} \times e^{-0.1\times16.48} = 10^{20}\frac{1}{e^{1.648}}$$

[Calculation of $e^{1.648}$

$\log_{10}(e^{1.648})$ =(1.648)$\log_{10}e$

$\log_{10}(e^{-1.68})$ =(1.648) × 0.434 = 0.7155

antilog (0.7155) = 5.194

thus value of $e^{1.648}$ = 5.194]

$\therefore N_X = 10^{20} \times \frac{1}{5.194} = 1.925 \times 10^{19}$

From equation (4)

$$\lambda_X N_X = \lambda_y N_y$$
$$N_Y = N_X \frac{\lambda_X}{\lambda_y}$$
$$N_Y = 1.925 \times 10^{19} \times \frac{0.1}{1/30} = 3 \times 1.925 \times 10^{19} = 5.772 \times 10^{19}$$

Now $N_Z = N_0 - N_X - N_Y$

$N_z = (10\times10^{19}) - (1.925\times10^{19}) - (5.772\times10^{19}) = 2.303\times10^{19}$

Q) In a mixture of two elements A and B having decay constants 0.1 day^{-1} and 0.2 day^{-1} respectively; initially the activity of A is 3 times that of B. If the initial activity of the mixture is 2mCi, find the activity of it after 10 days

Solution:

λ_A= 0.1 day^{-1} λ_B=0.2 day^{-1}

$(I_0)_A = 3(I_0)_B$

At time t = 0, activity of mixture is

$I_0 = (I_0)_A + (I_0)_B$

$I_0 = 3(I_0)_B + (I_0)_B$

$2 = 4(I_0)_B$
$(I_0)_B = 0.5$ mCi
$(I_0)_A = 1.5$ mCi
At time t , activity of A is

$$I_A = (I_0)_A e^{-\lambda_A t} = (1.5)e^{-(0.1)(10)}$$
$$I_A = \frac{1.5}{e} = \frac{1.5}{2.718} = 0.552\ mCi$$

At time t , activity of B is

$$I_B = (I_0)_B e^{-\lambda_B t} = (0.5)e^{-(0.2)(10)}$$
$$I_B = \frac{1.5}{e^2} = \frac{1.5}{(2.718)^2} = 0.067\ mCi$$

At time t, total activity of the mixture
$I = I_A + I_B = 0.552 + 0.067 = 0.619$ mCi

Mean life (τ)

The time-interval, during which the number of nuclei of a radioactive element becomes equal to the e^{th} part of its original number, is called the mean life or average life τ of the element .
When $N = N_0/e$, we can put t = mean life = τ

$$\therefore \frac{N_0}{e} = N_0 e^{-\lambda\tau}$$

$$e = e^{\lambda\tau}$$

$$\tau = \frac{1}{\lambda}$$

Thus mean life is equal to the reciprocal of the decay constant
Relation between $T_{1/2}$ and mean life

$$T_{1/2} = \frac{0.693}{\lambda}$$

$$T_{1/2} = 0.693\tau$$

Solved Numerical

Q) A radioactive sample emits n β particles in 2 second. In next 2 seconds, it emits 0.75n β particles. What is the mean life of the sample
Solution:
Disintegration of one nucleon give one β particle
If n β-particles are emitted then N-n nucleons are not disintegrated thus
$N-n = Ne^{-\lambda 2}$
$n = N(1-e^{-2\lambda})$ ----eq(1)

In 4 seconds total emission o is n +0.75n = (1.75)n thus
(1.75)n =N(1-$e^{-4\lambda}$) ---eq(2)
Dividing eq(2) by eq(1)

$$1.75 = \frac{1-e^{-4\lambda}}{1-e^{-2\lambda}}$$
$$\frac{7}{4} = \frac{1-e^{-4\lambda}}{1-e^{-2\lambda}}$$
$$7-7e^{-2\lambda} = 4-4e^{-4\lambda}$$
$$4e^{-4\lambda}-7e^{-2\lambda}+3=0$$
$$4e^{-4\lambda}-4e^{-2\lambda}-3e^{-2\lambda}+3=0$$
$$4e^{-2\lambda}\left(e^{-2\lambda}-1\right)-3\left(e^{-2\lambda}-1\right)=0$$
$$\left(\mathrm{e}^{-2\lambda}-1\right)\left(4e^{-2\lambda}-3\right)=0$$
$$\text{But } \mathrm{e}^{-2\lambda}-1\neq 0$$
$$\therefore 4e^{-2\lambda}-3=0$$
$$e^{-2\lambda}=\frac{3}{4}$$
$$e^{2\lambda}=\frac{4}{3}$$
$$2\lambda = ln\left(\frac{4}{3}\right)$$
$$\frac{1}{\lambda}=\frac{2}{ln\left(\frac{4}{3}\right)}=\frac{2}{ln4-ln3}$$
$$\tau=\frac{2}{ln4-ln3}$$

Radioactive displacement law

During a radioactive disintegration, the nucleus which undergoes disintegration is called a parent nucleus and that which remains after the disintegration is called a daughter nucleus. In 1913,Soddy and Fajan framed the displacement laws governing radioactivity.

α-decay

When a radioactive nucleus disintegrates by emitting an α-particle, the atomic number decreases by two and mass number decreases by four. The α-decay can be expressed as
$_{Z}X^{A} \rightarrow {}_{Z-2}Y^{A-4} + {}_{2}He^{4}$
example, when $_{92}U^{238}$ undergoes alpha-decay, it transforms to $_{90}Th^{234}$

$$\mathbf{{}_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_{2}He^{4}}$$

The alpha-decay of $_{92}U^{238}$ can occur spontaneously (without an external source of energy) because the total mass of the decay products $_{90}Th^{234}$ and 2He4 is less than the mass of the original $_{92}U^{238}$.

Thus, the total mass energy of the decay products is less than the mass energy of the original nuclide.
The difference between the initial mass energy and the final mass energy of the decay products is called the *Q value* of the process or the *disintegration energy*.
Thus, the *Q* value of an alpha decay can be expressed as

$$Q = (m_X - m_Y - m_{He})\,c^2$$

This energy is shared by the daughter nucleus and the alpha-particle, in the form of kinetic energy. Alpha-decay obeys the radioactive law

β–*decay*

In the process of β-decay, a nucleus spontaneously emits electron or positron. Positron has the same charge as that of electron but it is positive and its other properties are exactly identical to those of electron. Thus positron and the antiparticle of electron. Positron and electron are respectively written as β^+ and β^- or ${}_{+1}e^0$ and ${}_{-1}e^0$ and e^+ and e^-

β^- emission

$${}_{15}P^{32} \rightarrow {}_{16}S^{32} + {}_{-1}e^0 + \bar{\upsilon} \quad (\bar{\upsilon} \text{ is called antineutrino})$$

Compared to parent element, the atomic number of daughter element is one unit more in β^- decay
In this reaction neutron disintegrates into proton can be sated as

$$N^0 \rightarrow P^{+1} + \beta^- + \bar{\upsilon}$$

β^+ emission

$${}_{11}Na^{22} \rightarrow {}_{10}Ne^{23} + {}_{+1}e^0 + \upsilon \ (\upsilon \text{ is called neutrino})$$

Compared to parent element, the atomic number of daughter element is one unit less in β^+ decay
In this reaction Proton disintegrates into Neutron can be sated as

$$P^{+1} \rightarrow N^0 + \beta^+ + \upsilon$$

Neutrino and anti-neutrino are the anti particles of each other. They are electrically neutral and their mass is extremely small as compared to even that of electron. Their interaction with other particles is negligible and hence it is extremely difficult to detect them. They can pass without interaction even through very large matter (even through the entire earth). They have h/2π spin

γ–decay

There are energy levels in a nucleus, just like there are energy levels in atoms. When a nucleus is in an excited state, it can make a transition to a lower energy state by the emission of electromagnetic radiation.
As the energy differences between levels in a nucleus are of the order of MeV, the photons emitted by the nuclei have MeV energies and are called **gamma rays**.
Most radio-nuclides after an alpha decay or a beta decay leave the daughter nucleus in an excited state. The daughter nucleus reaches the ground state by a single transition or sometimes by successive transitions by emitting one or more gamma rays.
A well-known example of such a process is that of ${}_{27}Co^{60}$.
By beta emission, the ${}_{27}Co^{60}$ nucleus transforms into ${}_{28}Ni^{60}$ nucleus in its excited state.
The excited ${}_{28}Ni^{60}$ nucleus so formed then de-excites to its ground state by successive emission of 1.17 MeV and 1.33 MeV gamma rays.

Nuclear reactions

By bombarding suitable particles of suitable energy on a stable element, that element can be transformed into another element. Such a reaction is called artificial nuclear transmutation. Example

$$\mathbf{{}_{7}N^{14} + {}_{2}He^{4} \rightarrow {}_{8}O^{17} + {}_{1}H^{1} + Q}$$

Such process, in which change in the nucleus takes place are called nuclear reactions. Here Q is called Q –value of the nuclear reaction and it shows that the energy released in the process. If Q>0, the reaction is exoergic and if Q<0 then reaction is endoergic
Reaction can be symbolically represented as

$$\mathbf{A + a \rightarrow B + b + Q}$$

A : is called the target nucleus
a: is called projectile partile
B: is called product nucleus
b: is called emitted particle

The energy liberated $Q = [m_A + m_a - m_B - m_b]\, c^2$, here m represents the mass of respective particle

Nuclear Fission

The process of breaking up of the nucleus of a heavier atom into two fragments with the release of large amount of energy is called nuclear fission.

The fission is accompanied of the release of neutrons.
The fission reactions with ${}_{92}U^{235}$ are represented as

$$\mathbf{{}_{92}U^{235} + {}_{0}n^{1} \rightarrow {}_{56}Ba^{141} + {}_{36}Kr^{92} + 3\ {}_{0}n^{1} + Q\ ...(1)}$$
$$\mathbf{{}_{92}U^{235} + {}_{0}n^{1} \rightarrow {}_{54}Xe^{140} + {}_{38}Sr^{94} + 2\ {}_{0}n^{1} + Q\ ...(2)}$$
$$\mathbf{{}_{92}U^{235} + {}_{0}n^{1} \rightarrow {}_{51}Sb^{140} + {}_{41}Nb^{99} + 4\ {}_{0}n^{1} + Q\ ...(3)}$$

The product nuclei obtained by the fission are called the fission fragments, the neutrons are called the fission neutrons and energy is called fission energy. In the above reaction 60 different nuclei are obtained as fission fragment, having Z value between 36 and 56. The probability is maximum for formation of nucli with A = 95 and A = 140. The fission fragments are radio active and by successive emission of β^- particles results in stable nucli. The disintegration energy in fission events first appears as the kinetic energy of the fragments and neutrons. Eventually it is transferred to the surrounding matter appearing as heat.

Solved Numerical

Q In the reaction ${}_{Z}X^{A} \rightarrow {}_{Z-2}Y^{A-4} + {}_{2}He^{4}$ + Q of the nucleus X at rest, taking the ratio of mass of α-particle M_α and mass of Y-nucleus as

$$\frac{M_\alpha}{M_\beta} = \frac{4}{A-4}$$

Show that the Q-value of the reaction is given by

$$Q = K_\alpha\left(\frac{A}{A-4}\right)$$

K_α = kinetic energy of α particles

Solution:

Q-value of reaction = energy equivalent to mass-difference

$$Q = (M_X - M_Y - M_\alpha)c^2$$

Q = increase in kinetic energy

Q = ($K_\alpha + K_\beta$) – 0 (∴ X was steady)

$$Q = \frac{1}{2}M_\alpha v_\alpha^2 + \frac{1}{2}M_\beta v_\beta^2 \quad ----eq(1)$$

From conservation of momentum

$$M_Y\vec{v}_Y + M_Y\vec{v}_Y = 0$$

$M_Y v_Y = M_\alpha v_\alpha$ (in magnitude)

$$v_Y = \left(\frac{M_\alpha}{M_Y}\right)v_\alpha$$

Substituting this value in equation (1)

$$Q = \frac{1}{2}M_\alpha v_\alpha^2 + \frac{1}{2}M_Y\left(\frac{M_\alpha}{M_Y}\right)^2 {v_\alpha}^2$$

$$Q = \frac{1}{2}M_\alpha v_\alpha^2\left[\frac{M_\alpha}{M_Y} + 1\right]$$

$$Q = K_\alpha\left[\frac{4}{A-4}+1\right]$$
$$Q = K_\alpha\left(\frac{A}{A-4}\right)$$

Chain reaction

Consider a neutron causing fission in a uranium nucleus producing three neutrons. The three neutrons in turn may cause fission in three uranium nuclei producing nine neutrons. These nine neutrons in turn may produce twenty seven neutrons and so on. A chain reaction is a self propagating process in which the number of neutrons goes on multiplying rapidly almost in a geometrical progression.

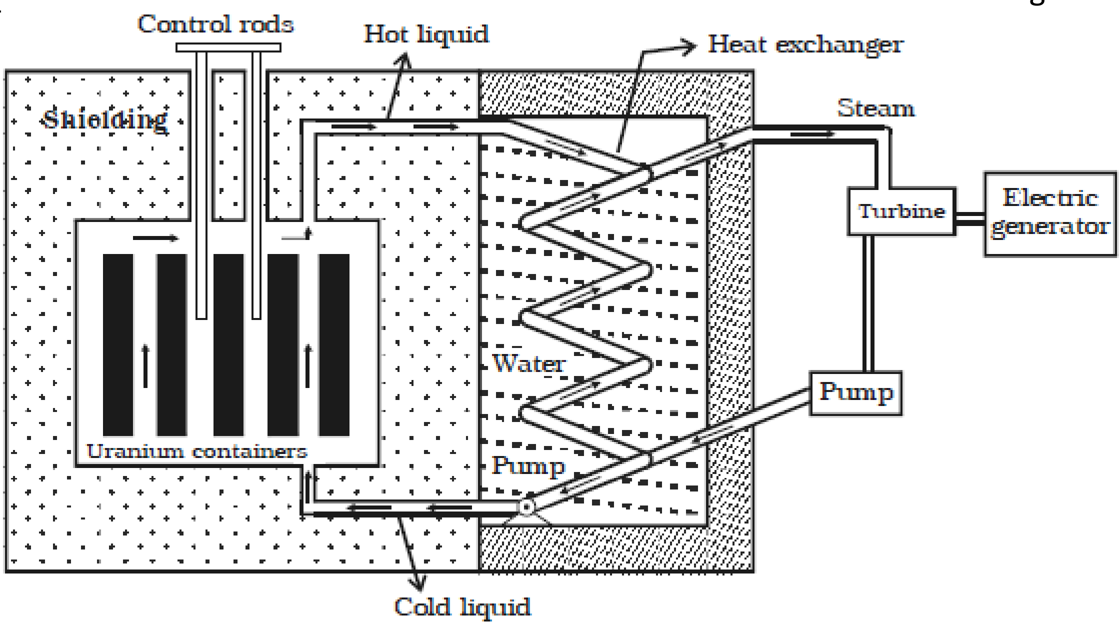
Critical size

Critical size of a system containing a fissile material is defined as the minimum size in which at least one neutron is available for further fission reaction. The mass of the fissile material at the critical size is called critical mass. The chain reaction is not possible if the size is less than the critical size.

Nuclear reactor

A nuclear reactor is a device in which the nuclear fission reaction takes place in a self sustained and controlled manner

The schematic diagram of a nuclear reactor is shown in Fig In such a reactor, water is used both as the moderator and as the heat transfer medium. In the *primary-loop*, water is circulated through the reactor vessel and transfers energy at high temperature and pressure (at about 600 K and 150 atm) to the steam generator, which is part of the *secondary-loop*. In the steam generator, evaporation provides high-pressure steam to operate the turbine that drives the electric generator. The low-pressure steam from the turbine is cooled and condensed to water and forced back into the steam generator



(i) Fissile material or fuel

The fissile material or nuclear fuel generally used is $_{92}U^{235}$. But this exists only in a small amount (0.7%) in natural uranium. Natural uranium is enriched with more number of $_{92}U^{235}$ (2 – 4%) and this low enriched uranium is used as fuel in some reactors. Other than U^{235}, the fissile isotopes U^{233} and Pu^{239} are also used as fuel in some of the reactors.

(ii) Moderator

The function of a moderator is to slow down fast neutrons produced in the fission process having an average energy of about 2 *MeV* to thermal neutrons with an average energy of about 0.025 *eV*, which are in thermal equilibrium with the moderator. Ordinary water and heavy water(D_2O) are the commonly used moderators. A good moderator slows down neutrons by elastic collisions and it does not remove them by absorption. The moderator is present in the space between the fuel rods in a channel. Graphite is also used as a moderator in some countries. In fast breeder reactors, the fission chain reaction is sustained by fast neutrons and hence no moderator is required.

(iii) Neutron source

A source of neutron is required to initiate the fission chain reaction for the first time. A mixture of beryllium with plutonium or radium or polonium is commonly used as a source of neutron.

(iv) Control rods

The control rods are used to control the chain reaction. They are very good absorbers of neutrons. The commonly used control rods are made up of elements like boron or cadmium. The control rods are inserted into the core and they pass through the space in between the fuel tubes and through the moderator. By pushing them in or pulling out, the reaction rate can be controlled. In our country, all the power reactors use boron carbide (B4C), a ceramic material as control rod.

Because of the use of control rods, it is possible that the ratio, *K*, of number of fission produced by a given generation of neutrons to the number of fission of the preceeding generation may be greater than one. This ratio is called the *multiplication factor;* it is the measure of the growth rate of the neutrons in the reactor. For *K* = 1, the operation of the reactor is said to be *critical*, which is what we wish it to be for steady power operation. If *K* becomes greater than one, the reaction rate and the reactor power increases exponentially. Unless the factor *K* is brought down very close to unity, the reactor will become supercritical and can even explode.

In addition to control rods, reactors are provided with *safety rods* which, when required, can be inserted into the reactor and *K* can be reduced rapidly to less than unity.

(v) The cooling system

The cooling system removes the heat generated in the reactor core. Ordinary water, heavy water and liquid sodium are the commonly used coolants. A good coolant must possess large specific heat capacity and high boiling point. The coolant passes through the tubes containing the fuel bundle and carries the heat from the fuel rods to the steam generator through heat exchanger. The steam runs the turbines to produce electricity in power reactors.
Being a metal substance, liquid sodium is a very good conductor of heat and it remains in the liquid state for a very high temperature as its boiling point is about 1000^{o} C.

(vi) Neutron reflectors
Neutron reflectors prevent the leakage of neutrons to a large extent, by reflecting them back. In pressurized heavy water reactors the moderator itself acts as the reflector.
In the fast breeder reactors, the reactor core is surrounded by depleted uranium (uranium which contains less than 0.7% of ${}_{92}U^{235}$) or thorium (${}_{90}Th^{232}$) which acts as neutron reflector. Neutrons escaping from the reactor core convert these materials into Pu^{239} or U^{233} respectively.

(vii) Shielding
As a protection against the harmful radiations, the reactor is surrounded by a concrete wall of thickness about 2 to 2.5 m.

Nuclear fusion – energy generation in stars

Energy can be released if two light nuclei combine to form a single larger nucleus, a process called *nuclear fusion*.
Some examples of such energy liberating reactions are

$$\mathbf{{}_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + e^+ + \nu + 0.42\ MeV \quad \text{-- (a)}}$$
$$\mathbf{1H^2 + 1H^2 \rightarrow {}_2He^3 + n + 3.27\ MeV \quad \text{---(b)}}$$
$$\mathbf{1H^2 + 1H^2 \rightarrow {}_1He^3 + {}_1H^1 + 4.03\ MeV \quad \text{--- (c)}}$$

In all these reactions, we find that two positively charged particles combine to form a larger nucleus.
It must be realized that such a process is hindered by the Coulomb repulsion that acts to prevent the two positively charged particles from getting close enough to be within the range of their attractive nuclear forces and thus 'fusing'.
The height of this *Coulomb barrier* depends on the charges and the radii of the two interacting nuclei. The temperature at which protons in a proton gas would have enough energy to overcome the coulomb's barrier is about 3×10^9 K.
To generate useful amount of energy, nuclear fusion must occur in bulk matter. What is needed is to raise the temperature of the material until the particles have enough energy – due to their thermal motions alone – to penetrate the coulomb barrier. This process is called *thermonuclear fusion*.

The fusion reaction in the sun is a multi-step process in which hydrogen is burned into helium, hydrogen being the 'fuel' and helium the 'ashes'.
The *proton-proton* (p, p) *cycle* by which this occurs is represented by the following sets of reactions:

$${}_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + e^+ + \nu + 0.42\ \text{MeV} \quad \text{--- (d)}$$
$$e^+ + e^- \rightarrow \gamma + 1.02\ \text{MeV} \quad \text{--- (e)}$$
$${}_1H^2 + {}_1H^1 \rightarrow {}_2He^3 + \gamma + 5.49\ \text{MeV} \quad \text{--- (f)}$$
$${}_2H^3 + {}_2H^3 \rightarrow {}_2He^4 + {}_1H^1 + {}_1H^1 + 12.86\ \text{MeV} \quad \text{---(g)}$$

For the fourth reaction to occur, the first three reactions must occur twice, in which case two light helium nuclei unite to form ordinary helium or nucleus.
the net effect is

$$4\,{}_1H^1 + 2e^- \rightarrow {}_2He^4 + 2\nu + 6\gamma + 26.7\text{MeV}$$

Thus, four hydrogen atoms combine to form an ${}_2He^4$ atom with a release of 26.7 MeV of energy.
Calculations show that there is enough hydrogen to keep the sun going for about the same time into the future. In about 5 billion years, however, the sun's core, which by that time will be largely helium, will begin to cool and the sun will start to collapse under its own gravity. This will raise the core temperature and cause the outer envelope to expand, turning the sun into what is called a *red giant*
If the core temperature increases to 10^8 K again, energy can be produced through fusion once more – this time by burning helium to make carbon. As a star evolves further and becomes still hotter, other elements can be formed by other fusion reactions.
The energy generation in stars takes place via thermonuclear fusion.

Solved Numerical

Q) By the fusion of 1Kg deuterium ($1H^2$) according the reaction
${}_1H^2 + {}_1H^2 \rightarrow {}_2He^4 + {}_0n^1 + 3.27$ Mev, how long can a bulb of 100 W give light?
Molecular wt of deuterium is 2g
Thus number of moles of deuterium in 1kg = 500 moles
Number of nucli of deuterium = $500\times6.02\times10^{23} = 3.01\times10^{26}$
Now Two nucli gives energy of 3.27 MeV = $3.27\times10^6\times1.6\times10^{-19}$ J=5.23×10^{-13} J
Thus 3.01×10^{26} nucli will release energy of

$$\frac{3.01\times10^{26}\times5.23\times10^{-13}}{2} = 7.87\times10^{13} J$$

If a bulb of 100 W glows for t seconds, then energy consumed = 100t J

$$\therefore 100t = 7.87 \times 10^{13}$$

t = 7.874×10^{11} sec

$$t = \frac{7.874 \times 10^{11}}{3.16 \times 10^{7\,s/year}} = 24917\ Yr$$

Dual Nature of Radiation and Matter

Emission of electrons:

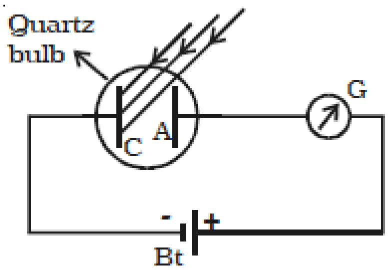
We know that metals have free electrons (negatively charged particles) that are responsible for their conductivity.
However, the free electrons cannot normally escape out of the metal surface. If an electron attempts to come out of the metal, the metal surface acquires a positive charge and pulls the electron back to the metal.
The free electron is thus held inside the metal surface by the attractive forces of the ions. Consequently, the electron can come out of the metal surface only if it has got sufficient energy to overcome the attractive pull. A certain minimum amount of energy is required to be given to an electron to pull it out from the surface of the metal.
This minimum energy required by an electron to escape from the metal surface is called the *work function* of the metal. It is generally denoted by ϕ_0 and measured in eV (electron volt).
The work function (ϕ_0) depends on the properties of the metal and the nature of its surface. These values are approximate as they are very sensitive to surface impurities.
The work function of platinum is the highest (ϕ_0 = 5.65 eV) while it is the lowest (ϕ_0 = 2.14 eV) for caesium.
The minimum energy required for the electron emission from the metal surface can be supplied to the free electrons by any one of the following physical processes:
(i) Thermionic emission: By suitably heating, sufficient thermal energy can be imparted to the free electrons to enable them to come out of the metal.
(ii) Field emission: By applying a very strong electric field (of the order of 108 V m^{-1}) to a metal, electrons can be pulled out of the metal, as in a spark plug.
(iii) Photo-electric emission: When light of suitable frequency illuminates a metal surface, electrons are emitted from the metal surface. These photo(light)-generated electrons are called photoelectrons.

Photoelectric effect

When an electromagnetic radiation of enough high frequency is incident on a cleaned surface, electrons can be liberated from the metal surface. This phenomenon is known as the photoelectric effect and the electron emitted are known as Photo electrons.
To have photo emission, the frequency of incident light should be more than some minimum frequency. This minimum frequency is called the threshold frequency (f_o).
It depends on the type of the metal.
For most of the metals (e.g. Zn, Cd, Mg) threshold frequency lies in the ultraviolet region of electromagnetic spectrum. But for alkali metals (Li, K, Na, Rb) it lies in the visible region

Hallwachs' and Lenard's observations
Hallwachs, in 1888, undertook the study further and connected a negatively charged zinc plate to an electroscope. He observed that the zinc plate lost its charge when it was illuminated by ultraviolet light. Further, the uncharged zinc plate became positively charged when it was irradiated by ultraviolet light. Positive charge on a positively charged zinc plate was found to be further enhanced when it was illuminated by ultraviolet light. From these observations he concluded that negatively charged particles were emitted from the zinc plate under the action of ultraviolet light

Lenard (1862-1947) observed that when ultraviolet radiations were allowed to fall on the emitter plate of an evacuated glass tube enclosing two electrodes (metal plates), current flows in the circuit as shown in figure. As soon as the ultraviolet radiations were stopped, the current flow also stopped. These observations indicate that when ultraviolet radiations fall on the emitter plate C, electrons are ejected from it which are attracted towards the positive, collector plate A by the electric field. The electrons flow through the evacuated glass tube, resulting in the current flow. Thus, light falling on the surface of the emitter causes current in the external circuit. The emission of electrons causes flow of electric current in the circuit. The potential difference between the emitter and collector plates is measured by a voltmeter (V) whereas the resulting photo current flowing in the circuit is measured by a microammeter (μA). The photoelectric current can be increased or decreased by varying the potential of collector plate A with respect to the emitter plate C. The intensity and frequency of the incident light can be varied, as can the potential difference *V* between the emitter C and the collector A.



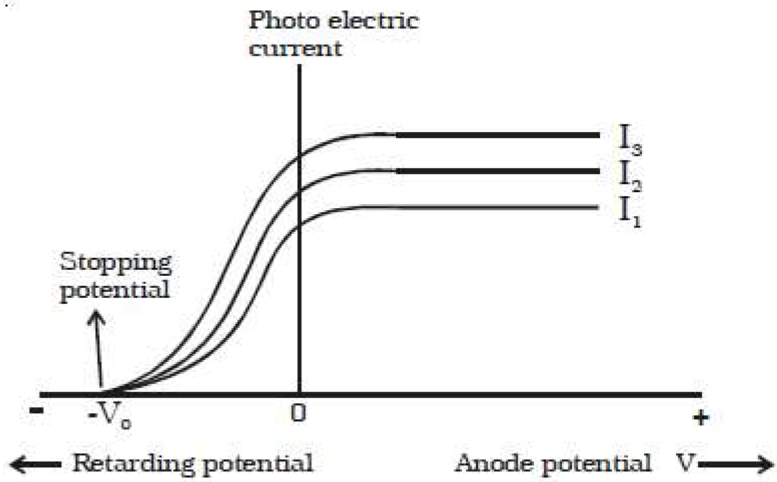
The amount of current passing through the ammeter gives an idea of the number of photoelectrons. At some value of positive potential difference, when all the emitted electrons are collected, increasing the potential difference further has no effect on the current.

Effect of potential on photoelectric current

When the collector (A) is made negative with respect to C, the emitted electrons are repelled and only those electrons which have sufficient kinetic energy to overcome the repulsion may reach to the collector(A) and constitute current.
So the current in ammeter falls. On making Collector (A) more negative , number of photoelectrons reaching the collector further decreases.

For specific negative potential of the collector, even the most energetic electrons are unable to reach collector and photoelectric current becomes zero.
It remains zero even if the potential is made further negative than the specific value of negative potential.
This minimum specific negative potential of the collector with respect to the emitter (photo sensitive surface) at which photo-electric current becomes zero is known as the Stopping Potential (V_O) for the given surface.
It is thus the maximum kinetic energy $\frac{1}{2}mv^2$ of the emitted photoelectrons. If charge and mass of an electron are e and m respectively then

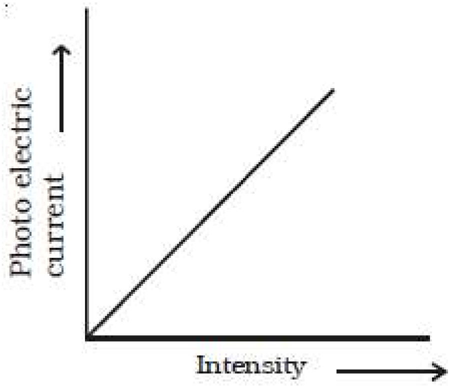
$$\frac{1}{2}mv^2 = eV_o$$



We can now repeat this experiment with incident radiation of the same frequency but of higher intensity I_2 and I_3 ($I_3 > I_2 > I_1$). We note that the saturation currents are now found to be at higher values. This shows that more electrons are being emitted per second, proportional to the intensity of incident radiation. But the stopping potential remains the same as that for the incident radiation of intensity I_1, as shown graphically in Fig.
Thus, for a given frequency of the incident radiation, the stopping potential is independent of its intensity. In other words, the maximum kinetic energy of photoelectrons depends on the light source and the emitter plate material, but is independent of intensity of incident radiation.

Effect of intensity of incident radiation on photo electric current

Keeping the frequency of the incident radiation and the potential difference between the collector(A) and the Surface (C) at constant values, the intensity of incident radiation is varied. The corresponding photoelectric current is measured in the micro-ammeter.
It is found that the photo electric current increases linearly with the intensity of incident radiation (Fig).



Since the photoelectric current is directly proportional to the number of photoelectrons emitted per second, it implies that the number of photoelectrons emitted per second is proportional to the intensity of incident radiation.

Effect of frequency of incident radiation on stopping potential

Keeping the photosensitive plate (C) and intensity of incident radiation a constant, the effect of frequency of the incident radiations on stopping potential is studied.

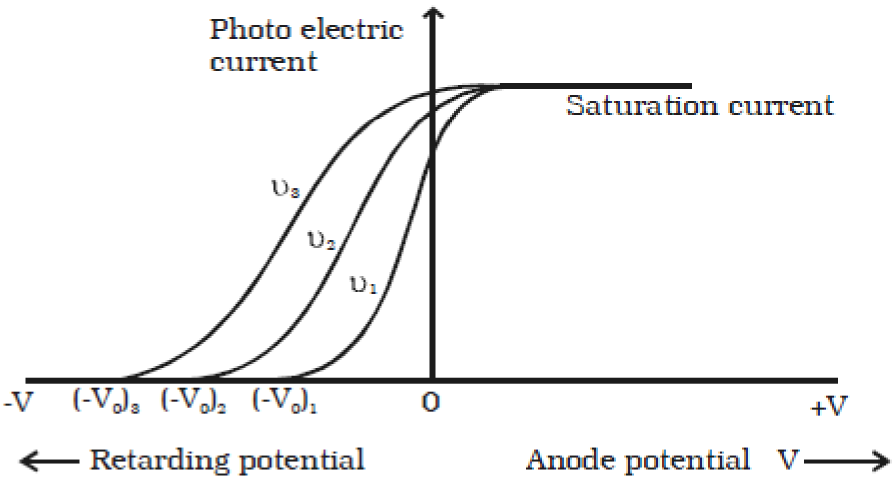
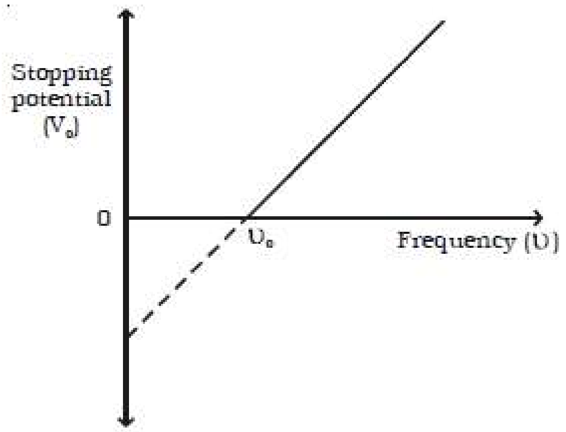


Fig shows the variation of the photo electric current with the applied potential difference *V* for three different frequencies. From the graph, it is found that higher the frequency of the incident radiation, higher is the value of stopping potential *Vo.* For frequencies $\nu_3 > \nu_2 > \nu_1$, the corresponding stopping potentials are in the same order $(V_o)_3 > (V_o)_2 > (V_o)_1$.
It is concluded from the graph that, the maximum kinetic energy of the photoelectrons varies linearly with the frequency of incident radiation but is independent of its intensity.
If the frequency of the incident radiation is plotted against the corresponding Stopping potential, a straight line is obtained as shown in Fig



From this graph, it is found that at a frequency ν_o, the value of the stopping potential is zero. This frequency is known as the threshold frequency for the photo metal used. The photoelectric effect occurs above this frequency and ceases below it.
Therefore, threshold frequency is defined as the minimum frequency of incident radiation, below which the photoelectric emission is not possible completely. The threshold frequency is different for different metals.

Laws of photoelectric emission:

The experimental observations on photoelectric effect may be summarized as follows, which are known as the fundamental laws of photoelectric emission.
(i) For a given photo sensitive material, there is a minimum frequency called the threshold frequency, below which emission of photoelectrons stops completely, however great the intensity may be.
(ii) For a given photosensitive material, the photo electric current is directly proportional to the intensity of the incident radiation, provided the frequency is greater than the threshold frequency.
(iii) The photoelectric emission is an instantaneous process. i.e. there is no time lag between the incidence of radiation and the emission of photo electrons.
(iv)The maximum kinetic energy of the photo electrons is directly proportional to the frequency of incident radiation, but is independent of its intensity.

Wave theory fails to explain the photoelectric effect as:

(1) According to the wave theory of light, energy and intensity of wave depend on its amplitude. Hence intense radiation has higher energy and on increasing intensity, energy of photoelectrons should increase. But experimental results show that photoelectric effect is independent of intensity of light, but depends on the frequency of light. According to wave theory of light, energy of light has nothing to do with frequency. Hence change in energy of photoelectrons with change in frequency cannot be explained

(2) Photons are emitted immediately (within 10^{-9}s) on making light incident on metal surface. Since the free electrons within metal are withheld under the effect of certain forces, and to bring them out, energy must be supplied

Now if the incident energy is showing wave nature, free electrons in metal get energy gradually and when accumulates energy at least equal to work function then they escape from metal.

Thus electrons get emitted only after some time

(3) According to wave theory of light , less intense light is 'weak' in terms of energy. To liberate photoelectron with such light one has to wait long till electron gather sufficient energy. Whereas experimental result shows that phenomenon depends on frequency and for low intensity light of appropriate frequency photoelectrons are emitted instantly

Solved Numerical

Q) Let an electron requires 5×10^{-19} joule energy to just escape from the irradiated metal. If photoelectron is emitted after 10^{-9}s of the incident light, calculate the rate of absorption of energy. If this process is considered classically, the light energy is assumed to be continuously distributed over the wave front. Now, the electron can only absorb the light incident within a small area, say 10^{-19} m^2. Find the intensity of illumination in order to see the photoelectric effect

Solution:

Rate of absorption of energy is power

$$P = \frac{E}{t} = \frac{5\times10^{-19}}{10^{-9}} = 5\times10^{-10}\frac{J}{s}$$

From the definition of intensity of light

$$I = \frac{Power}{Area} = \frac{5\times10^{-10}}{10^{-19}} = 5\times10^{9}\frac{J}{s.m^2}$$

Since, practically it is impossibly high energy, which suggest that explanation of photoelectric effect in classical term is not possible

Q) Work function is 2eV. Light of intensity 10^{-5} W m^{-2} is incident on 2cm^2 area of it. If 10^{17} electrons of these metals absorb the light, in how much time does the photo electric effect start? Consider the waveform of incident light

Solution:

Intensity of incident light is 10^{-5} W m^{-2}

Now intensity

$$I = \frac{E}{A\cdot t}$$

$$E = IAt$$

$$E = 10^{-5}\times2\times10^{-4}\times1 = 2\times10^{-9}J$$

This energy is absorbed by 10^{17} electrons

Average energy absorbed by each electron = $2\times10^{-9}/10^{17} = 2\times10^{-26}$ J
Now, electron may get emitted when it absorbs energy equal to the work function of its metal = 2eV = 3.6×10^{-19} J
Thus time required to absorb energy = 3.6×10^{-19} J / 2×10^{-26} J = 1.6×10^{7} s

Light waves and photons

The electromagnetic theory of light proposed by Maxwell could not explain photoelectric effect. But, Max Planck's quantum theory successfully explains photoelectric effect. According to Planck's quantum theory, light is emitted in the form of discrete packets of energy called 'quanta' or photon. The energy of each photon is E = $h\nu$, where h is Planck's constant. Photon is neither a particle nor a wave. In the phenomena like interference, diffraction, polarization, the photon behaves like a wave. Energy of n photon E = $nh\nu$
In the phenomena like emission, absorption and interaction with matter (photo electric effect) photon behaves as a particle. Hence light photon has a dual nature.

Solved Numerical

Q) If the efficiency of an electric bulb is of 1 watt is 10%, what is the number of photons emitted by it in one second? The wave length of light emitted by it is 500nm,
h = 6.625×10^{-34}
Solution:
As the bulb is of 1W, if its efficiency is 100%, it may emit 1 J radiant energy in 1s. But here the efficiency is 10%, hence it emits 10^{-1} J energy in the form of light in 1 s , and remaining in the form of heat.
∴Radiant energy obtained from bulb in 1s = 10^{-1} J
If it consists of n photons then
E= $nh\nu$

$$E = nh\frac{c}{\lambda}$$

$$n = \frac{E\lambda}{hc}$$

$$n = \frac{10^{-1}\times 500\times 10^{-9}}{6.625\times 10^{-34}\times 3\times 10^{8}}$$

n = 2.53×10^{17} photons

Einstein's photoelectric equation

In 1905, Albert Einstein successfully applied quantum theory of radiation to photoelectric effect.
Plank had assumed that emission of radiant energy takes place in the quantized form, the photon, but once emitted it propagate in the form of wave. Einstein further assumed that not only the emission, even the absorption of light takes place in the form of photons.
According to Einstein, the emission of photo electron is the result of the interaction between a single photon of the incident radiation and an electron in the metal. When a photon of energy $h\nu$ is incident on a metal surface, its energy is used up in two ways:

(i) A part of the energy of the photon is used in extracting the electron from the surface of metal, since the electrons in the metal are bound to the nucleus. This energy W spent in releasing the photo electron is known as photoelectric work function of the metal. The work function of a photo metal is defined as the minimum amount of energy required to liberate an electron from the metal surface.
(ii) The remaining energy of the photon is used to impart kinetic energy to the liberated electron. If *m* is the mass of an electron and *v*, its velocity then
Energy of the incident photon = Work function + Kinetic energy of the electron

$$h\nu = \phi_0 + \frac{1}{2}mv^2$$

If the electron does not lose energy by internal collisions, as it escapes from the metal, the entire energy $(h\nu - \phi_0)$ will be exhibited as the kinetic energy of the electron.
Thus, $(h\nu - \phi_0)$ represents the maximum kinetic energy of the ejected photo electron. If V_{max} is the maximum velocity with which the photoelectron can be ejected, then

$$h\nu = \phi_0 + \frac{1}{2}mv_{max}^2 - -(1)$$

This equation is known as Einstein's photoelectric equation.
When the frequency (ν) of the incident radiation is equal to the threshold frequency (ν_o) of the metal surface, kinetic energy of the
electron is zero. Then equation (1) becomes,
$h\nu_o = \phi_0$ ...(2)
Substituting the value of *W* in equation (1) we get,

$$h\nu - h\nu_0 = \frac{1}{2}mv_{max}^2 - -(3)$$

Or $K_{max} = h\nu - \phi_0$ or $eV_O = h\nu - \phi_0$ –--(4)
This is another form of Einstein's photoelectric equation.

Solved numerical

Q) A beam of photons of intensity 2.5 W m^{-2} each of energy 10.6eV is incident on 1.0×10^{-4} m^2 area of the surface having work function 5.2eV. If 0.5% of incident photons emits photo-electrons, find the number of photons emitted in 1s. Find the minimum and maximum energy of photo-electrons.

Solution:
Intensity I

$$I = \frac{E}{A \cdot t}$$

But E = nhν, here n is number of photons

$$I = \frac{nh\nu}{At}$$

$$n = \frac{IAt}{h\nu}$$

Energy of each photon = hν =10.6eV =$10.6\times1.6\times10^{-19}$ J

$$n = \frac{2.5 \times 1 \times 10^{-4} \times 1}{10.6 \times 1.6 \times 10^{-19}} = 1.47 \times 10^{14}$$

As 0.5% of these photons emits electrons
Number of photo electrons emitted N = $1.47\times10^{14} \times (0.5/100) = 7.35\times10^{11}$

The minimum energy of photo electron is = 0 J. Such photoelectron spend all its energy gained from the photon against work function
Maximum energy of photo electron:
E = hν $-\phi_0$ = 10.6 eV – 5.2eV = 5.4 eV

Q) U.V light of wavelength 200nm is incident on polished surface of Fe. Work function of Fe is 4.5eV Find
1) Stopping potential
2) maximum kinetic energy of photoelectrons
3) Maximum speed of photoelectrons
m = 9.11×10^{-31} kg ,
Solution
Work function = 4.5eV = 4.5e J

$$eV_O = h\nu - \phi_0$$

$$eV_0 = h\frac{c}{\lambda} - \phi_0$$

$$V_0 = \frac{h}{e}\frac{c}{\lambda} - \frac{\phi_0}{e}$$

$$V_0 = \frac{6.625 \times 10^{-34}}{1.6 \times 10^{-19}} \times \frac{3 \times 10^8}{200 \times 10^{-9}} - \frac{e \times 4.5}{e}$$

V_0 = 6.21 – 4.5 = 1.71 V

Maximum kinetic energy = eV_0

$$\frac{1}{2} m v_{max}^2 = eV_0$$

$$v_{max} = \sqrt{\frac{2eV_0}{m}}$$

$$v_{max} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 1.71}{9.11 \times 10^{-31}}}$$

V_{max} = 7.75 $\times 10^5$ m/s

Experimental verification of Einstein's photoelectric equation

(1) According to Eq. (4), K_{max} depends linearly on ν, and is independent of intensity of radiation, in agreement with observation.

This has happened because in Einstein's picture, photoelectric effect arises from the absorption of a single quantum of radiation by a single electron. The intensity of radiation (that is proportional to the number of energy quanta per unit area per unit time) is irrelevant to this basic process.

(2) Since K_{max} must be non-negative, Eq. (4) implies that photoelectric emission is possible only if $h\nu > \phi_0$

Or $\nu > \nu_0$

Where $\nu_0 = \phi_0 / h$ –eq(5)

Equation (4) shows that the greater the work function ϕ_0, the higher the minimum or threshold frequency ν_0 needed to emit photoelectrons. Thus, there exists a threshold frequency ν_0 (= ϕ_0/h) for the metal surface, below which no photoelectric emission is possible, no matter how intense the incident radiation may be or how long it falls on the surface

(3) Intensity of radiation as noted above is proportional to the number of energy quanta per unit area per unit time.

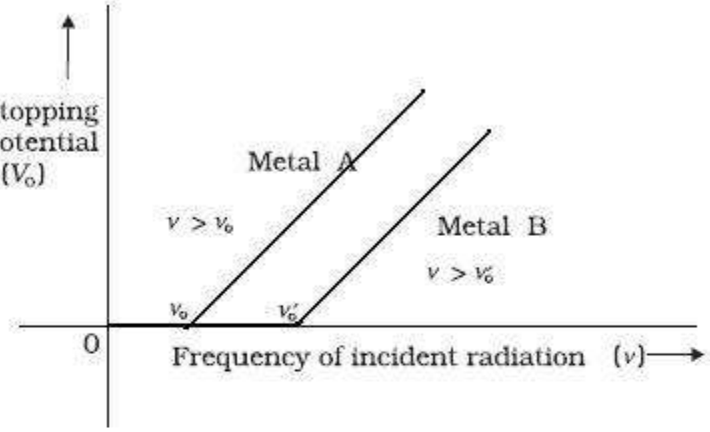
The greater the number of energy quanta available, the greater is the number of electrons absorbing the energy quanta and greater, therefore, is the number of electrons coming out of the metal (for ν > ν0). This explains why, for $\nu > \nu_0$,photoelectric current is proportional to intensity.

(4) The photoelectric equation, Eq. (3), can be written as

$$eV_0 = h\nu - \phi_0$$

$$V_0 = \frac{h}{e}\nu - \frac{\phi_0}{e}$$

This is an important result. It predicts that the V_0 versus ν curve is a straight line with slope = (*h*/*e*), independent of the nature of the material.
During 1906-1916, Millikan performed a series of experiments on photoelectric effect, aimed at disproving Einstein's photoelectric equation. He measured the slope of the straight line obtained for sodium, similar to that shown in Fig.



Using the known value of *e*, he determined the value of Planck's constant *h*. This value was close to the value of Planck's constant (= 6.626 × 10–34J s) determined in an entirely different context.
In this way, in 1916, Millikan proved the validity of Einstein's photoelectric equation, experimentally and found that it is in harmony with the observed facts.

PARTICLE NATURE OF LIGHT: THE PHOTON

(i) In interaction of radiation with matter, radiation behaves as if it is made up of particles called photons.
(ii) Each photon has energy *E* (=*h*ν) and momentum *p* (= *h* ν/*c*), and speed *c*, the speed of light.
(iii) All photons of light of a particular frequency ν, or wavelength λ, have the same energy *E* (=*h*ν = *hc*/λ) and momentum *p* (= *h*ν/*c* = *h*/λ),
whatever the intensity of radiation may be. By increasing the intensity of light of given wavelength, there is only an increase in the number of photons per second crossing a given area, with each photon having the same energy. Thus, photon energy is independent of intensity of radiation.
(iv) Photons are electrically neutral and are not deflected by electric and magnetic fields.
(v) In a photon-particle collision (such as photon-electron collision), the total energy and total momentum are conserved. However, the number of photons may not be conserved in a collision. The photon may be absorbed or a new photon may be created.
(v) Mass of photon $m = E/c^2$

WAVE NATURE OF MATTER

The radiant energy has dual aspects of particle and wave, hence a natural question arises, if radiation has a dual nature, why not the matter. In 1924, a French Physicist Louis de Broglie put forward the bold hypothesis that moving particles should possess wave like properties under suitable conditions. He reasoned this idea, from the fact, that nature is symmetrical and hence the basic physical entities– matter and energy should have symmetrical characters. If radiation shows dual aspects, so should matter.

de Broglie's wavelength of matter waves

de Broglie equated the energy equations of Planck (wave) and Einstein (particle).
For a wave of frequency ν, the energy associated with each photon is given by Planck's relation,
$E = h\nu$ ...(1)
where h is Planck's constant.
According to Einstein's mass energy relation, a mass m is equivalent to energy,
$E = mc^2$ ...(2)
where c is the velocity of light.
If, $h\nu = mc^2$

$$\therefore \frac{hc}{\lambda} = mc^2 or\ \lambda = \frac{h}{mc} - -(3)$$

For a particle moving with a velocity v, if $c = v$
from equation (3)

$$\lambda = \frac{h}{mv} = \frac{h}{p} - -(4)$$

where $p = mv$, the momentum of the particle. These hypothetical matter waves will have appreciable wavelength only for very light particles.

de Broglie wavelength of an electron

When an electron of mass m and charge e is accelerated through a potential difference V, then the energy eV is equal to kinetic energy of the electron.

$$\frac{1}{2}mv^2 = eV$$

$$v = \sqrt{\frac{2eV}{m}} - -(1)$$

The de Broglie wavelength is ,

$$\lambda = \frac{h}{mv}$$

Substituting the value of v,

$$\lambda = \frac{h}{m\sqrt{\frac{2eV}{m}}} = \frac{h}{\sqrt{2meV}} - -(2)$$

Substituting the known values in equation (2),

$$\lambda = \frac{12.27}{\sqrt{V}}\text{Å}$$

If V = 100 volts, then λ = 1.227 Å i.e., the wavelength associated with an electron accelerated by 100 volts is 1.227 Å.
Since E = eV is kinetic energy associated with the electron, the equation (2) becomes,

$$\lambda = \frac{h}{\sqrt{2mE}}$$

Wave packet:

Classically, particle men as a point like object endowed with a precise position and momentum.
The de Broglie's hypothesis, which also supports wave-like behavior of matter, question about how to measure accurately position and momentum of a material particle.
A pure harmonic wave extended in space obviously cannot represent a point like particle. This suggest that the wave activity of a wave representing a particle must be limited to the space occupied by the particle. For this reason an idea of wave packet, a wave which is confined to small region of space is introduced. Wave packet may be considered as superposition of many harmonic wave of slightly different wavelength
If the concept of wave packet is used to represent particle, position of the particle is more and is proportional to the size of the wave-packet. But as several waves of different wave lengths are used to represent a particle, its momentum is no longer unique and become uncertain.
In general, the matter wave associated with the electron is not extended all over space. It is a wave packet extending over some finite region of space. In that case Δx is not infinite but has some finite value depending on the extension of the wave packet. Wave packet of finite extension does not have a single wavelength. It is built up of wavelengths spread around some central wavelength.

Heisenberg's uncertainty Principle:
According to Heisenberg's uncertainty principle, if the uncertainty in the x-coordinate of the position of a particle is Δx and uncertainty in the x-component of momentum is Δp (i.e. in one dimension) them

$$\Delta x \cdot \Delta p \geq \frac{h}{2\pi}$$

Similarly

$$\Delta E \cdot \Delta t \geq \frac{h}{2\pi}$$

Solved numerical

Q) Find the certainty with which one can locate the position of
1) A bullet of mass 25g 2) An electron moving with speed 500 m/s accurate to 0.01%. .
Mass of electron is 9.1×10^{-31}kg

Solution

1) Uncertainty in measurement of momentum of bullet is 0.01% of its exact value i.e.
Δp = 0.01% of mv

$$\Delta p = \left(\frac{0.01}{100}\right) \times (25 \times 10^{-3})(500)$$
$$\Delta p = 1.25 \times 10^{-3} kgms^{-1}$$

Therefore, corresponding uncertainty in position

$$\Delta x = \frac{h}{2\pi p}$$
$$\Delta x = \frac{6.625 \times 10^{-34}}{2 \times 3.14 \times 1.25 \times 10^{-3}}$$

$$\Delta x = 8.44 \times 10^{-32} m$$

(2) Uncertainty in measurement o momentum of an electron is

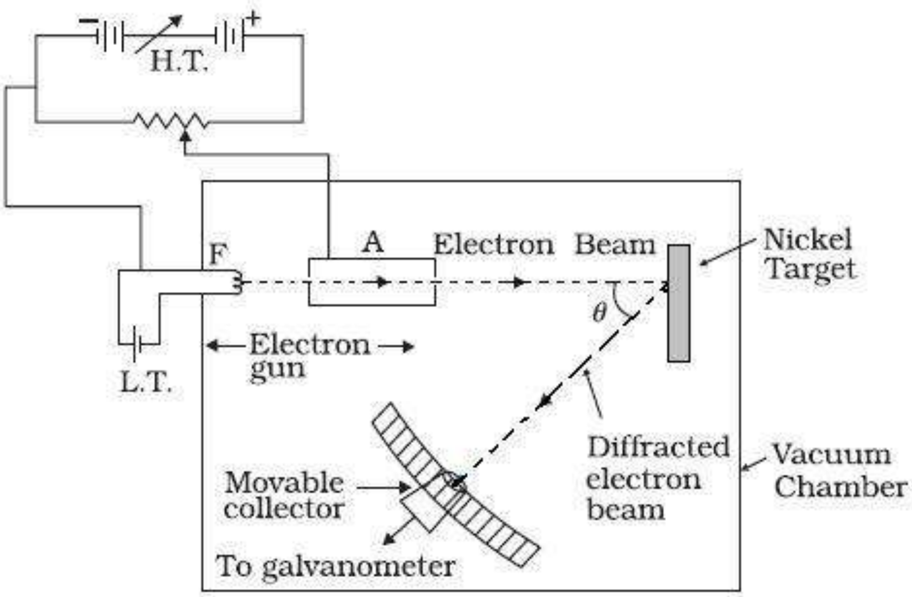
$$\Delta p = \frac{0.01}{100} \times (9.1 \times 10^{-31})(500) = 4.55 \times 10^{-32} kgms^{-1}$$
$$\Delta x = \frac{6.625 \times 10^{-34}}{2 \times 3.14 \times 4.55 \times 10^{-32}} = 0.23\ mm$$

Conclusion: The value of Δx is too small compared to the dimension of the bullet, and can be neglected. That is, position of the bullet is determined accurately

DAVISSON AND GERMER EXPERIMENT

The wave nature of electrons was first experimentally verified by C.J. Davisson and L.H. Germer in 1927 and independently by G.P. Thomson, in 1928, who observed diffraction effects with beams of electrons scattered by crystals.

The experimental arrangement used by Davisson and Germer is schematically shown in Fig.



It consists of an electron gun which comprises of a tungsten filament F, coated with barium oxide and heated by a low voltage power supply (L.T. or battery).
Electrons emitted by the filament are accelerated to a desired velocity by applying suitable potential/voltage from a high voltage power supply (H.T. or battery).
They are made to pass through a cylinder with fine holes along its axis, producing a fine collimated beam.
The beam is made to fall on the surface of a nickel crystal. The electrons are scattered in all directions by the atoms of the crystal.
The intensity of the electron beam, scattered in a given direction, is measured by the electron detector (collector). The detector can be moved on a circular scale and is connected to a sensitive galvanometer, which records the current.
The deflection of the galvanometer is proportional to the intensity of the electron beam entering the collector. The apparatus is enclosed in an evacuated chamber.
By moving the detector on the circular scale at different positions, the intensity of the scattered electron beam is measured for different values of angle of scattering θ which is the angle between the incident and the scattered electron beams.
The variation of the intensity (*I*) of the scattered electrons with the angle of scattering θ is obtained for different accelerating voltages.
The experiment was performed by varying the accelerating voltage from 44 V to 68 V.
It was noticed that a strong peak appeared in the intensity (*I*) of the scattered electron for an accelerating voltage of 54V at a scattering angle $\theta = 50^o$
The appearance of the peak in a particular direction is due to the constructive interference of electrons scattered from different layers of the regularly spaced atoms of the crystals.
From the electron diffraction measurements, the wavelength of matter waves was found to be 0.165 nm.
The de Broglie wavelength λ associated with electrons, using

$$\lambda = \frac{12.27}{\sqrt{V}}\text{Å}$$

$$\lambda = \frac{12.27}{\sqrt{54}}\text{Å}$$

λ=1.67Å

Thus, there is an excellent agreement between the theoretical value and the experimentally obtained value of de Broglie wavelength.
The de Broglie hypothesis has been basic to the development of modern quantum mechanics. It has also led to the field of electron optics. The wave properties of electrons have been utilized in the design of electron microscope which is a great improvement, with higher resolution, over the optical microscope.

Solved numerical

Q) An electron is at a distance of 10m from a charge of 10C. Its total energy is 15.6×10^{-10} J. Find its de Broglie wavelength at this point

$m_e = 9.1\times10^{-31}$ kg

Solution:

Potential energy of an electron

$$U = k\frac{qe}{r}$$

$$U = -\frac{9\times10^{9}\times10\times1.6\times10^{-19}}{10}$$

$$U = -14.4\times10^{-10}J$$

Total energy = Kinetic energy (K) + Potential energy

K = E – U

$K = 15.6\times10^{-10} + 14.4\times10^{-10} = 30\times10^{-10}$

But

$$K = \frac{p^2}{2m_e}$$

$$p = \sqrt{2Km_e}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2Km_e}}$$

$$\lambda = \frac{6.625\times10^{-34}}{\sqrt{2\times30\times10^{-10}\times9.1\times10^{-31}}}$$

$\lambda = 8.97\times10^{-15}$ m

----------------END--------------

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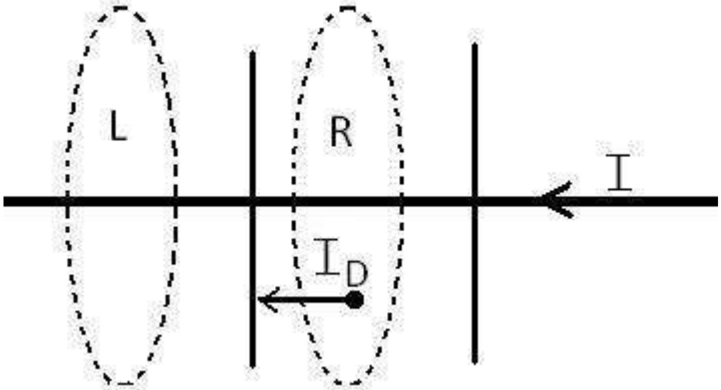
ELECTROMAGNETIC WAVES

DISPLACEMENT CURRENT

Current in capacitors

Consider the charging capacitor in the figure.



We have drawn two loops name as L which is outside the loop and Loop R which is in between the parallel plates of capacitor.
The capacitor is in a circuit that transfers charge (on a wire external to the capacitor) from the left plate to the right plate, charging the capacitor and increasing the electric field between its plates. The same current enters the right plate (say I) as leaves the left plate.
Although current is flowing through the capacitor, no actual charge is transported through the vacuum between its plates.
Ampere's circuital is not applicable for loop L and we can find magnetic field at point P using Ampere's circuital law

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I$$

Now if we consider an imaginary cylindrical surface. No conduction current enters cylinder surface *R*, while current *I* leaves through surface *L*. Thus Ampere's law is not applicable and magnetic field at point P must be zero. So we have a *contradiction*; calculated one way, there is a magnetic field at a point P; calculated another way, the magnetic field at P is zero.
Nonetheless, a magnetic field exists between the plates as though a current were present there as well.
For consistency of Ampere's Circuital law requires a displacement current $I_D = I$ to flow across surface *R*.
The explanation is that a *displacement current* I_D flows in the vacuum, and this current produces the magnetic field in the region between the plates according to Ampere's law
If Q is the charge on capacitor plate and area of plates of capacitor is A
Electric field between plates is

$$E = \frac{Q}{\varepsilon_0 A}$$

When capacitor is getting charged rate of change in electric field is

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt}$$

$$\varepsilon_0 A\frac{\partial E}{\partial t} = I_D$$

Here I_D is called displacement current
In integral form

$$\varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}\overrightarrow{da} = I_D$$

$$\varepsilon_0 \int \frac{d\phi_E}{dt} = I_D$$

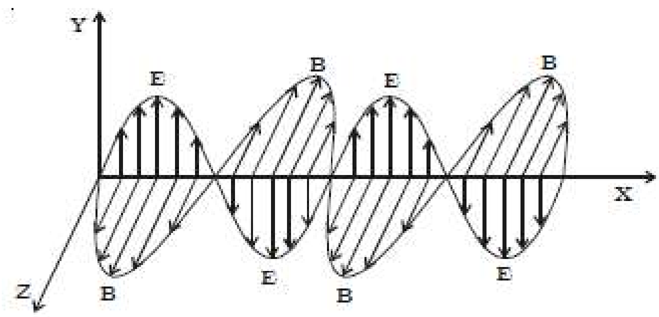
This current does not have significance in the sense of being the motion of charges.
The generalization made by Maxwell then is the following. The source of a magnetic field is not *just* the conduction electric current due to flowing charges, but also the time rate of change of electric field. More precisely, the total current *I* is the sum of the conduction current denoted by I_C and the displacement current denoted by I_D
Adding integral form of displacement current in Ampere's law we get

$$\oint \vec{B}\cdot\overrightarrow{dl} = \mu_0 I_C + \mu_0\varepsilon_0 \int \frac{d\phi_E}{dt}$$

and is known as Ampere-Maxwell law.

Electromagnetic waves

According to Maxwell, an accelerated charge is a source of electromagnetic radiation.
In an electromagnetic wave, electric and magnetic field vectors are at right angles to each other and both are at right angles to the direction of propagation.
They possess the wave character and propagate through free space without any material medium. These waves are transverse in nature. Fig shows the variation of electric field E along Y direction and magnetic field B along Z direction and wave propagation in + X direction



According to Maxwell's theory, these electric and magnetic field do not come into existence instantaneously. In the region closer to the oscillating change, the phase difference between electric field **E** and Magnetic field **B** is π/2 and their magnitude quickly decreases as $1/r^3$ (where r = distance from source) these components are called *Inductive component.*

At larger distance **E** and **B** are in phase and the decrease in their magnitude is comparatively slower with distance, as per 1/r. These components are called radiated components

Characteristics of Electromagnetic waves

(1) Representation in form of equations:
Electromagnetic wave shown in figure at time t, the y component is E_Y of electric field given by equation $E_Y = E_0 \sin(\omega t - kx)$
In vector form **E** = E_Y**j** = [$E_0 \sin(\omega t - kx)$] **j**
Similarly Magnetic component is given as **B** =[B_0 ($\omega t - kx$)] **k**

(2) Relation between magnitude of **E** and **B** is E = Bc
Here c is velocity of light

(3) The velocity of electromagnetic waves in vacuum

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

The velocity of electromagnetic waves in medium

$$v = \frac{1}{\sqrt{\varepsilon \mu}}$$

Here ε = permittivity of the medium and μ = permeability of the medium

From definition of refractive index

$$n = \frac{c}{v}$$

$$n = \frac{\sqrt{\varepsilon \mu}}{\sqrt{\varepsilon_0 \mu_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}$$

Since $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = k$ dielectric constant of medium and relative permeability $\mu_r = \frac{\mu}{\mu_0}$

$$n = \sqrt{\varepsilon_r \mu_r}$$

(4) Electromagnetic waves are transverse in nature

(5) Electromagnetic waves posses energy and they carry energy from one place to the other .

(6) Electromagnetic waves exerts pressure on a surface when they are incident on it, called radiation pressure
If ΔU is the energy of electromagnetic waves incident on a surface of area A in time Δt , in direction normal to the surface and if all energy is absorbed then change in momentum

$$\Delta p = \frac{\Delta U}{c}$$

(7) Energy density of electromagnetic wave

$\rho = \varepsilon_0 E_{rms}^2$ and $\rho = \frac{B_{rms}^2}{\mu_0}$

(8)The intensity of radiation (I) is defined as the radiant energy passing through unit area normal to the direction of propagation in one second

$$I = \frac{Energy}{(time)(area)} = Power$$

If radiation is passing through unit area with velocity c then volume in one second = c
Thus energy volume = ρc from the value of ρ *we get*

$$I = \varepsilon_0 c E_{rms}^2$$

Similarly

$$I = \frac{cB_{rms}^2}{\mu_0}$$

(9) E ×B gives the direction of propagation of the electromagnetic wave

ELECTROMAGNETIC SPECTRUM

Sr. No.	Name	Source	Wavelength in (m)	Frequency range (Hz)
1	γ – rays	Radioactive nuclei, nuclear reactions	10^{-14} to 10^{-10}	3×10^{22} to $3x\ 10^{18}$
2	x – rays	High energy electrons suddenly stopped by a metal target	1×10^{-10} to 3×10^{-8}	3×10^{18} to 1×10^{16}
3	Ultra−violet	Atoms and molecules in electrical discharge	$6 x\ 10^{-10}$ to 4×10^{-7}	$5 x\ 10^{17}$ to 8×10^{14}
4	Visible light	incandescent solids, Fluorescent, lamps	$4 x\ 10^{-7}$ to $8 x\ 10^{-7}$	$8 x\ 10^{14}$ to $4 x\ 10^{14}$
5	Infra−red (IR)	molecules of hot bodies	$8 x\ 10^{-7}$ to $3x\ 10^{-5}$	$4 x\ 10^{14}$ to 1×10^{13}
6	Microwaves	Electronic device (Vacuum tube)	10^{-3} to 0.3	$3 x\ 10^{11}$ to $1 x\ 10^{9}$
7	Radio frequency wave	Charges accelerated through conducting wires	10 to 10^4	3 x 107 – 3 x 104

Electromagnetic spectrum covers a wide range of wavelengths (or) frequencies. The whole electromagnetic spectrum has been classified into different parts and sub parts, in order of increasing wavelength and type of excitation. All electromagnetic waves travel with the velocity of light. The physical properties of electromagnetic waves are determined by their wavelength and not by their method of excitation.

The overlapping in certain parts of the spectrum shows that the particular wave can be produced by different methods.

Uses of electromagnetic spectrum

The following are some of the uses of electromagnetic waves.

1. Radio waves *:* These waves are used in radio and television communication systems.
AM band is from 530 kHz to 1710 kHz.
Higher frequencies upto 54 MHz are used for short waves bands.
Television waves range from 54 MHz to 890 MHz.
FM band is from 88 MHz to 108 MHz.
Cellular phones use radio waves in ultra high
frequency (UHF) band.

2. Microwaves : Due to their short wavelengths, they are used in radar communication system.
Microwave ovens are an interesting domestic application of these waves.

3. Infra red waves :

(i) Infrared lamps are used in physiotherapy.
(ii) Infrared photographs are used in weather forecasting.
(iii) As infrared radiations are not absorbed by air, thick fog, mist etc, they are used to take photograph of long distance objects.
(iv) Infra red absorption spectrum is used to study the molecular structure.

4. Visible light *:* Visible light emitted or reflected from objects around us provides information about the world. The wavelength range of visible light is 4000 Å to 8000 Å.

5. Ultra−violet radiations

(i) They are used to destroy the bacteria and for sterilizing surgical instruments.
(ii) These radiations are used in detection of forged documents, fingerprints in forensic laboratories.
(iii) They are used to preserve the food items.
(iv) They help to find the structure of atoms.

6. X rays :

(i) X rays are used as a diagnostic tool in medicine.
(ii) It is used to study the crystal structure in solids.

7. γ–rays : Study of γ rays gives useful information about the nuclear structure and it is used for treatment of cancer

Solved Numerical

Q) A 1000 W bulb is kept at the centre of a spherical surface and is at a distance of 10 m from the surface. Calculate the force acting on the surface of the sphere by the electromagnetic waves, along with E_0, B_0 and intensity I. Take the working efficiency of the bulb to be 2.5% and consider it as a point source, , calculate the energy density on the surface .

Solution:

The energy consumed every second by a 1000W bulb = 1000J

As the working efficiency of the bulb is equal to 2.5%, the energy radiated by the bulb per second

$$\Delta U = 1000 \times \frac{2.5}{100}$$
$$\therefore \Delta U = 25\, Js^{-1}$$

Considering, the bulb at the centre of the sphere, surface area of the sphere

A = $4\pi R^2$ = (4)(3.14)(10^2) = 1256 m^2

Intensity I

$$I = \frac{Energy}{(time)(area)} = \frac{25}{1256} = 0.02 Wm^{-2}$$
$$I = \varepsilon_0 c E_{rms}^2 = 0.02$$
$$\therefore E_{rms} = \left[\!\left[\frac{0.02}{8.85 \times 10^{-12} \times 3.0 \times 10^5}\right]\!\right]^{1/2} = 2.74\, Vm^{-1}$$

NOW

$$B_{rms} = \frac{E_{rms}}{c}$$
$$B_{rms} = \frac{2.74}{3.0 \times 10^8} = 9.13 \times 10^{-9} T$$
$$E_0 = \sqrt{2} E_{rms}$$
$$E_0 = 1.41 \times 2.74 = 3.86\, Vm^{-1}$$

$$B_0 = \sqrt{2} B_{rms}$$
$$B_0 = 1.41 \times 9.13 \times 10^{-9} = 1.29 \times 10^{-8}\, T$$

The total energy incident on the surface = 25J

∴ The momentum (Δp) imparted to the surface in one second (= force)

$$\Delta p = \frac{\Delta U}{c} = F = \frac{25}{3 \times 10^8} = 8.33 \times 10^{-8} N$$

From I = ρc, energy density

$$\rho = \frac{I}{c} = \frac{0.02}{3 \times 10^8} = 6.67 \times 10^{-11} Jm^{-3}$$

Q) The maximum electric field at a distance of 10 m from an isotropic point source of light is 3.0 V/m. Calculate (a) the maximum value of magnetic field (b) average intensity of the light at that place and (c) the power of the source

$\varepsilon_o = 8.854 \times 10^{-12}\ C^2N^{-2}\ m^{-2}$

Solution

(a)maximum value of magnetic field

E = Bc

$$B = \frac{E}{c} = \frac{3.0}{3.0 \times 10^8} = 10^{-8}\mathrm{T}$$

(b) average intensity of the light at that place

From formula

$$I = \varepsilon_0 c E_{rms}^2 = \varepsilon_0 c \times \frac{E_0^2}{2}$$

$$I = 8.854 \times 10^{-12} \times 3.0 \times 10^8 \times \frac{(3.0)^2}{2}$$

$$I = 1.195 \times 10^{-2}\ wm^{-2}$$

(c) Power

Power = I × Area = I ×4πr^2

Power = 1.195×10^{-2}×4×3.14×(10)^2 = 15 w

----------------END--------------

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RAY OPTICS AND OPTICAL INSTRUMENTS

SECTION I

REFLECTION OF LIGHT

Nature of light

- Light is an electromagnetic radiation which causes sensation in eyes.
- Wavelength of visible light is 400 nm to 750 nm
- Speed of light in vacuum is highest speed attainable in nature 3.0×10^8 m/s
- Wavelength of light is very small compared to the size of the ordinary objects, thus light wave is considered to travel from one point to another along straight line joining two points.
- The straight path joining two points is called ray of light.
- Bundle of rays is called a beam of light
- Light show optical phenomenon such as reflection , refraction, interference and diffraction

REFLECTION OF LIGHT BY SPHERICAL MIRRORS

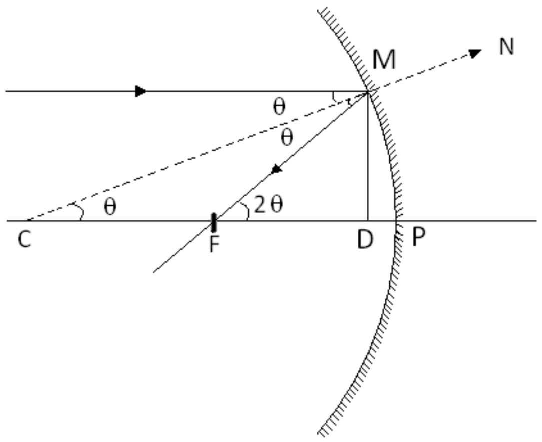
Law of reflection. I) The angle of incidence (angle between incident ray and normal to the surface) and angle of reflection (angle between reflected ray and normal to the surface) are equal.

ii) Incident ray, reflected ray and normal to the reflecting surface at the point of incidence lie in the same plane.

Note : Normal to the curved surface always passes through the centre of curvature

Sign convention: (i) All distances are measured from the pole of the mirror.
(ii) The distance in the direction of incidence light is taken as positive.
(iii) Distance in opposite to the direction of incident light is taken as negative
(iv) Distance above the principal axis is taken as positive.
(v) Distance below the principal axis taken as negative.

FOCAL LENGTH OF SPHERICAL MIRROR



Let P the pole of concave mirror, F be focal point and C be the centre of curvature.
Consider incident light parallel to principal axis strikes the mirror at point M and reflected rays passes through focal point F.
MD is perpendicular from M on principal axis.
Let $\angle MCP = \theta$
Then from geometry of figure, $\angle MFP = 2\theta$.
Now $\tan\theta = \frac{MD}{CD}$ and $\tan 2\theta = \frac{MD}{FD}$ –eq(1)

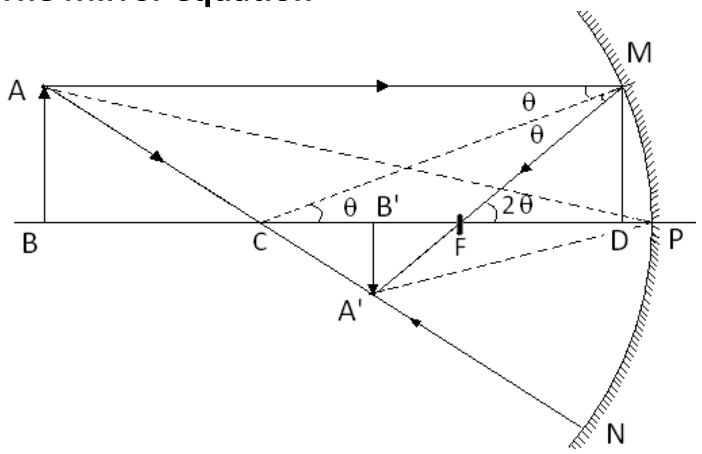
For small θ , tan θ = θ and tan2 θ = 2 θ

Therefore from eq(1) $2\frac{\mathrm{MD}}{\mathrm{CD}} = \frac{\mathrm{MD}}{\mathrm{FD}}$

Thus FD = CD/2 But CD = R and FD = f

Thus f = R/2

The Mirror equation



As shown in figure AB is object while A'B' is image of the object . AM and AN are two incident rays emitted from point A. Let F be focal point and FD = f focal length

For small aperture , DP will be very small neglecting DP we will take FD = FP = f

I) In Δ A'B'F and ΔMDF are similar as ∠MDF = ∠ A'B'F

$$\frac{A'B'}{MD} = \frac{B'F}{FD}$$

As MD = AB and FD = FP

$$\frac{A'B'}{AB} = \frac{B'F}{FP}$$

ii) In Δ A'B'P and Δ ABP are also similar. Therefore

$$\frac{A'B'}{AB} = \frac{PB'}{PB}$$

Thus comparing above two equations

$$\frac{B'F}{FP} = \frac{PB'}{PB}$$

But B'F = PB'-PF

$$\frac{PB' - PF}{FP} = \frac{PB'}{PB} \quad - eq(1)$$

From sign convention

Focal length = PF = -f

Image distance PB' = -v

Object distance = PB = -u

Putting the values in equation (1)

$$\frac{-v - (-f)}{-f} = \frac{-v}{-u}$$

$$\frac{v - f}{f} = \frac{v}{u}$$

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

This relation is known as mirror equation

Magnification

In Δ A'B'P and Δ ABP are also similar. Therefore

$$\frac{A'B'}{AB} = \frac{PB'}{PB}$$

From sign convention

PB' = image distance =-v

PB = object distance = -u

A'B' = size of image = -h '

AB = size of object = h

$$\frac{-h'}{h} = \frac{-v}{-u}$$

$$\frac{h'}{h} = -\frac{v}{u}$$

Magnification, m = size of the image / size of object = h'/h

$$m = -\frac{v}{u}$$

Note : If m is negative , image is real and inverted

If m is positive image is virtual and inverted

If |m| = 1 , size of the object = size of image

If |m| >1 size of image > size of the object

If |m| < 1 size of the image < size of the object

Solved Numerical

Q) An object is placed in front of concave mirror at a distance of 7.5 cm from it. If the real image is formed at a distance of 30 cm from the mirror, find the focal length of the mirror. What should be the focal length if the image is virtual?

Solution: Case I: When the image is real

U = -7.5 cm; v=-30cm; f= ?

We know that

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{f} = \frac{1}{-30} + \frac{1}{-7.5}$$

$$\frac{1}{f} = \frac{-5}{30}$$

f = -6 cm

The negative sign shows that the spherical mirror is convergent or concave

Case II : When image is virtual

u = -7.5 cm ; v = +30 cm

We know

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{f} = \frac{1}{30} + \frac{1}{-7.5}$$

$$\frac{1}{f} = \frac{-3}{30} = \frac{-1}{10}$$

f = -10 cm

Q) An object 0.5 cm high is placed 30 cm from convex mirror whose focal length is 20 cm. Find the position, size and nature of the image.

Solution: We have

U = -30 cm, f = +20 cm

Form mirror formula

$$\frac{1}{20} = \frac{1}{v} + \frac{1}{-30}$$

$$\frac{1}{v} = \frac{1}{12}$$

V = 12 cm

The image is formed 12 cm behind the mirror. It is virtual and erect

m = h'/h = -v/u = -12/30

$$m = \frac{h'}{h} = \frac{-v}{u} = \frac{-12}{-30} = 0.4$$

h' = mh = 0.4 ×0.5 = 0.2 cm

positive sign of m indicate image is erect.

Q) A thin rod AB of length 10 cm is placed on the principal axis of a concave mirror such that its end B is at a distance of 40 cm from the mirror and end A is further away from the mirror. If the focal length of the mirror is 20cm, find the length of the image of rod

Solution: given f = -20 cm, distance of B = u_1 =- 40 cm,

Since B is at centre of curvature image will be formed at -40 cm

Distance of A u_2 = -50 cm

$$\frac{1}{-20} = \frac{1}{v} + \frac{1}{-50}$$

V = -33.3 cm

Image of A is also on the side of object , Now length of image = 40-33.3 = 6.70 cm

SECTION II
REFRACTION OF LIGHT

REFRACTION

When an obliquely incident ray of light travels from one transparent medium to other it changes it direction of propagation at the surface separating two medium. This phenomenon is called as refraction. This phenomenon is observed as light have different velocity in different medium.

Laws of refraction

(i) The incident ray, refracted ray and the normal to the interface at the point of incidence, all lie in the same plane

(ii) The ratio of the sine of the angle of incidence to the sine of angle of refraction is constant.

$$\frac{sini}{sinr} = n_{21}$$

Where n_{21} is constant, called refractive index of the second medium with respect to first medium. Equation is known as Snell's law of refraction

Case i) $n_{21} > 1$, $r < i$, refracted ray bends towards normal , then medium 2 is called optically denser medium

Case ii) $n_{21} < 1$, $r > i$, refracted ray bends away from normal, then medium 2 is called optically rarer medium

If n_{21} is refractive index of medium 2 with respect to medium 1 and n_{12} is the refractive index of medium 1 with respect to medium 2 then

$$n_{21} = \frac{1}{n_{12}}$$

Similarly

$$n_{32} = n_{31} \times n_{12}$$

General form of Snell's law

$$n_{21} = \frac{v_1}{v_2}$$

Absolute refractive index is given by formula

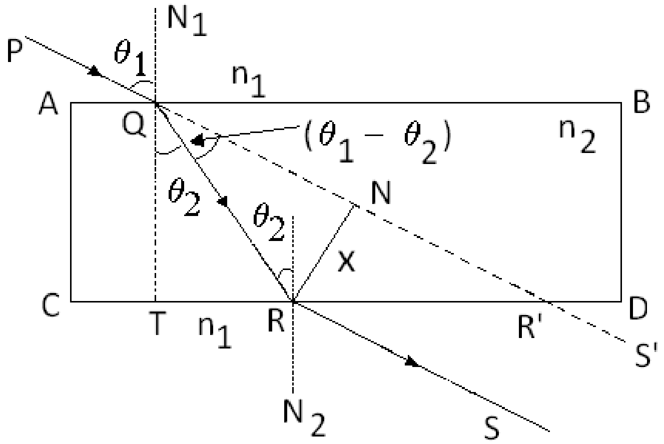
$$n = \frac{C}{v}$$

Thus $n_1 = C/v_1$ and $n_2 = C/v_2$

$v_1 = C/n_1$ and $v_2 = C/n_2$

$$n_{21} = \frac{C/n_1}{C/n_2} = \frac{n_2}{n_1}$$
$$\frac{sini}{sinr} = \frac{n_2}{n_1}$$
$$n_1 sini = n_2 sinr$$

Lateral shift



As shown in figure light ray undergoes refraction twice, once at top (AB) and then from bottom (CD) surfaces of given homogeneous medium.
The emergent ray is parallel to PQR'S' ray. Here PQR'S' is the path of light in absence of the other medium.
Since emergent ray is parallel to the incident ray but shifted sideways by distance RN. This RN distance is called lateral shift (x)

Calculation of lateral shift

Let n_1 and n_2 be the refractive indices of the rarer and denser medium, respectively. Also $n_1 < n_2$ From figure $\angle RQN = (\theta_1 - \theta_2)$, $RN = X$

i) From ΔQRN, $\sin(\theta_1 - \theta_2) = RN/QR = X/QR$ --eq(1)

ii) In ΔQTR, $\cos\theta_2 = QT/QR$

$\therefore QR = QT/\cos\theta_2$; $QR = t/\cos\theta_2$

From equation (1)

$$\sin(\theta_1 - \theta_2) = \frac{x}{\frac{t}{\cos\theta_2}}$$

$$x = \frac{t\sin(\theta_1 - \theta_2)}{\cos\theta_2}$$

If angle of incidence θ_1 is every small, θ_2 will also be small

$\sin(\theta_1 - \theta_2) \approx (\theta_1 - \theta_2)$ and $\cos\theta_2 \approx 1$

$$x = t(\theta_1 - \theta_2)$$

$$x = t\theta_1\left(1 - \frac{\theta_2}{\theta_1}\right)$$

From Snell's law

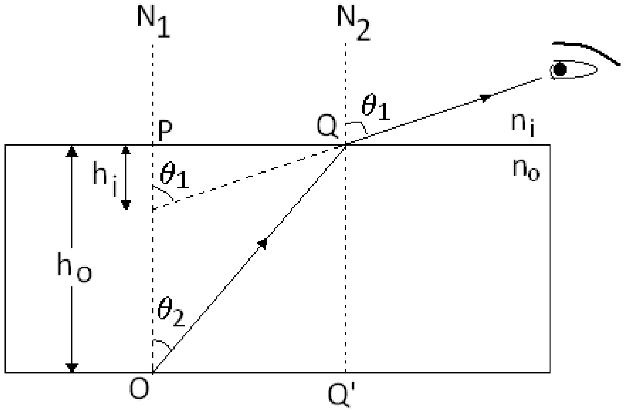
$$\frac{sin\theta_1}{sin\theta_2} = \frac{\theta_1}{\theta_2} = \frac{n_2}{n_1}$$

Thus

$$x = t\theta_1\left(1 - \frac{n_1}{n_2}\right)$$

Real Depth and Virtual depth

Is Another example of lateral shift.



n_i is the refractive index of observer medium, n_o is the refractive index of object medium. h_o is the real depth and h_i is virtual depth.
Applying Snell's law at point Q,
$n_i sin\theta_1 = n_o sin\theta_2$ For normal incidence θ_1 and θ_2 are very small.
$\therefore sin\theta \approx \theta \approx tan\theta$
$n_i tan\theta_1 = n_o tan\theta_2$ . .---- Eq(1) But from figure

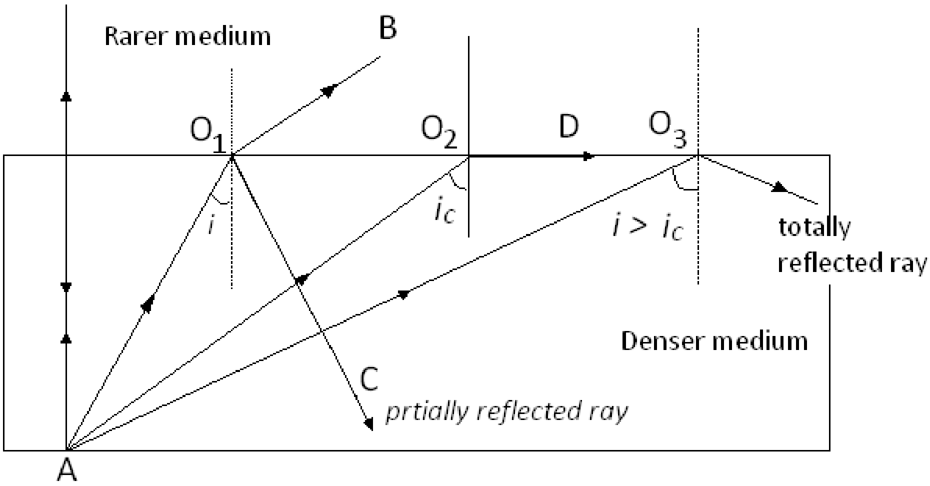
$$tan\theta_1 = \frac{PQ}{PI} = \frac{PQ}{h_i} \text{ and } tan\theta_2 = \frac{PQ}{PO} = \frac{PQ}{h_o}$$

Thus from equation (1)

$$n_i\frac{PQ}{h_i} = n_o\frac{PQ}{h_o}$$

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

Total Internal Reflection



When light travels from optically denser medium to rarer medium at the interface, it is partly reflected back into same medium and partly refracted to the second medium. This reflection is called the *internal reflection*.

When a ray of light enters from a denser medium to a rarer medium, it bends away from the normal.

for example. As shown in figure , The incident ray AO_1 is partially reflected (O_1 C) and partially transmitted (O_1 B) or refracted.

the angle of refraction (r) being larger than the angle of incidence (i).

As the angle of incidence increases, so does the angle of refraction, till for the ray AO_2 the angle of refraction is $\pi/2$. The refracted ray is bent so much away from the normal that it grazes the surface at the interface between the two media.

This is shown by the ray AO_2 D in Fig.

If the angle of incidence is increased still further (e.g, the ray AO_3), refraction is not possible. And the incident ray is totally reflected. This is called total internal reflection.

The angle of incidence for which angle of refraction is $\pi/2$ is called critical angle

From Snell's law $\sin i / \sin r = n_2 / n_1$ If n_2 is air then $n_1 = 1$

When $i = i_C$ (critical angle) then $r = \pi/2$. Let n_1 refractive index of denser medium = n Then

$\sin i_C = 1/n$

Solved Numerical

Q) A ray of light is incident at angle of 60^O on the face of a rectangular glass slab of the thickness 0.1m and refractive index 1.5 . Calculate the lateral shift

Solution: Here $i = 60^O$; $n = 1.5$ and $t = 0.1$ m

From Snell's law

$$n = \frac{\sin i}{sinr}$$

$$sinr = \frac{sini}{n} = \frac{sin60}{1.5} = \frac{0.866}{1.5} = 0.5773$$

Thus r = $35^{O}15'$

Now lateral shift

$$\mathrm{x} = \frac{\mathrm{tsin}(\theta_1 - \theta_2)}{\cos\theta_2} = \frac{0.1 \times sin(60^{O} - 35^{O}15')}{cos35^{O}15'} = 0.0512\, m$$

Q) A tank filled with water to a height of 12.5cm. The apparent depth of needle lying at the bottom of the tank is measured by a microscope to be 9.4 cm. What is the refractive index 1.63 up to the same height, by what distance would the microscope have to be moved to focus on the needle again?

Solution:

Case I : When the tank is filled with water:

Real depth h_O = 12.5 cm Apparent depth h_1 = 9.4 cm, n_i = air = 1

From formula

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

$$\frac{1}{n_o} = \frac{9.4}{12.5}$$

n_O = 1.33

Case II: When tank filled with the liquid of refractive index = n_O = 1.63

$$\frac{1}{1.63} = \frac{h_i}{12.5}$$

$$h_i = \frac{12.5}{1.63} = 7.67\ cm$$

Therefore the distance through which the microscope to be moved = 9.4 -767 = 1.73 cm

Q) A fish rising vertically to the surface of water in a lake uniformly at the rate of 3 m/s observes kingfisher bird diving vertically towards water at the rate 9 m/s vertically above it. If the refractive index of water is 4/3, find the actual velocity of the dive of the board.

Solution: Velocity of bird with respect to stationary fish in water = 6m/s

Thus apparent displacement of bird in 1 sec h_i = 6 m

n_i = 4/3, n_O = 1 (air)

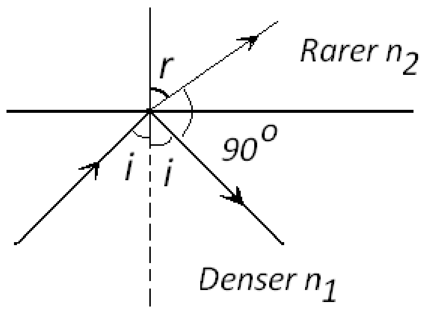
Now from formula

$$\frac{n_i}{n_o} = \frac{h_i}{h_o}$$

$$\frac{\frac{4}{3}}{1} = \frac{6}{h_o}$$

$$h_O = \frac{6 \times 3}{4} = 4.5\ m/s$$

Q) A ray of light from a denser medium strikes a rarer medium at an angle of incidence i. If the reflected and the refracted rays are mutually perpendicular to each other, what is the value of the critical angle?



Solution: From Snell's law, we have

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} = n_{21}$$

According to given problem

i+r+90 = 190

R = 90-i

Thus from above equation

$$n_{21} = \frac{sin(90 - i)}{sini} = coti$$

By definition for critical angle

$$i_c = sin^{-1}\left(\frac{1}{n_{21}}\right)$$

$$i_c = sin^{-1}\left(\frac{1}{cot\ i}\right)$$

$$i_c = sin^{-1}(\tan i)$$

Examples of total internal reflection

(i) **Mirage**: In summer, air near the surface becomes hotter than the layer above it. Thus as we go away from the surface air becomes optically denser.

Ray of light travelling from the top of tree or building towards the ground it passes through denser air layer to rarer layers of air, as a result refracted rays bend away from normal and angle of incidence on consecutive layers goes on increasing, at a particular layer angle of incidence is more than critical angle rays gets internally reflected and enters the eye of observer and observer observes inverted image of the object. Such inverted image of distant tall object causes an optical illusion to observer. This phenomenon is called mirage.

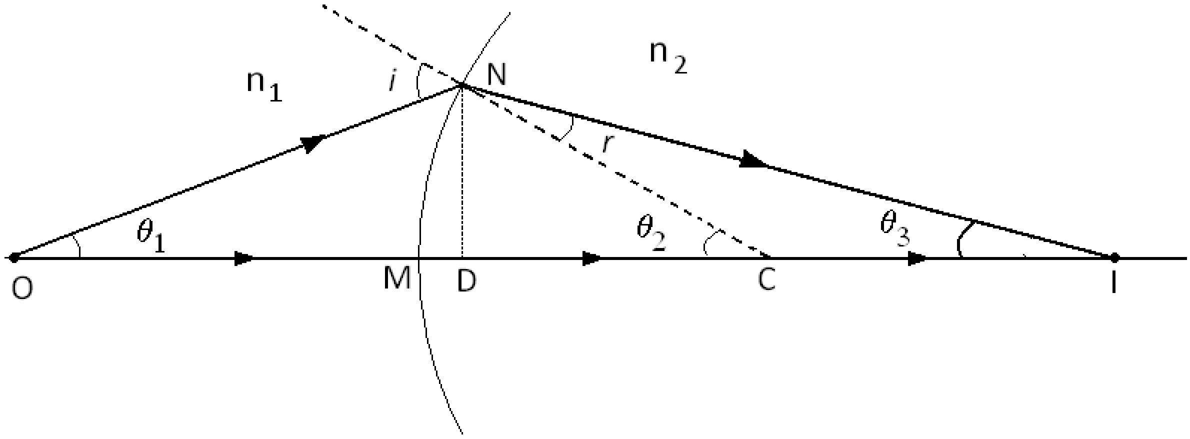
(ii) Diamond: Brilliance of diamond is due the internal reflection of light. Critical angle of diamond- air interface is (24.4^O) is very small, it is very likely to undergo total internal reflection inside it. By cutting the diamond suitably, multiple total internal reflections can be made to occur.

(iii) Prism: Prism designed to bend light by 90^O or 180^O make use of total internal reflection. Such a prism is also used to invert images without changing the size.

(iv) Optical fibres. Optical fibres make use of the phenomenon of total internal reflection. When signal in the form of light is directed at one end of the fibre at suitable angle, it undergoes repeated total internal reflections along the length of the fibre and finally comes out of the other end.

Optical fibres are fabricated with high quality composite glass/quartz fibres. Each fibre consists of a core and cladding. The refractive index of the material of the core is higher than that of the cladding. The main requirement in fabrication of optical fibres is that there should be very little absorption of light as it travels for long distances inside them. This has been achieved by purification and special preparation of materials such as as quartz. In silica fibres, it is possible to transmit more than 95 of light over a fibre length of 1km

Refraction at spherical surface and by lenses



MO is object distance = - u
MC is radius of curvature = +R
MI is image distance = +v
The ray incident from a medium of refractive index $n_{1+,}$ to another of refractive index n_2.
We take the aperture of the surface so small that we can neglect the distance MD.

$$tan\theta_1 = \frac{ND}{OD} = \frac{MN}{OM}$$

$$tan\theta_2 = \frac{ND}{CD} = \frac{MN}{CM}$$

$$tan\theta_3 = \frac{ND}{ID} = \frac{MN}{IM}$$

θ_1, θ_2, θ_3 are very small , for small θ , tan θ = θ

$$\theta_1 = \frac{ND}{OD} = \frac{MN}{OM} = \frac{MN}{-u}$$

$$\theta_2 = \frac{ND}{CD} = \frac{MN}{CM} = \frac{MN}{R}$$

$$\theta_3 = \frac{ND}{ID} = \frac{MN}{IM} = \frac{MN}{v}$$

From figure i is exterior angle i = $\theta_1 + \theta_2$ and

θ_2 is exterior angle thus $\theta_2 = r + \theta_3$ or $r = \theta_2 - \theta_3$
Applying Snell's law at point N we get
$n_1 \sin i = n_1 \sin r$
for small angles of I and r
$n_1 i = n_2 r$
$n_1 (\theta_1 + \theta_2) = n_2 (\theta_2 - \theta_3)$

$$n_1\left(\frac{MN}{-u}+\frac{MN}{R}\right)=n_2\left(\frac{MN}{R}-\frac{MN}{v}\right)$$

$$n_1\left(\frac{1}{-u}+\frac{1}{R}\right)=n_2\left(\frac{1}{R}-\frac{1}{v}\right)$$

By rearranging the terms

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{R}$$

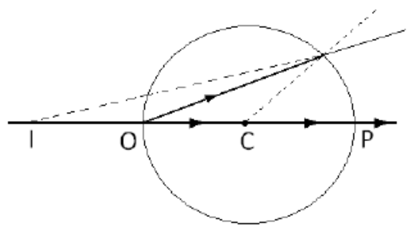
Above equation gives relation between object distance and image distance in terms of refractive index of the medium and radius of curvature of the spherical surface. It holds good for any spherical surface.

curved surface magnification

$$m=\frac{n_1}{n_2}\frac{v}{u}$$

Solved Numerical

Q) If a mark of size 0.2 cm on the surface of a glass sphere of diameter 10cm and n = 1.5 is viewed through the diametrically opposite point, where will the image be seen and of what size?



Solution:
As the mark is on one surface, refraction will occur on other surface (which is curved). Further refraction is taking place from glass to air so,
$n_2 = 1$ (air) $n_1 = 1.5$ (glass)object distance = diameter of sphere = -10cm and R = -5 cm
Using formula

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{R}$$

$$\frac{1}{v} - \frac{1.5}{-10} = \frac{1-1.5}{-5}$$

V = -20 cm

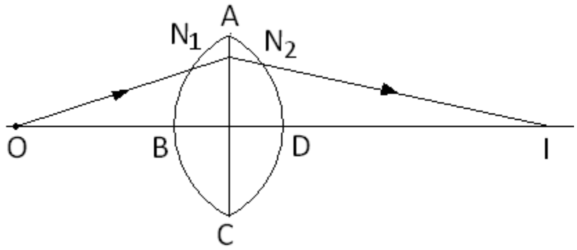
Hence image is at a distance of 20 cm from P towards Q

In case of refraction at curved surface magnification

$$m = \frac{n_1}{n_2}\frac{v}{u} = \frac{1}{1.5} \times \frac{-20}{-10} = +3$$

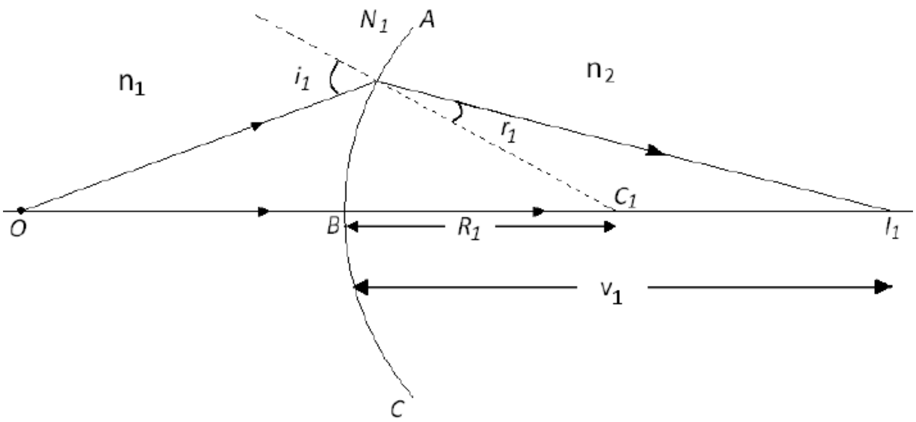
So, the image is virtual erect and of size h' = m×h = 3×0.2 = 0.6 cm

Refraction by lens



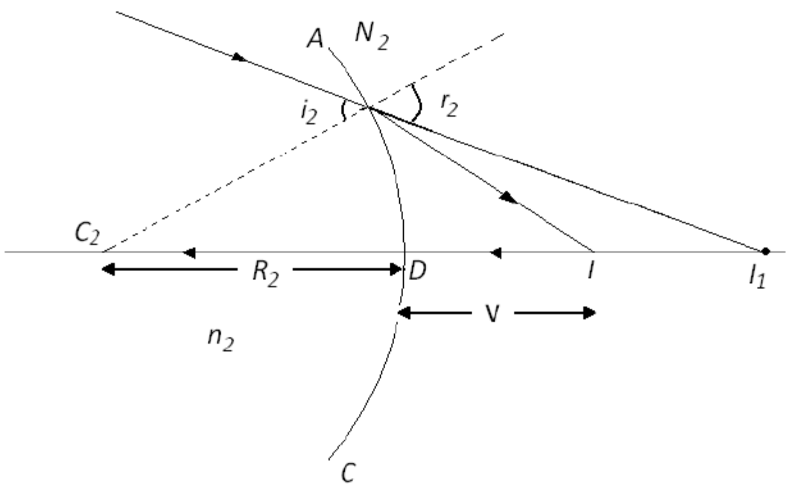
Lens shown in diagram have two curved surfaces. Surface ABC (N_1) have radius of curvature R_1 . Another surface ADC (N_2) have radius of curvature R_2. Image formed by Surface N_1 acts as an object for surface N_2 and final image is formed at I

Consider surface ABC as shown in figure here object O is in medium of refractive index n_1 and form image at I_1 at a distance v_1 in medium of refractive index n_2. from formula for refraction at curved surface.



$$\frac{n_2}{v_1} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1} \quad ..eq(1)$$

Now image I_1 which is in medium of refractive index n_2 acts like a object for surface ADC (N_2) and forms image in medium of refractive n_1 at I



here u = v_1 and radius of curvature = R_2
From formula for refraction at curved surface

$$\frac{n_1}{v} - \frac{n_2}{v_1} = \frac{n_1 - n_2}{R_2} \quad ..eq(2)$$

On adding equation (1) and (2) we get

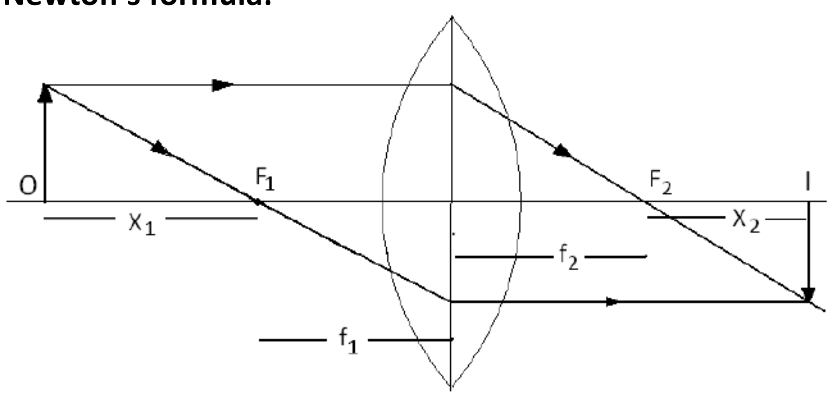
$$\frac{n_1}{v} - \frac{n_1}{u} = (n_2 - n_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(3)$$

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(4)$$

Here $\frac{n_2}{n_1}$ is refractive index of lens material with respect to surrounding.
Lens-Makers's formula
If u = ∞ the image will be formed at f thus from equation 4

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad ..eq(5)$$

Magnification
m = v/u
Newton's formula:



x_1 and x_2 are known as extra focal distance and extra focal image distance.
$x_1 . x_2 = f_1f_2$
if $f_1 = f_2 = f$ then
$x_1 . x_2 = f^2$

Power of lens: If defined as the tangent of the angle by which it converges or diverges a beam of light falling at unit distant from the optical centre

$$tan\delta = \frac{h}{f}$$

If h = 1 then

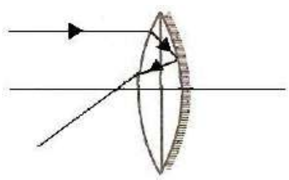
$$tan\delta = \frac{1}{f}$$

For small value of δ

$$\delta = \frac{1}{f}$$

Thus P = 1 / f SI unit for power of lens is diopter (D) 1D = 1 m^{-1}

Lens with one surface silvered

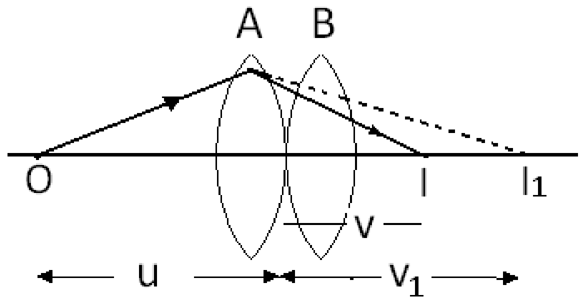


When one surface is silvered, the rays are reflected back at this silvered surface and set up acts as a spherical mirror
Focal length is given by formula

$$\frac{1}{f} = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

Here f_l is focal length of lens , f_m is focal length of mirror
In the above formula, **the focal length of converging lens or mirror is taken positive and that of diverging lens or mirror is taken as negative.**

Combination of lenses in contact



Consider two lenses A and B of focal length f_1 and f_2 placed in contact with each other. Let the object be placed at a point O beyond the focus of the first lens A.
The first lens produces the image at I_1. Since the image is real. It serves as object for second lens B produces image at I
The direction of rays emerging from the first lens gets modified in accordance with the angle at which they strike the second lens.
We assume the optical centres of the thin lens coincident.
For image formed by lens A. we get

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}$$

For the image formed by second lens B, we get

$$\frac{1}{v}-\frac{1}{v_1}=\frac{1}{f_2}$$

Adding above equations

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}$$

If two lens-system is regarded as equivalent to a single lens of focal length f, we have

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

So we have

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$$

The derivation is valid for any number of thin lenses in contact

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}\ldots\ldots\ldots$$

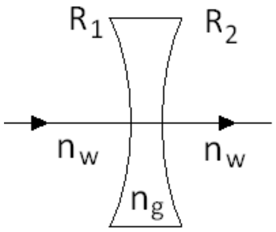
In terms of equivalent power

$P = P_1 + P_2 + P_3 + \ldots\ldots$

Image form by first lens is object for second lens it implies that total magnification for combination of lenses is

$m = m_1\ m_2\ m_3 \ldots\ldots$ (here m_1 and m_2…etc are magnification of individual lenses)

Solved Numerical



Q) Calculate the focal length of a concave lens in water (Refractive index = 4/3) if the surface have radii equal to 40cm and 30 cm (refractive index of glass = 1.5)

Solution: R_1 = -30cm R_2 = +40cm

We have

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{1}{f}=\left(\frac{1.5}{4/3}-1\right)\left(\frac{1}{-30}-\frac{1}{40}\right)$$

$$\frac{1}{f}=-\frac{960}{7}=-131.7\ cm$$

Q) A plano-covex lens has focal length 12cm and is made up of glass with refractive index 1.5. Find the radius of curvature of its curved side

Solution: Let R_1 be the radius of curvature, $R_2 = \infty$, f = +12cm

From formula for focal length

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$
$$\frac{1}{12}=\left(\frac{1.5}{1}-1\right)\left(\frac{1}{R_1}-\frac{1}{\infty}\right)$$
R_1 = 6 cm

Q) A magnifying lens has a focal length of 10cm (i) Where should the object be placed if the image is to be 30cm from the lens? (ii) What will be the magnification
Solution: For a convergent lens , If image formed on object side ,
Thus v = -30 cm f = +10cm
Let x be the distance of object
Using lens formula

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$
$$\frac{1}{10}=\frac{1}{-30}-\frac{1}{-x}$$
x= 7.5 cm

(ii) magnification m = v/u = -30/-7.5 = +4
Thus image is virtual erect and four times the size of the object.

Q) An object 25 cm high is placed in front of a convex lens of focal length 30 cm. If the height of the image formed is 50 cm, find the distance between the object and the image
Solution
We have
|m| = h'/h = v/u = 50/25 = 2
There are two possibilities
(i) if the image is inverted (i.e.) real
M = V/u = -2 or V = -2u
Let x be the object distance, in this case
We have u = -x, v = +2x f = + 30cm
Using lens formula

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$
$$\frac{1}{30}=\frac{1}{2x}-\frac{1}{-x}$$
x = 45 cm then v = 90 cm

Hence distance between object and image is = 45 + 90 = 135

(ii) If the image is erect (i.e virtual)
m = v/u = +2
Let x' be the object distance , we have
U = -x'; v = -2x' ; f = 30 cm

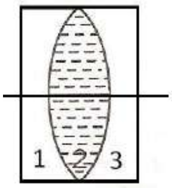
Using the lens formula, we have

$$\frac{1}{30} = \frac{1}{-2x'} - \frac{1}{-x'}$$

x' = 15 cm

Hence distance between object and image = 15 cm

Q) Two plano-concave lenses made of glass of refractive index 1.5 have radii of curvature 20 cm and 30cm. they are placed in contact with curved surface towards each other and the space between them is filled with liquid of refractive index 4/3. Find the focal length of the system



Solution: As shown in figure, the system is equivalent to combination of three lenses in contact

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

By lens maker's formula

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_1} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{\infty} - \frac{1}{20}\right) = \frac{1}{40} cm$$

$$\frac{1}{f_2} = \left(\frac{4}{3} - 1\right)\left(\frac{1}{20} - \frac{1}{-30}\right) = \frac{5}{180} cm$$

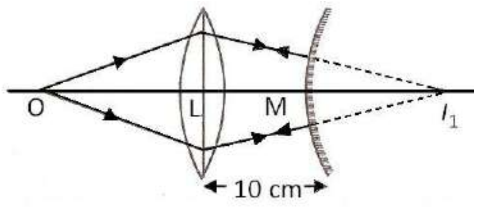
$$\frac{1}{f_3} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{-30} - \frac{1}{\infty}\right) = \frac{-1}{60} cm$$

Now

$$\frac{1}{F} = \frac{1}{40} + \frac{1}{180} + \frac{-1}{60} = \frac{-1}{72}$$

F = -72 cm

Q) A point object is placed at a distance of 12 cm on the axis of convex lens of focal length 10cm. On the other side of the lens, a convex mirror is placed at the distance of 10 cm from the lens such that the image formed by the combination coincides with the object itself. What is the focal length of the mirror.?



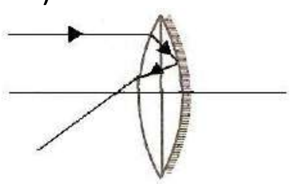
Solution :
For the refraction at the convex lens, we have u= -12 cm, v = ? f = + 10 cm
Using lens formula, we have

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$

$$\frac{1}{10}=\frac{1}{v}-\frac{1}{-12}$$

V = +60cm
Thus, in the absence of convex mirror, convex lens will form the image at I_1 , at a distance 60 cm behind the mirror. This image will act as object for mirror t, object distance for mirror = 60-10 = +50cm
Now as the final image I is formed at the object itself, indicates that the rays on the mirror are incident normally i.e image I_1 is at the centre of curvature of mirror
So R = 50 cm and f = R/2 = 50/2 = +25cm

Q) One face of an equiconvex lens of focal length 60cm made of glass (Refractive index 1.5) is silvered. Does it behave like a concave mirror or convex mirror



Solution : Let x be the radius of curvature of each surface from lens maker's formula

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{1}{60}=(1.5-1)\left(\frac{1}{x}-\frac{1}{-x}\right)$$

$$\frac{1}{60}=0.5\times\frac{2}{x}$$

X = 60 cm

Let f be the focal length of the equivalent spherical mirror. Then

$$\frac{1}{f}=\frac{1}{f_l}+\frac{1}{f_m}+\frac{1}{f_l}$$

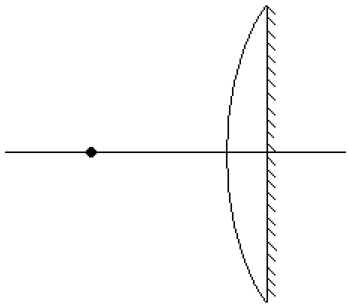
Since lens is equiconvex

$$\frac{1}{f}=\frac{2}{f_l}+\frac{1}{f_m}$$

Here f_l = +60 cm (convex lens) , f_m =R/2 = +30 (concave mirror)

$$\frac{1}{f} = \frac{2}{60} + \frac{1}{30} = \frac{1}{15}$$

F = +15cm cm
The positive sign indicates that the resulting mirror is concave



Q) The plane surface of a plano-convex lens of focal length 60 cm is silvered . A point object is placed at a distance 20cm from the lens. Find the position and nature of the final image formed

Solution:
Let f be the focal length of the equivalent spherical mirror
We have

$$\frac{1}{f} = \frac{1}{f_l} + \frac{1}{f_m} + \frac{1}{f_l}$$

Or

$$\frac{1}{f} = \frac{2}{f_l} + \frac{1}{f_m}$$

Here f_l = + 60 cm, f_m = ∞

$$\frac{1}{f} = \frac{2}{60} + \frac{1}{\infty} = \frac{1}{30}$$

Or f = + 30 cm
The problem is reduced to a simple case where a point object is placed in front of a cocave mirror/
Now, using the mirror formula

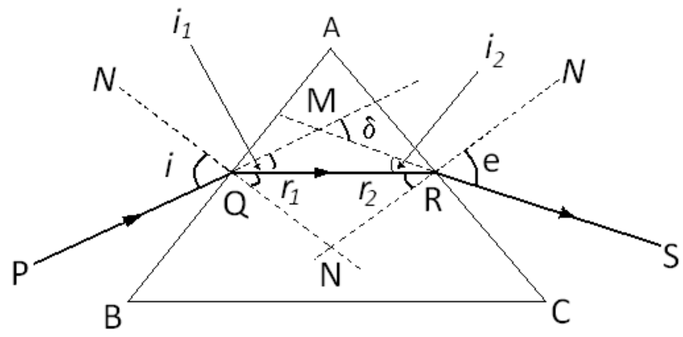
$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{-30} = \frac{1}{v} + \frac{1}{-20}$$

V = 60 cm

SECTION III
REFRACTION THROUGH A PRISM

REFRACTION THROUGH A PRISM



PQ is incident ray and angle made by PQ with normal is i. QR is refracted ray at surface BA and angle of refraction is r_1

For surface AC ray QR is incident ray and angle of incidence is r_2 .. RS refracted ray and angle is called angle of emergence

The angle between the emergent ray and RS and the direction of incident ray PQ is called the angle of deviation δ.

In quadrilateral AQNR, two angles (at vertices Q and R) are right angles. Therefore the sum of other angles of quadrilateral is 180^{O}

$\angle A + \angle QNR = 180^{O}$ --eq(1)

From the triangle QNR

$r_1 + r_2 + \angle QNR = 180^{O}$ --eq(2)

from equation (1) and (2)

$r_1 + r_2 = A$ --eq(3)

Now $\delta = i_1 + i_2$ (as δ is exterior angle) –eq(4)

Now $i_1 = (i - r_1)$ and $i_2 = (e - r_1)$ (from geometry of figure)

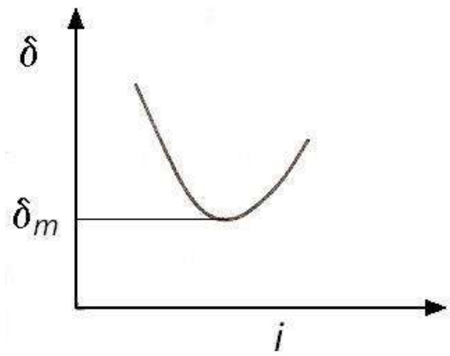
From equation 3

$\delta = (i - r_1) + (e - r_1)$

$\delta = (i + e) - (r_1 + r_1)$

$\delta = (i + e) - A$ --eq(5)

Thus , the angle of incidence depends on the angle of incidence.



From the graph δ vs i we can find that for every angle of deviation there are two values of angle of incidence. It means that vale of i and e are interchangeable. Except i=e . At i =e the angle of deviation is minimum denoted by δ_m or D_m, at angle of minimum deviation ray inside the prism becomes parallel.

Calculation of refractive index of Prism

By applying Snells law at the surface BA we get

$n_1 \sin i = n_2 \sin r_1$

$$n_{21} = \frac{n_2}{n_1} = \frac{sini}{sinr}$$

At angle of minimum deviation I = e implies $r_1 = r_2 = r$ From equation (3)

2r = A or r = A/2 ---eq(6)
From equation (5)
D_m= 2i – A or i = (A+D_m)/2 –eq(7)

Substituting values from equation 6 and 7

$$n_{21} = \frac{sin\frac{A+D_m}{2}}{sin\frac{A}{2}}$$

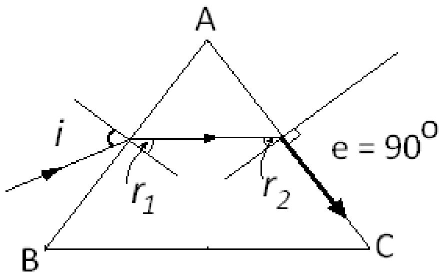
For small angle prism D_m is also very small and we get

$$n_{21} = \frac{sin\frac{A+D_m}{2}}{sin\frac{A}{2}} = \frac{\frac{A+D_m}{2}}{\frac{A}{2}} = \frac{A+D_m}{A}$$

$D_m = (n_{21} – 1) A$
It implies that, thin prism do not deviate light much

Solved Numerical

Q) A ray of light falls on one side of a prism whose refracting angle is 60^O . Find the angle of incidence in order that the emergent ray just graze the other side (Refractive index = 3/2)



Solution: Given A = 60^O, e = 90^O
∴ r_2 = C the critical angle of the prism
Now n = 1 /sinC
Or sinC = 1/n = 2/3
C = 41^{O}49'
Again, A = r_1 + r_2
r_1 = A - r_2 = 60^O – 41^{O}49' = 18^O 11'
For the refraction at surface AB we have

$$n = \frac{\sin i}{\sin r_1}$$

sin i = n sinr_1
sin i = 1.5 × sin 18^O 11'
sin i = 1.5 × 0.312 = 0.468
i = 27^{O}55'

Q) The refractive index of the material of a prism of refracting angle 45^O is 1.6 for a certain monochromatic ray. What should be the minimum angle of incidence of this ray on the prism so that no that no total internal reflection takes place as they come out of the prism
Solution:
Given A = 45^O , n = 1.6
We have
Sin C = 1/n
sinC = 1/1.6

$C = 38.68^{O}$

For total internal reflection not take place at the surface AC, we have

$r_2 \leq C$

or $(r_2)_{max} = C$

Now $r_1 + r_2 = A$

Or $r_1 = (A - r_2)$

Or $(r_1)_{mini} = A - (r_2)_{max} = 45^{O} - 38..68^{O} = 6.32^{O}$

For the refraction at the first face

We have

$$n = \frac{sini_1}{sinr_1}$$

Or $sini_1 = n\ sinr_1$

Sin $i_1 = 1.6 \times \sin(6.32^{O}) = 0.176$

$i_1 = 10.14^{O}$

Q) Find the minimum and maximum angle of deviation for a prism with angle $A = 60^{O}$ and $n = 1.5$

Solution : Minimum deviation:

The angle of minimum deviation occurs when i = e and $r_1 = r_2$ and is given by

$$n = \frac{sin\frac{(A+\delta_m)}{2}}{sin\frac{A}{2}}$$

$$\delta_m = 2sin^{-1}\left(nsin\frac{A}{2}\right) - A$$

Substituting n = 1.5 and $A = 60^{O}$, we get

$\delta_m = 2sin^{-1}(0.75) - 60 = 37^{O}$

Maximum deviation

The deviation is maximum when $i = 90^{O}$ or $e = 90^{O}$

Let $i = 90^{O}$

$r_1 = C = sin^{-1}(1/n)$

$r_1 = sin^{-1}(2/3) = 42^{O}$

$r_2 = A - r_1 = 60^{O} - 42^{O} = 18^{O}$

Using

$$\frac{sinr_2}{sine} = \frac{1}{n}$$

Sine = $nsinr_2$

$e = 28^{O}$

$\therefore$ Deviation = $\delta_{max} = (I + e) - A = 90 + 2\ 8 - 60 = 58^{O}$

Dispersion by a prism

The phenomenon of splitting of light into its component colours is known as dispersion.

Different colours have different wavelengths and different wavelengths have different velocity in transparent medium.
Longer wave length have more velocity compared to shorter wavelength. Thus refractive index is more for shorter vale length than longer wavelength.
Red colour light have highest wave length in visible light spectrum and hence lowest refractive index.
Violet colour light have smallest wave length in visible light spectrum., it have maximum refractive index.
Thick lens could be considered as maid of many prism, therefore, thick lens shows chromatic aberration due to dispersion of light

Some natural phenomena due to sunlight

THE RAINBOW

Primary rainbow

Rainbow is formed due to two times refraction and one time total internal reflection in water drop present in atmosphere

Sunlight is first refracted as it enters a raindrop, which causes the different wavelengths of white light to separate. Longer wavelength of light (red) are bent the least while the shorter wavelength (violet) are bent the most. Next, these component rays strike the inner surface of the water droplet and gets internally reflected if the angle of incidence is greater than the critical angle. The reflected light is again refracted as it comes out of the drop.

It is found that violet light emerges at an angle 40^{O} related to the incoming sunlight and red light emerges at an angle of 42^{O} . For other colours , angle lies between these two values.

Secondary rainbow

When light rays undergoes two internal refraction inside the rain drop, a secondary raindrop is formed. Its intensity is less compared to primary due to four stem process. Violet ray makes an angle of 53^{O} and red light rays makes 50^{O} with related to incoming sunlight. The order of colours is reversed

Rayleigh scattering

The amount of scattering is inversely proportional to the fourth power of the wavelength. This is known as **Rayleigh scattering**

Why sky is blue

As sunlight travels through the earth's atmosphere, it gets scattered (changes its direction) by the atmospheric particles. Light of shorter wavelengths is scattered much more than light of longer wavelengths. The amount of scattering is inversely proportional to the fourth power of the wavelength. This is known as **Rayleigh scattering**. Hence, the bluish colour predominates in a clear sky, since blue has a shorter wavelength than red and is scattered much more strongly. In fact, violet gets scattered even more than blue, having a shorter wavelength. But since our eyes are more sensitive to blue than violet, we see the sky blue.

Why clouds look white
Large particles like dust and water droplets present in the atmosphere behave differently. The relevant quantity here is the relative size of the wavelength of light λ, and do not follow **Rayleigh scattering** . For large scattering objects(for example, raindrops, large dust or ice particles) all wavelengths are scattered nearly equally. Thus, clouds which have droplets of water with larger particle are generally white.

At sunset or sunrise, Sun looks reddish
Sun rays have to pass through a larger distance in the atmosphere (Fig. 9.28).Most of the blue and other shorter wavelengths are removed by scattering. The least scattered light reaching our eyes, therefore, the sun looks reddish. This explains the reddish appearance of the sun and full moon near the horizon.

SECTION III
OPTICAL INSTRUMENTS

OPTICAL INSTRUMENTS
Eye
Function of ciliary muscles: The shape (curvature) and therefore the focal length of the lens can be modified somewhat by the ciliary muscles
For example, when the muscle is relaxed, the focal length is about 2.5 cm and objects at infinity are in sharp focus on the retina. When the object is brought closer to the eye, in order to maintain the same image-lens distance (≅ 2.5 cm), the focal length of the eye lens becomes shorter by the action of the ciliary muscles.
Accommodation: Property of eye to change the focal length as required is called accommodation.
Retina: The retina is a film of nerve fibers covering the curved back surface of the eye. The retina contains rods and cones which sense light intensity and colour, respectively, and transmit electrical signals via the optic nerve to the brain which finally processes this information.. For example, when the muscle is relaxed, the focal length is about 2.5 cm and objects at infinity are in sharp focus on the retina. When the object is brought closer to the eye, in order to maintain the same image-lens distance (≅ 2.5 cm), the focal length of the eye lens becomes shorter by the action of the ciliary muscles.
Least distance of distinct vision: If the object is too close to the eye, the lens cannot curve enough to focus the image on to the retina, and the image is blurred. The closest distance for which the lens can focus light on the retina is called the least distance of distinct vision, or the near point. The standard value for normal vision is taken as 25 cm. (Often the near point is given the symbol D.) This distance increases with age, because of the decreasing effectiveness of the ciliary muscle and the loss of flexibility of the lens. The near point may

be as close as about 7 to 8 cm in a child ten years of age, and may increase to as much as 200 cm at 60 years of age.

Presbyopia: If an elderly person tries to read a book at about 25 cm from the eye, the image appears blurred. This condition (defect of the eye) is called presbyopia. It is corrected by using a converging lens for reading.

Myopia: the light from a distant object arriving at the eye-lens may get converged at a point in front of the retina. This type of defect is called nearsightedness or myopia. This means that the eye is producing too much convergence in the incident beam. To compensate this, we interpose a concave lens between the eye and the object, with the diverging effect desired to get the image focused on the retina

Simple Microscope

A simple magnifier or microscope is a converging lens of small focal length.

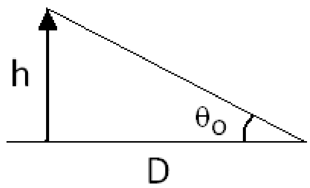
The least distance at which a small object can be seen clearly with comfort is known as near point (D) or distance of most distinct vision. For normal eye this distance is 25 cm.

Suppose a linear object with height h_O is kept at near point (i.e. u = D = 25 cm) from eye. Let it subtend an angle θ_O with eye .

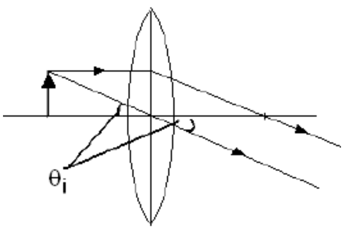
The magnification when the image is at infinity.

Now, if object is kept is kept at the focal length (f) of a convex lens such that its virtual image is formed at a infinity. In this case we will have to obtained the *angular* magnification.

Suppose the object has a height *h*. The maximum angle it can subtend, and be clearly visible (without a lens), is when it is at the near point, i.e., a distance *D.* The angle subtended is then given by



$$tan\theta_O \approx \theta_O = \frac{h}{D}$$



We now find the angle subtended at the eye by the image when the object is at *u*. From the relations $m = \frac{h'}{h} = \frac{v}{u}$

$$h' = \frac{v}{u}h$$

we have the angle subtended by the image

$$tan\theta_i \approx \theta_i = \frac{h'}{-v} = \frac{v}{u}h\left(\frac{1}{-v}\right) = \frac{h}{-u}$$

The angle subtended by the object, when it is at *u* = –*f*.

$$\theta_i = \frac{h}{f}$$

The angular magnification is, therefore

$$m = \frac{\theta_i}{\theta_O} = \frac{D}{f}$$

The magnification when the image is at the closest comfortable distance
If the object is at a distance slightly less than the focal length of the lens, the image is virtual and closer than infinity.
Although the closest comfortable distance for viewing the image is when it is at the near point (distance D ≅ 25 cm), it causes some strain on the eye. Therefore, the image formed at infinity is often considered most suitable for viewing by the relaxed eye. The linear magnification m, for the image formed at the near point D, by a simple microscope can be obtained by using the relation.

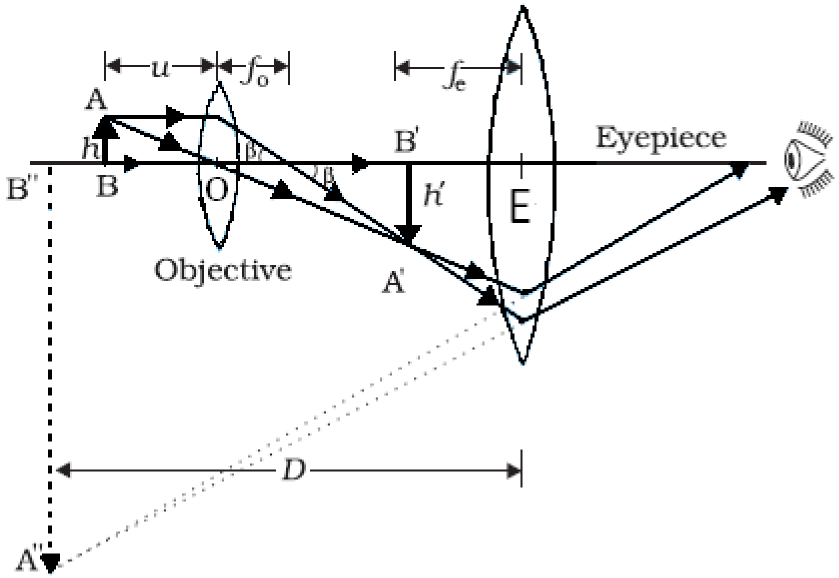
$$m = \frac{v}{u} = v\left(\frac{1}{v} - \frac{1}{f}\right) = \left(1 - \frac{v}{f}\right)$$

Now according to our sign convention, *v* is negative, and is equal in magnitude to *D.* Thus, the magnification is

$$m = \left(1 + \frac{D}{f}\right)$$

A simple microscope has a limited maximum magnification (≤ 9) for realistic focal lengths

Compound microscope



A simple microscope has a limited maximum magnification (≤ 9) for realistic focal lengths. For much larger magnifications, one uses two lenses, one compounding the effect of the other. This is known as a *compound microscope*.
A schematic diagram of a compound microscope is shown in Fig..The lens nearest the object, called the *objective*, And lens near to eye is called *eye piece*
Objective forms a real, inverted, magnified image of the object. A'B'.
This serves as the object for the *eyepiece*, which functions essentially like a simple microscope or magnifier, produces the final image, which is enlarged and virtual A"B"
the distance between the second focal point of the objective and the first focal point of the eyepiece (focal length f_e) is called the **tube length** (L) of the compound microscope

magnification

The ray diagram of Fig. shows that the (linear) magnification due to the objective,

$$tan\beta = \left(\frac{h}{f_o}\right) = \left(\frac{h'}{L}\right)$$

$$m_o = \frac{h'}{h} = \frac{L}{f_o}$$

Here h' is the size of the first image, the object size being h and f_o being the focal length of the objective.

The first image is formed near the focal point of the eyepiece. Magnification due to eye piece which behaves as simple microscope is given by

$$m_e = \left(1 + \frac{D}{f_e}\right)$$

When the final image is formed at infinity, the angular magnification due to the eyepiece

$$m_e = \left(\frac{D}{f_e}\right)$$

Thus, the total magnification, when the image is formed at infinity, is

$$m = m_o m_e = \left(\frac{L}{f_o}\right)\left(\frac{D}{f_e}\right)$$

Thus, the total magnification, when the image is formed at near point, is

$$m = m_o m_e = \left(\frac{L}{f_o}\right)\left(1 + \frac{D}{f_e}\right)$$

Telescope

The telescope is used to provide angular magnification of distant objects.

Lens towards object is objective have larger focal length than eye piece.

Light from a distant object enters the objective and a real image is formed in the tube at its second focal point. The eyepiece magnifies this image producing a final inverted image.

The magnifying power *m* is the ratio of the angle β subtended at the eye by the final image to the angle α which the object subtends at the lens or the eye. Hence

$$m \approx \frac{\beta}{\alpha} = \frac{h}{f_e}\frac{f_o}{h} = \frac{f_o}{f_e}$$

In this case, the length of the telescope tube is *fo + fe .*

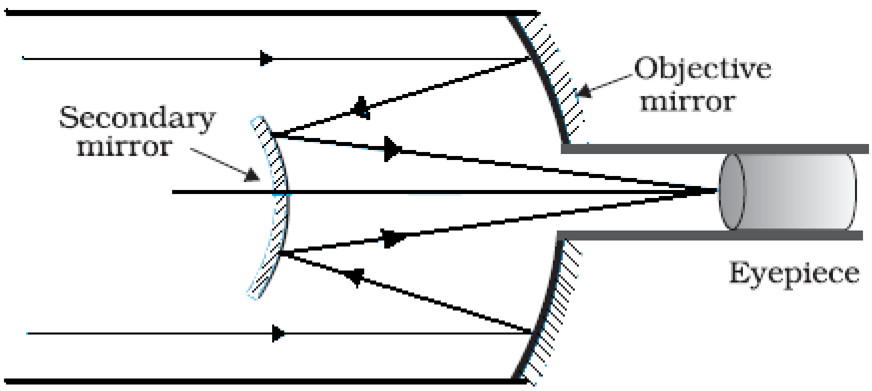
Terrestrial telescopes have, in addition, a pair of inverting lenses to make the final image erect. Refracting telescopes can be used both for terrestrial and astronomical observations.

If the final image is formed at the least distance of distinct vision, then magnifying power is

$$m = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right)$$

The main considerations with an astronomical telescope are its light gathering power and its resolution or resolving power. The former clearly depends on the area of the objective. With larger diameters, fainter objects can be observed.
The **resolving power**, or the ability to observe two objects distinctly, which are in very nearly the same direction, also depends on the diameter of the objective. So, the desirable aim in optical telescopes

Reflecting telescope



Telescopes with mirror objectives are called *reflecting* telescopes.
They have several advantages over refracting telescope
First, there is no chromatic aberration in a mirror.
Second, if a parabolic reflecting surface is chosen, spherical aberration is also removed.
Mechanical support is much less of a problem since a mirror weighs much less than a lens of equivalent optical quality, and can be supported over its entire back surface, not just over its rim.
One obvious problem with a reflecting telescope is that the objective mirror focuses light inside the telescope tube. One must have an eyepiece and the observer right there, obstructing some light (depending on the size of the observer cage). This is what is done in the very large 200 inch (~5.08 m) diameters, Mt. Palomar telescope, California.
The viewer sits near the focal point of the mirror, in a small cage. Another solution to the problem is to deflect the light being focused by another mirror. One such arrangement using a convex secondary mirror to focus the incident light, which now passes through a hole in the objective primary mirror, is shown

Solved Numerical

Q) A compound microscope with an objective of 2.0 cm focal length and an eye piece of 4.0 cm focal length, has a tube length of 40cm. Calculate the magnifying power of the microscope, if the final image is formed at the near point of the eye

Solution Formula for magnification when image is formed at near point of eye

$$m = m_o m_e = \left(\frac{L}{f_o}\right)\left(1+\frac{D}{f_e}\right)$$

$$m = \left(\frac{40}{2}\right)\left(1+\frac{25}{4}\right) = 145$$

Q) A compound microscope consists of an objective of focal length 1cm and eyepiece of focal length 5cm separated by 12.2 cm (a) At what distance from the objective should an object should be placed to focus it properly so that the final image is formed at the least distance of clear vision (25cm)? (b) Calculate the angular magnification in this case.

Solution : From lens formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

Here f = 1 cm ,; u= u_e , v = - 25cm

$$\frac{1}{5} = \frac{1}{25} + \frac{1}{u_e}$$

U_e = -4.2 cm

V_O = L - | u_e| = (12.2-4.2) cm = 8 cm

From lens formula

$$\frac{1}{1} = \frac{1}{8} + \frac{1}{u_O}$$

U_O = -1.1 cm

From formula for angular magnification =

$$m = \left(\frac{v_O}{u_O}\right)\left(1+\frac{D}{f_e}\right)$$

$$m = \left(\frac{8}{-1.1}\right)\left(1+\frac{25}{5}\right) = -43.6$$

Q) The separation between the objective (f = 0.5cm) and the eyepiece (f = 5cm) of a compound microscope is 7cm. Where should a small object be placed so that the eye is least strained to see the image? Find the angular magnification produced by the microscope

Solution: The eye will strain least if the final image is formed at infinity> In this case the image formed by the objective shall fall at the focus of the eye piece.

Now v_O = L - v_e =(7-5) cm = 2cm

From lends formula for objective lens

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

$$\frac{1}{0.5} = \frac{1}{2} + \frac{1}{u_o}$$

u_O = -2/3 cm
angular magnification for image at infinity

$$m = \left(\frac{v_O}{u_O}\right)\left(\frac{D}{f_e}\right)$$

$$m = \left(\frac{2 \times 3}{-2}\right)\left(\frac{25}{5}\right) = -15$$

Q) An astronomical telescope, in normal adjustment position, has magnifying power 5. The distance between the objective and the eyepiece is 120cm. Calculate the focal lengths of the objective and of the eyepiece.
Solution: m = f_O / f_e
f_O = m × f_e
f_O = 5 f_e
f_O + f_e = 120
thus f_e = 20 cm and f_e = 100 cm
Q) The focal length of the objective of an astronomical telescope is 75 cm and that of the eye piece is 5 cm. If the final image is formed at the least distance of distinct vision from eye, calculate the magnifying power of telescope
Solution:

$$m = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right)$$

$$m = \frac{75}{5}\left(1 + \frac{5}{25}\right) = 18$$

----------------END--------------

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Wave Optics

Wave front

The wave front at any instant is defined as the locus of all the particles of the medium which are in the same state of vibration.

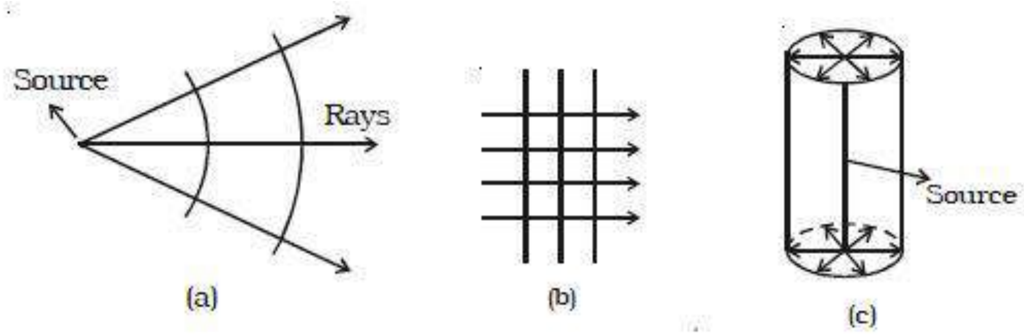
Or

Aan imaginary surface passing through particles oscillating with same phase is known as wavefront

A point source of light at a finite distance in an isotropic medium emits a spherical wave front (Fig a).

A point source of light in an isotropic medium at infinite distance will give rise to plane wavefront (Fig. b).

A linear source of light such as a slit illuminated by a lamp, will give rise to cylindrical wavefront (Fig c).



HUYGENS PRINCIPLE

Huygen's principle states that,

(i) every point on a given wave front may be considered as a source of secondary wavelets which spread out with the speed of light in that medium and

(ii) the new wavefront is the forward envelope of the secondary wavelets at that instant

Huygen's construction for a spherical and plane wavefront:

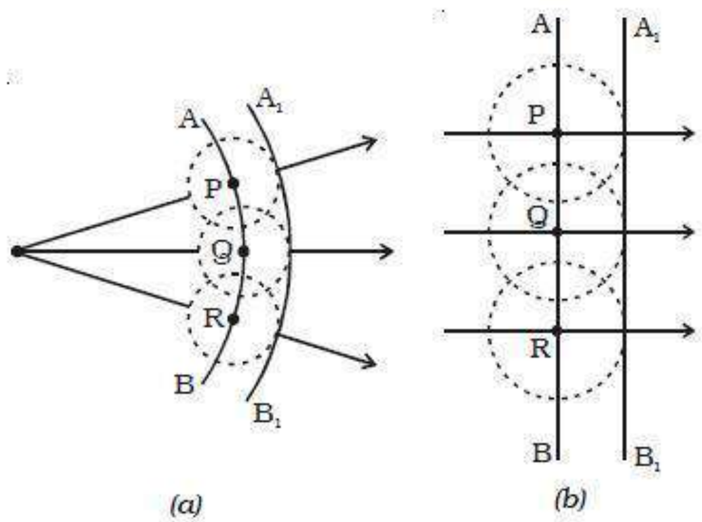
Huygen's construction for a spherical and plane wavefront is shown in Fig.a.

Let AB represent a given wavefront at a time $t = 0$. According to Huygen's principle, every point on AB acts as a source of secondary wavelets which travel with the speed of light c.

To find the position of the wave front after a time t, circles are drawn with points P, Q, R ... etc as centres on AB and radii equal to ct.

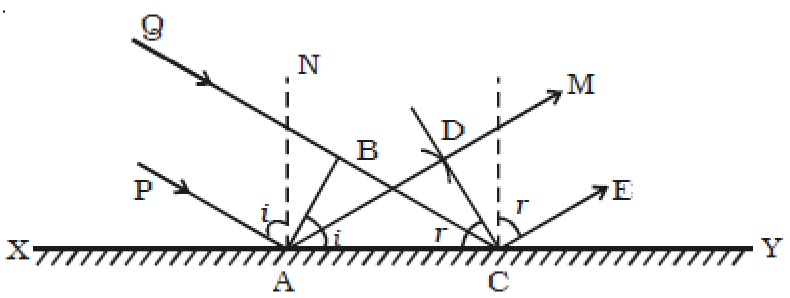
These are the traces of secondary wavelets. The arc A_1B_1 drawn as a forward envelope of the small circles is the new wavefront at that instant.

If the source of light is at a large distance, we obtain a plane wave front A_1 B_1 as shown in Fig b.



Reflection of a plane wave front at a plane surface

Let XY be a plane reflecting surface and AB be a plane wavefront incident on the surface at A. PA and QBC are perpendiculars drawn to
AB at A and B respectively. Hence they represent incident rays. AN is the normal drawn to the surface.
The wave front and the surface are perpendicular to the plane of the paper (Fig.).
According to Huygen's principle each point on the wavefront acts as the source of secondary wavelet.
By the time, the secondary wavelets from B travel a distance BC, the secondary wavelets from A on the reflecting surface would travel the same distance BC after reflection.
Taking A as centre and BC as radius an arc is drawn.
From C a tangent CD is drawn to this arc. This tangent CD not only envelopes the wavelets from C and A but also the wavelets from all the points between C and A.



Therefore CD is the reflected plane wavefront and AD is the reflected ray.

Laws of reflection
(i) The incident wavefront AB, the reflected wavefront CD and the reflecting surface XY all lie in the same plane.
(ii) Angle of incidence i = ∠ PAN = 90^0 − ∠ NAB = ∠BAC

Angle of reflection r = ∠ NAD = 90^0 − ∠ DAC = ∠DCA

$\angle B = \angle D = 90^O$

BC = AD and AC is common

∴ The two triangles are congruent

$\angle$ BAC = $\angle$DCA

i.e. i = r

Thus the angle of incidence is equal to angle of reflection.

Refraction of a plane wavefront at a plane surface

Let XY be a plane refracting surface separating two media 1 and 2 of refractive indices μ_1 and μ_2 (Fig). The velocities of light in these two media are respectively v_1 and v_2.

Consider a plane wave front AB incident on the refracting surface at A. PA and QBC are perpendiculars drawn to AB at A and B respectively. Hence they represent incident rays. NAN_1 is the normal drawn to the surface. The wave front and the surface are perpendicular to the plane of the paper.

According to Huygen's principle each point on the wave front act as the source of secondary wavelet.

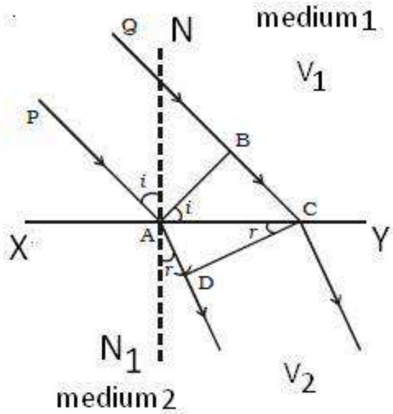
By the time, the secondary wavelets from B, reaches C, the secondary wavelets from the point A would travel a distance AD = v_2t, where t is the time taken by the wavelets to travel the distance BC.

∴BC = C_1t and AD = C_2t

Taking A as centre and C_2t as radius an arc is drawn in the second medium. From C a tangent CD is drawn to this arc.

.

Therefore CD is the refracted plane wavefront and AD is the refracted ray



Laws of refraction

(i) The incident wave front AB, the refracted wave front CD and the refracting surface XY all lie in the same plane.

(ii) From figure for Δ ABC and Δ ACD

$$\frac{sini}{sinr} = \frac{BC/_{AC}}{AD/_{AC}} = \frac{BC}{AD} = \frac{v_1t}{v_2t} = \frac{v_1}{v_2} = n_{21}$$

Constant n_{21} in above equation is known as refractive index of medium 2 with respect to medium also represented as $_1\mu_2$

This is Snell's law of refraction

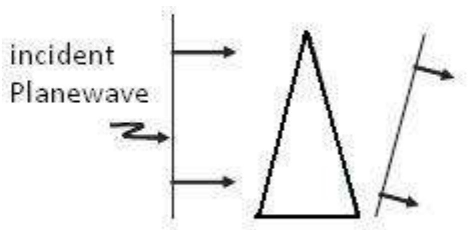
Further, if λ_1 and λ_2 denote the wavelengths of light in medium 1 and medium 2, respectively and if the distance BC is equal to λ_1 then the distance AE will be equal to λ_2 (because if the crest from B has reached C in time τ, then the crest from A should have also reached *E* in time τ); thus

$$\frac{\lambda_1}{\lambda_2} = \frac{BC}{AE} = \frac{v_1}{v_2}$$

The above equation implies that when a wave gets refracted into a denser medium ($v_1 > v_2$) the wavelength and the speed of propagation decrease but the *frequency* f (= v/λ) *remains the same*.

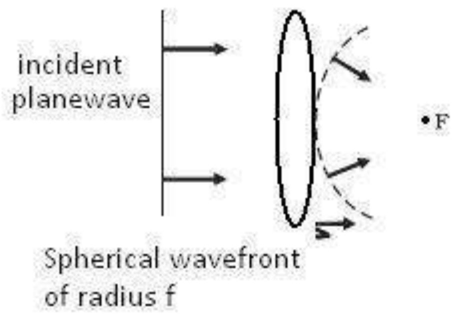
Refraction of a plane wave by a thin prism

we consider a plane wave passing through a thin prism. Clearly, since the speed of light waves is less in glass, the lower portion of the incoming wavefront (which travels through the greatest thickness of glass) will get delayed resulting in a tilt in the emerging wavefront as shown in the figure.



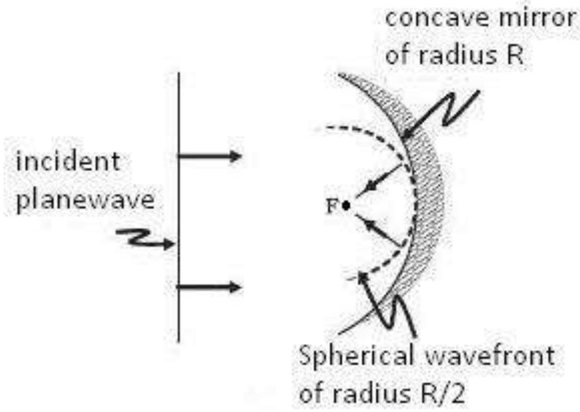
(b) a convex lens.

We consider a plane wave incident on a thin convex lens; the central part of the incident plane wave traverses the thickest portion of the lens and is delayed the most. The emerging wavefront has a depression at the centre and therefore the wavefront becomes spherical and converges to the point F which is known as the *focus*.



(c) Reflection of a plane wave by a concave mirror

a plane wave is incident on a concave mirror and on reflection we have a spherical wave converging to the focal point F.

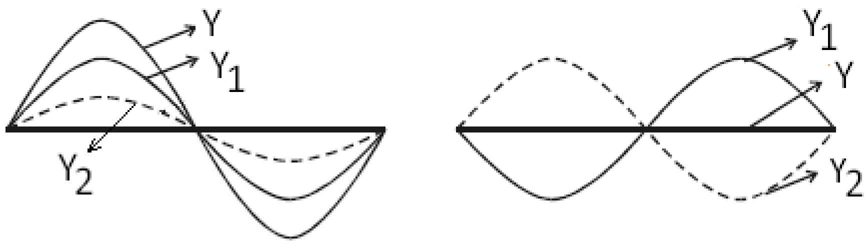


Coherent and incoherent sources

Two sources are said to be coherent if they emit light waves of the same wave length and start with same phase or have a constant phase difference.
Two independent monochromatic sources, emit waves of same wave length. But the waves are not in phase. So they are not coherent.
This is because, atoms cannot emit light waves in same phase and these sources are said to be incoherent sources.

Superposition principle

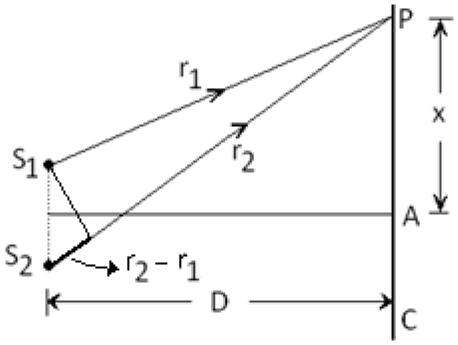
When two or more waves simultaneously pass through the same medium, each wave acts on every particle of the medium, as if the other waves are not present. The resultant displacement of any particle is the vector addition of the displacements due to the individual waves. This is known as principle of superposition. If $\mathbf{Y_1}$ and $\mathbf{Y_2}$ represent the individual displacement then the resultant displacement is given by $\mathbf{Y} = \mathbf{Y_1} + \mathbf{Y_2}$



Thus, superposition principle describes a situation when more than one waves superpose (i.e. interfere) at a point.
" The effect produced by superposition of two or more wave is called interference".

Interference due to two waves

Suppose two harmonic waves having initial phase φ_1 and φ_2 are emitted from two point like sources S_1 and S_2 respectively. They superimpose simultaneously (i.e. at the same time t) at a point P as shown in figure.



Visible perception of light is produced only by electric field, and therefore, in the present case we write light waves produced by source S_1 and S_2 in terms of electric fields (**e**) only.
Due to S_1 source,

$$\overrightarrow{e_1} = \overrightarrow{E_1} sin(\omega_1 t - k_1 r_1 + \varphi_1)$$

And due to source S_2

$$\overrightarrow{e_2} = \overrightarrow{E_2} sin(\omega_2 t - k_2 r_2 + \varphi_2)$$

Here $\mathbf{E_1}$ and $\mathbf{E_2}$ represent amplitude of electric fields, ω_1 and ω_2 denotes angular frequencies of waves, and k_1 and k_2 are wave vectors.
Let $\delta_1 = \omega_1 t - k_1 r_1 + \varphi_1$ and $\delta_2 = \omega_2 t - k_2 r_2 + \varphi_2$
Then $\mathbf{e_1} = \mathbf{E_1} \sin \delta_1$ and $\mathbf{e_2} = \mathbf{E_2} \sin \delta_2$
Now according to principle of superposition
$\mathbf{e} = \mathbf{e_1} + \mathbf{e_2}$
magnitude of resultant vector **e**

$$e^2 = e_1^2 + e_2^2 + 2\overrightarrow{e_1}\overrightarrow{e_2}$$

If at a instant of time E_1 and E_2 amplitude of waves then, resultant amplitude E is

$$E^2 = E_1^2 + E_2^2 + 2E_1 E_2 cos(\delta_1 - \delta_2)$$

The average intensity of light is proportional to square of amplitude
$I \propto E^2$ thus equation becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

In above equation I_1 and I_2 are the average intensities due to each wave. They are independent of time.
The last term in above equation is known as *interference term* which depends on time
Now

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_{t=0}^{t=T} cos\,(\delta_1 - \delta_2) dt$$

Here t is period of electric field oscillation. On substituting value of δ_2 and δ_1 in above equation

$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_{t=0}^{t=T} cos\,\{(\omega_1 t - \omega_2 t) + (k_2 r_2 - k_1 r_1) + (\varphi_2 - \varphi_1)\} dt$ --(1)

Case I: Incoherent sources
If angular frequency of both source is not same thus $\cos(\delta_1 - \delta_2)$ is time dependent and average value is zero. Thus superposed two waves produce the average intensity $I_1 + I_2$ at point P
Case II: Coherent sources:
For sources to be coherent there angular frequency should be same thus $\omega_1 = \omega_2 = \omega$ (say) Also since both waves are travelling in same medium there speed will be also same thus wave length is same thus $k_1 = k_2 = k$ (say) for sake of simplicity we will consider $\varphi_2 = \varphi_1$.
From equation (1) ignoring negative sign of cos

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}\int_0^T cos\{k(r_2 - r_1)\}dt$$

$$\langle cos(\delta_1 - \delta_2)\rangle = \frac{1}{T}cos\{k(r_2 - r_1)\}\int_0^T dt$$

$\langle cos(\delta_1 - \delta_2)\rangle = cos\{k(r_2 - r_1)\}$-- eq(2)
Further we will assume that amplitude of both waves is equal $I_1 = I_2 = I'$ then
From equation (1) and eq(2) we get

$$I = I' + I' + 2\sqrt{I'I'}cosk(r_2 - r_1)$$

$$I = 2I' + 2I'cosk(r_2 - r_1)$$

$$I = 2I'\{1 + cosk(r_2 - r_1)\}$$

$$I = 2I'\left[2cos^2\left\{\frac{k(r_2 - r_1)}{2}\right\}\right]$$

[Use trigonometric identity $cos^2u = \frac{1+cos2u}{2}$]

$$I = 4I'cos^2\left\{\frac{k(r_2 - r_1)}{2}\right\}$$

Here $r_2 - r_1 = \delta$ is known as the path difference between superposing waves

$$I = 4I'cos^2\left\{\frac{k\delta}{2}\right\}$$

Special Cases

Case I : Constructive Interference
For $I = 4I' = I_0$ maximum intensity of light

term $cos^2\left\{\frac{k(r_2-r_1)}{2}\right\}$ Should be equal to one, It is possible if

$$\frac{k\delta}{2} = n\pi$$

$$\frac{2\pi\delta}{2\lambda} = n\pi \ \because k = \frac{2\pi}{\lambda}$$

$$\delta = n\lambda$$

Here n = 0, 1, 2, 3, ……

"If the path difference between superposing waves is nλ (n = 0, 1, 2, 3, …..) intensity at a superposing point is maximum. Such interference is called constructive interference"

From equation

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

For constructive interference cos(δ_2-δ_1) = 0 thus

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

Or

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

$$I_{max} \propto (A_1 + A_2)^2$$

Case II: Destructive Interference

For intensity I = 0

term $cos^2\left\{\frac{k(\delta)}{2}\right\}$ Should be equal to zero, It is possible if

$$\frac{k\delta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \dots ….$$

Or

$$\frac{k\delta}{2} = (2n-1)\frac{\pi}{2}$$

As k = 2π/λ

$$\frac{2\pi\delta}{2\lambda} = (2n-1)\frac{\pi}{2}$$

$$\delta = (2n-1)\frac{\lambda}{2}$$

Here n = 1, 2, 3, 4, …..

" If phase difference between superposing waves is (2n-1)π intensity at a superposing point is minimum. This interference is called destructive interference"

"If path difference between superposing wave is (2n-1) (λ/2) intensity at superposed point is minimum. Such interference is known as destructive interference"

Here n = 1, 2, 3, 4, ….

From equation

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

For destructive interference cos($\delta_1 - \delta_2$) = -1 thus

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

Or

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

$$I_{min} \propto (A_1 - A_2)^2$$

Condition for sustained interference

The interference pattern in which the positions of maximum and minimum intensity of light remain fixed with time, is called sustained or permanent interference pattern. The conditions for the formation of sustained interference may be stated as :

(i) The two sources should be coherent

(ii) Two sources should be very narrow

(iii) The sources should lie very close to each other to form distinct and broad fringes

Solved numerical

Q) Two sources of intensity I and 3I are used in an interference experiment. Find the intensity at a point where the waves from the two sources superimpose with a phase difference (1) Zero (2) π/2

<u>Solution:</u>

In case of interference

$$I' = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta_1 - \delta_2)\rangle$$

$$I' = I_1 + I_2 + 2\sqrt{I_1 I_2}\langle cos(\delta)\rangle$$

(1) As δ = 0, cosδ =1

$$\therefore I' = 3I + I + 2\sqrt{(3I)(I)} \times 1$$

$$I' = 4I + 2\sqrt{3}I$$

(2) As δ = π/2, cosδ = 0

$$\therefore I' = 3I + I + 2\sqrt{(3I)(I)} \times 0$$

$$\therefore I' = 4I$$

Q) Ratio of the intensities of rays emitted from two different coherent sources is α. For the interference pattern formed by them, prove that

$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{1 + \alpha}{2\sqrt{\alpha}}$$

I_{max} : Maximum of intensity in the interference fringe

I_{min} : Minimum of intensity in the interference fringe

Solution:

Given $I_1 = \alpha I_2$
Since $I \propto A^2$
Thus $A_1 = \sqrt{\alpha}\, A_2$
Now

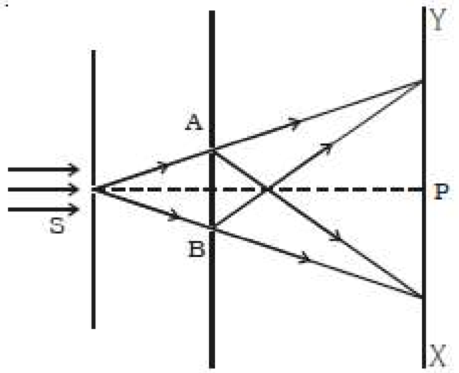
$$I_{max} \propto (A_1 + A_2)^2$$
$$I_{max} \propto \left(\sqrt{\alpha}A_2 + A_2\right)^2$$
$$I_{max} \propto A_2^2\left(\sqrt{\alpha} + 1\right)^2$$

And

$$I_{min} \propto (A_1 - A_2)^2$$
$$I_{min} \propto A_2^2\left(\sqrt{\alpha} - 1\right)^2$$
$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{A_2^2\left(\sqrt{\alpha}+1\right)^2 + A_2^2\left(\sqrt{\alpha}-1\right)^2}{A_2^2\left(\sqrt{\alpha}+1\right)^2 - A_2^2\left(\sqrt{\alpha}-1\right)^2}$$
$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{\left(\sqrt{\alpha}+1\right)^2 + \left(\sqrt{\alpha}-1\right)^2}{\left(\sqrt{\alpha}+1\right)^2 - \left(\sqrt{\alpha}-1\right)^2}$$
$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{\alpha + 1 + 2\sqrt{\alpha} + \alpha + 1 - 2\sqrt{\alpha}}{\alpha + 1 + 2\sqrt{\alpha} - \alpha - 1 + 2\sqrt{\alpha}}$$
$$\frac{I_{max} + I_{min}}{I_{max} - I_{min}} = \frac{2(\alpha+1)}{4\sqrt{\alpha}} = \frac{\alpha+1}{2\sqrt{\alpha}}$$

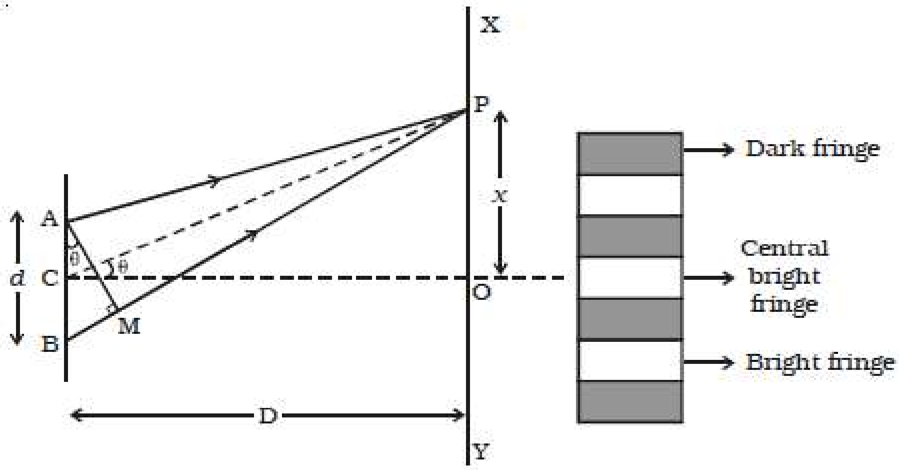
Young’s double slit experiment

The phenomenon of interference was first observed and demonstrated by Thomas Young in 1801. The experimental set up is shown in Fig.
Light from a narrow slit S, illuminated by a monochromatic source, is allowed to fall on two narrow slits A and B placed very close to each other.
The width of each slit is about 0.03 mm and they are about 0.3 mm apart. Since A and B are equidistant from S, light waves from S reach A and B in phase. So A and B act as coherent sources.
According to Huygen’s principle, wavelets from A and B spread out and overlapping takes place to the right side of AB. When a screen XY is placed at a distance of about 1 metre from the slits, equally spaced alternate bright and dark fringes appear on the screen. These are called interference fringes or bands.
Using an eyepiece the fringes can be seen directly. At P on the screen, waves from A and B travel equal distances and arrive in phase. These two waves constructively interfere and bright fringe is observed at P. This is called central bright fringe.
When one of the slits is covered, the fringes disappear and there is uniform illumination on the screen. This shows clearly that the bands are due to interference.



Expression path difference in terms of D and x

Let *d* be the distance between two coherent sources A and B of wavelength λ. A screen XY is placed parallel to AB at a distance D from the coherent sources. C is the mid point of AB. O is a point on the screen equidistant from A and B. P is a point at a distance *x* from O, as shown in Fig. Waves from A and B meet at P in phase or out of phase depending upon the path difference between two waves.



Draw AM perpendicular to BP The path difference δ = BP – AP AP = MP

∴ δ = BP – AP = BP – MP = BM

In right angled Δ ABM, BM = d sin θ

If θ is small, sin θ = θ

∴ The path difference δ = θ.d

In right angled triangle COP,

$$tan\theta = \frac{OP}{CO} = \frac{x}{D}$$

For small values of θ, tan θ = θ

∴ The path difference

$$\delta = \frac{xd}{D}$$

Location of bright and dark fringes on screen (x)

Bright fringes:

By the principle of interference, condition for constructive interference is the path difference = nλ

$$n\lambda = \frac{xd}{D}$$

$$x = \frac{n\lambda D}{d}$$

By substituting n = 0, 1, 2, 3…

If n = 0 then we get location central bright fringe

n=1, we get then location of first bright fringe

n=2, we get then location of second bright fringe

n=3, , we get then location of third bright fringe… etc

Dark fringes:

By the principle of interference, condition for destructive interference is the path difference

$$\delta = (2n-1)\frac{\lambda}{2}$$

$$(2n-1)\frac{\lambda}{2} = \frac{xd}{D}$$

$$x = \frac{D(2n-1)\lambda}{2d}$$

By substituting n = 1, 2, 3…

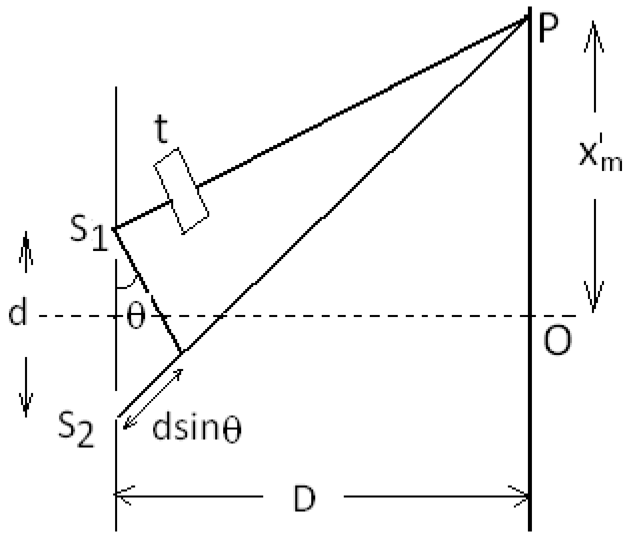
n=1, we get then location of first dark fringe

n=2, we get then location of second dark fringe

n=3, we get then location of third dark fringe… etc

Displacement of fringe pattern

If a thin transparent slab of thickness t and refractive index μ is placed in front of one of sources, for example, in front of S_1. This changes the path difference because light from S_1 now travels more optical path than earlier. *(Optical path in medium is equal to the product of refractive index of the medium to geometrical path length in air)*



Path difference before placing slab = $S_2P - S_1P = r_2 - r_1 = \delta$

On placing slab path length of thickness t effective path length $S_1P = (r_1 - t) + t\mu$

Thus effective path difference after placing slab $\delta' = r_2 - [(r_1 - t) + t\mu]$

$\therefore \delta' = (r_2 - r_1) + t(\mu - 1)$

From the geometry of figure $r_2 - r_1 = d\sin\theta$, since θ is very small

$$r_2 - r_1 = d\tan\theta = \frac{dx'_m}{D}$$

$$\therefore \delta' = \frac{dx'_m}{D} + (\mu - 1)t$$

$$x'_m = \frac{n\lambda D}{d} - (\mu - 1)\frac{tD}{d}$$

In absence of slab, the m^{th} maxima is given by

$$x_m = \frac{n\lambda D}{d}$$

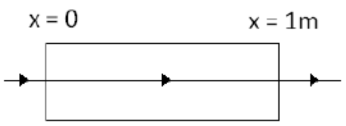
Therefore, the fringe shift is given by

$$x_0 = x_m - x'_m = (\mu - 1)\frac{tD}{d}$$

When a transparent slab is introduced, the fringe pattern shifts in the direction where the slab is placed.

Solved numerical

Q) A ray of light travels through a slab as shown in figure. The refractive index of the material of the slab varies as $\mu = 1.2 + x$, where $0 \le x \le 1$ m. What is the equivalent optical path of the glass slab?



Solution

Consider a small geometric path dx then optical path = μdx

Thus optical path op =

$op = \int_0^1 (1.2 + x)dx = \left[1.2x + \frac{x^2}{2}\right]_0^1$

$$op = 1.2 + \frac{1}{2} = \frac{3.4}{2} = 1.7$$

Distance between two consecutive bight or dark fringe
Or Width of fringe , Band width (β)

The distance between any two consecutive bright or dark bands is called bandwidth.
The distance between (n+1)th and nth order consecutive bright fringes from O is given by

$$\beta = x_{n+1} - x_n = (n+1)\frac{\lambda D}{d} - (n)\frac{\lambda D}{d} = \frac{\lambda D}{d}$$

Similarly, it can be proved that the distance between two consecutive dark bands is also equal to $\frac{\lambda D}{d}$

Angular fringe width or angular separation between fringes is

$$\theta = \frac{\lambda}{d}$$

Since bright and dark fringes are of same width, they are equi−spaced on either side of central maximum.

Solved numerical

Q) In Young's double slit experiment, angular width of a fringe formed on a distant screen is 0.1^{O}. the wavelength of the light used is 6000Å . What is the spacing between the slit. If above setup is immersed in liquid it is observed that angular fringe width is decreased by 30% find refractive index of liquid

Solution

Angular fringe width or angular separation between fringes is

$$\theta = \frac{\lambda}{d}$$

$$d = \frac{\lambda}{\theta} = \frac{6000 \times 10^{-10}}{0.1 \times \frac{\pi}{180}} = 3.44 \times 10^{-4} m$$

(ii)Given that when set up with first light is immersed in liquid angular width decreases by 30% thus wave length of first light in liquid = 4200Å

From formula for refractive index

$$\mu = \frac{\lambda_{air}}{\lambda_{liq}} = \frac{6000}{4200} = 1.428$$

Q) In Young's double slit experiment using monochromatic light the fringe pattern shifts by certain distance on the screen when a mica sheet of refractive index 1.6and thickness 1.964 micron is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the plane of the slit and the screen is doubled. It is found that the distance between successive maximum now is the same as the observed fringe shift upon the introduction of mica sheet. Calculate the wavelength of the light.

Solution

Due to introduction of mica sheet, the shift on the screen

$$x_0 = (\mu - 1)\frac{tD}{d}$$

Given that on removal of mica sheet and increasing the distance between screen and slit two times, then it is observed that distance between successive maximum now is the same as the *observed fringe shift* upon the introduction of mica sheet

Thus $x_o = \beta$ fringe width when distance is doubled

Now fringe width

$$\beta = \frac{\lambda(2D)}{d}$$

Therefore

$$(\mu - 1)\frac{tD}{d} = \frac{\lambda(2D)}{d}$$

$$\lambda = \frac{(\mu - 1)t}{2}$$

$$\lambda = \frac{(1.6 - 1)(1.964 \times 10^{-6})}{2} = 5892\text{Å}$$

Q) In Young's double slit experiment a beam of light of wavelength 6500Å and 5200Å is used . Find the minimum distance from the central bright fringe where bright fringe produced by both the wavelength get superposed. The distance between two slit is 0.5mm and the distance between the slits and the screen is 100cm.

Solution

Let n^{th} bright fringe due wavelength 6500Å to and m^{th} bright fringe due to 5200Å superposed at distance x from central bright spot

Thus $x_n = x_m$

$$\frac{n\lambda_1 D}{d} = \frac{m\lambda_2 D}{d}$$

$$n\lambda_1 = m\lambda_2$$

$$n6500 = m5200$$

$$\frac{m}{n} = \frac{6500}{5200} = \frac{5}{4}$$

That is 4^{th} bright fringe of 6500 Å and 3^{rd} bright fringe of 5200 Å superpose

Taking n =4 in equation we get the minimum distance from the central bright fringe where bright fringe produced by both the wavelength get superposed

$$x_n = \frac{n\lambda_1 D}{d}$$

$$x_n = \frac{4 \times 6500 \times 10^{-8} \times 100}{0.05} = 0.52cm$$

Q) The ratio of the intensities at minima to maxima in the interference pattern is 9:25. What will be the ratio of the widths of the two slits in young's double slit experiment
Solution
Intensity is proportional to width of slit, so amplitude is proportional to the square root of the width of the slit

$$\frac{A_1}{A_2} = \sqrt{\frac{\mathrm{w}_1}{\mathrm{w}_2}}$$

W_1 and w_2 represent the width of the two slits

$$\frac{I_{min}}{I_{max}} = \frac{(A_1 - A_2)^2}{(A_1 + A_2)^2}$$

$$\frac{I_{min}}{I_{max}} = \frac{\left(1 - \frac{A_2}{A_1}\right)^2}{\left(1 + \frac{A_2}{A_1}\right)^2}$$

$$\frac{9}{25} = \frac{\left(1 - \frac{A_2}{A_1}\right)^2}{\left(1 + \frac{A_2}{A_1}\right)^2}$$

$$\frac{3}{5} = \frac{1 - \frac{A_2}{A_1}}{1 + \frac{A_2}{A_1}}$$

$$8\frac{A_2}{A_1} = 2$$

$$\frac{A_1}{A_2} = \frac{4}{1}$$

Thus from equation (1)

$$\frac{4}{1} = \sqrt{\frac{\mathrm{w}_1}{\mathrm{w}_2}}$$

$$\frac{w_1}{w_2} = \frac{16}{1}$$

Condition for obtaining clear and broad interference bands

(i) The screen should be as far away from the source as possible.
(ii) The wavelength of light used must be larger.
(iii) The two coherent sources must be as close as possible.

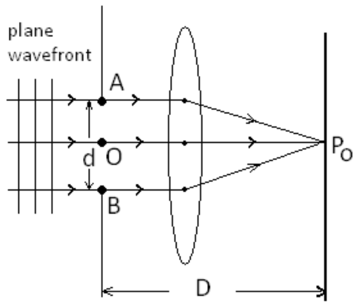
Diffraction

When waves encounters obstacles or openings like slits, they bend round the edges. This bending of wave is called diffraction. Diffraction is the effect produced by the limiting part of the wavefront.

Smaller is the width of the slit, more will be diffraction for given wavelength. It is also found that if the wavelength and the width of the slit are so changed that ratio (λ/d) remains constant, amount of bending or diffraction does not change.

If ratio λ/d is more ,then more is the diffraction

Diffraction due to single slit



Central Maxima:

Consider a plane wavefront arrive at a plane of slit, according to Huyen's principle all the point on the slit like AOB acts as secondary source having the same phase and produce secondary waves.

Those waves originated from each points of a slit and diffracted normal to the plane of the slit or we can say in the direction of incident wave will be concentrated at point P_o by lens.

In figure out of a many waves only three rays are shown in figure.

Screen is at focal length of lens .

Ray emitted from A and B are in phase and passes equal distance through air and lens thus they are in phase when get converged at P_o .

Now ray emitted from O travel less distance in air but more distance in lens, in lens velocity of light gets reduced thus optical path travelled by the ray emitted by O is equal to optical path due to ray A and B.. Thus all rays meeting at P^o are in phase produces central bright fringe.

(Optical path in medium is equal to the product of refractive index of the medium to geometrical path length in air)

First minimum

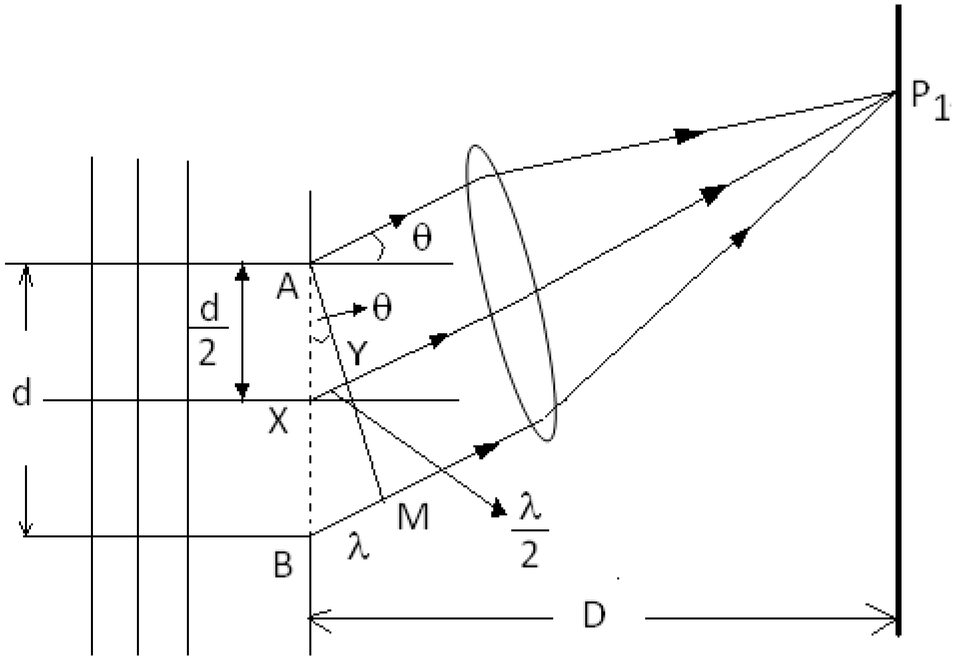
As shown in figure consider a waves which is diffracted an angle θ with respect to perpendicular bisector XP_o of the slit. Here, point X is the midpoint of slit AB. Therefore AX = Xb = d/2.

Here secondary waves originated from all points A, X, B of slit are through to be divided in two parts

Wave from AX and waves from X to B.

As per figure, all these waves diffracted at an angle θ are focused at point P_1 of a screen.

Draw AM $\perp$ BL. It is obvious that all the rays reaching from AM to P_1 have equal optical path

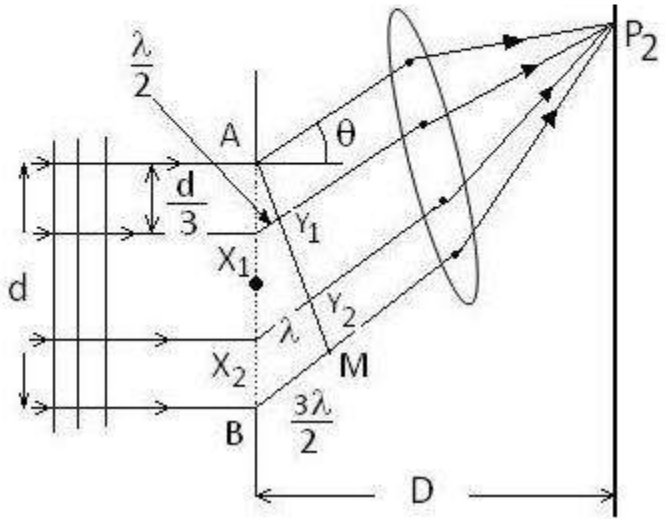


But rays going from A and X, and reaching to point P_1 have path difference of XY
Let assume diffraction angle be θ is such that XY = λ/2
In this situation, waves from A and X will follow the condition of destructive interference at point P_1 and resultant intensity will be zero
Further for all point between AX there exists a point between XB ,such that ray from point between XB have path difference of λ/2 with respect to rays from point between AX
Thus in totality, destructive interference will take place at point P_1
Point P_1 is known as first minimum

Condition for minima:

From geometry of figure dsinθ = λ
General equation is
dsinθ = nλ
For n = 1 we get first minima
n = 2 we get second minima

First Maxima:
As shown in figure suppose slit AB is assumed to be divided in three equal parts AX_1, X_1X_2, X_2B . Here $AX_1 = X_1X_2 = X_2B = d/3$.
Draw AM ⊥ BL. Wave reaching from AM to P_2 will have equal optical path
Waves starting from A and X_1 and imposing at point P_1 will have path difference X_1Y_1 .



Let us assume that diffraction θ is such that

$$X_1Y_1 = \frac{\lambda}{2}$$
$$X_2Y_2 = \lambda$$
$$BM = \frac{3\lambda}{2}$$

Since path difference between waves originated from A and X_1 and superimpose at point P_2 is λ/2, they interfere destructively. And intensity at point P_2 due to these waves will be zero.

In the same way, waves from every pair AX_1 and X_1X_2 will have path difference λ/2 and resultant intensity at point P_2 due to them is zero.

However, intensity of ray diffracted at an angle θ from section X_1B is not vanishing at point P_2. Therefore due to this section of the slit intensity at point P_2 will not be zero. And point P_2 will be bright

Here point P_2 is known as first maximum. It is obvious that intensity at point P_2 will be far less than central bright spot

Condition for minima:

From geometry of figure for fist maxima

$$d\sin\theta = \frac{3\lambda}{2}$$

General formula

$$d\sin\theta = \frac{(2n+1)\lambda}{2}$$

For n = 1 we get first maxima

n = 2 we get second maxima

Angular width

For first order minima dsinθ =λ or

$$sin\theta = \frac{\lambda}{d}$$

For small angle

$$\theta = \frac{\lambda}{d}$$

Also sinθ = tanθ
From geometry of figure

$$tan\theta = \frac{x}{D}$$

$$\therefore \frac{x}{D} = \frac{\lambda}{d}$$

∴ width of central maxima

$$2x = \frac{2\lambda D}{d}$$

Angular width of central maxima is given by

$$2\theta = \frac{2\lambda}{d}$$

Comparison between Interference and diffraction

(i) The interference pattern has a number of equally spaced bright and dark bands. The diffraction pattern has a central bright maximum which is twice as wide as the other maxima. The intensity falls as we go to successive maxima away from the centre, on either side.
(ii) We calculate the interference pattern by superposing two waves originating from the two narrow slits. The diffraction pattern is a superposition of a continuous family of waves originating from each point on a single slit.
(iii) For a single slit of width *a*, the first null of the interference pattern occurs at an angle of λ/*a.* At the same angle of λ/*a*, we get a maximum (not a null) for two narrow slits separated by a distance *a.*

Solved numerical

Q)Angular width of central maximum in diffraction obtained by single slit using light of wavelength 6000Å is measured. If light of another wavelength is used, the angular width of the central maximum is found to be decreased by 30%. Find
(i)The other wavelength (ii) If the experiment is repeated keeping the apparatus in a liquid, the angular width of central maxima decreases by the same amount (30%0, find its refractive index
Solution:
(i)Angular fringe width or angular separation between fringes is

$$2\theta = \frac{2\lambda}{d}$$

For first light

$$\theta_1 = \frac{\lambda_1}{d}$$

For second light

$$\theta_2 = \frac{\lambda_2}{d}$$

$$\frac{\theta_2}{\theta_1} = \frac{\lambda_2}{\lambda_1}$$

But θ_2 is 70% of θ_1
That is , $\theta_2 = 0.7\ \theta_1$

$$\therefore 0.7 = \frac{\lambda_2}{\lambda_1}$$

$$\lambda_2 = 0.7 \times 6000 = 4200\ \text{Å}$$

Q) A slit of width d is illuminated by white light. For what value of d will the first minimum for red light of wavelength λ_R =6500Å appear at θ =15^{O}? What is the situation for violet colour having wavelength λ_V = 4333Å at the same point. Sin15^{O} = 0.2588
Solution:
Since the diffraction occurs separately for each wavelength, we have to check condition for minia and maxima for each wavelength separately
For the first minimum of red colour n = 1, using condition
$d\sin\theta = n\lambda_R$

$$d = \frac{n\lambda_R}{sin\theta} = \frac{1 \times 6500 \times 10^{-10}}{sin15}$$

d=2.512×10^{-6} m
For violet colour wavelength is different. We have to check whether slit width above can give us minima or maxima
Thus $d\sin\theta = n'\lambda_V$

$$n' = \frac{dsin\theta}{\lambda_v} = \frac{2.512 \times 10^{-6} \times 0.2588}{4333 \times 10^{-10}} = 1.5$$

Here n' should be integer. Thus, for violet colour condition for minima does not satisfy
Now using condition for maxima

$$dsin\theta = (2n' + 1)\frac{\lambda_V}{2}$$

$$n' = \frac{dsin\theta}{\lambda_v} - \frac{1}{2} = 1.5 - 0.5 = 1.0$$

This result suggest that for violet colour first maximum is observed

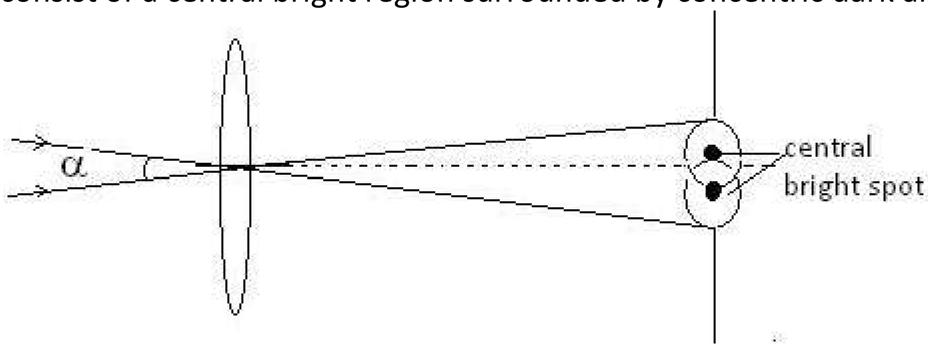
Resolving power of optical instrument

When a beam of light (light waves) from a point like object passes through the objective of an optical instruments, the lens acts like a circular aperture and produces a diffraction pattern instead of sharp point image.
If there are two point objects kept closed to each other, their diffraction pattern may overlap. Then it may be difficult to distinguish them as separate.
The criterion to get distinct and separate images of two closely placed point like objects was given by Rayleigh
" The images of two point like objects can be seen as separate if the central maximum in the diffraction pattern of one falls either on the first minimum of the diffraction pattern of the other or it is at grater separation"
For the case of circular aperture diffraction due to lens of diameter D. Rayleigh's criterion is given by

$$\sin\alpha \approx \alpha = \frac{1.22\lambda}{D}$$

Resolving power of telescope:

Consider a parallel beam of light falling on a convex lens. If the lens is well corrected for aberrations, then beam will get focused to a point.
However, because of diffraction, the beam instead of getting focused to a point gets focused to a spot of finite area. In this case the effects due to diffraction can be taken into account by considering a plane wave incident on a circular aperture followed by a convex lens.
Taking into account the effects due to diffraction, the pattern on the focal plane would consist of a central bright region surrounded by concentric dark and bright rings.



If two stars are very close to each other separated by angle α will be very small and the diffraction pattern of both stars will mingle with each other. In this situation it is difficult to see both the stars distinctly and clearly

"Ability of an optical instrument to produce distinctly separate images of two closely placed objects is called its resolving power"
It is clear that for optical instruments resolving power depends on angle α . is a minimum angle to see two images distinctly

$$\alpha_{min} = \frac{1.22\lambda}{D}$$

Here D is diameter of lens and λ is wavelength
Width of the central maxima or radius is given by

$$\alpha_{min} f = \frac{1.22\lambda}{D} f$$

Here α_{min} is known as angular resolution of the telescope, while its inverse is known as resolving power or geometrical resolution
Thus resolving power of telescope

$$\frac{1}{\alpha_{min}} = \frac{D}{1.22\lambda}$$

Solved numerical

Q) Calculate the useful magnifying power of a telescope of 11cm objective. The limit of resolution of eye is 2' and wavelength of light used is 5000Å
Solution
The magnifying power of a telescope is given by
M = D/d, where D is diameter of the objective and d is diameter of eye piece
For normal (useful) magnification, diameter of eyepiece should be equal to the diameter of the pupil d_e of the eye. Therefore, useful magnification is
M = D/d_e
From the equation of limit of resolution of telescope

$$d\theta = \frac{1.22\lambda}{D}$$

$$d\theta = \frac{1.22 \times 5500 \times 10^{-10}}{11 \times 10^{-2}} = 6.1 \times 10^{-6}\ rad$$

Limit of resolution of eye is given dθ'=2'

$$d\theta' = \frac{2 \times 3.14}{60 \times 180^{o}} = 5.815 \times 10^{-4}\ rad$$

∴ Useful magnification

$$\frac{d\theta'}{d\theta} = \frac{5.815 \times 10^{-4}}{6.1 \times 10^{-6}} = 95.3$$

Q) Huble space telescope is at a distance 600 km from earth's surface. Diameter of its primary lens (objective) is 2.4m. When a light of 550nm is used by this telescope, at what minimum angular distance two objects can be seen separately? Also obtain linear minimum distance between these objects. Consider these objects on the surface of earth and neglect effects of atmosphere.

Solution:

$$\alpha_{min} = \frac{1.22\lambda}{D} = \frac{1.22 \times 550 \times 10^{-9}}{2.4}$$

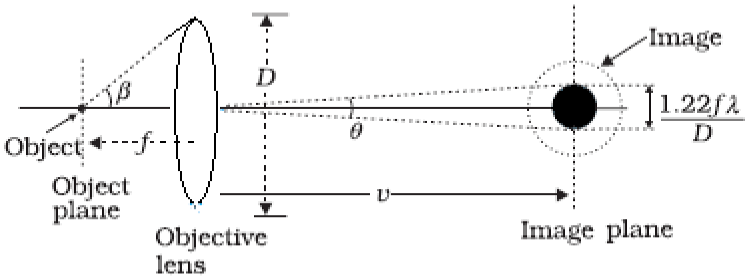
$$\alpha_{min} = 2.8 \times 10^{-7}\, rad$$

Linear distance between objects= $\alpha_{min}L$

Where L = distance between object telescope and object

Linear distance between objects = $2.8\times10^{-7}\times600\times10^{3}$ = 0.17m

Resolving power of microscope:



Let the diameter of lens be D and its focal length be f. As object distance is usually kept greater than that of f . Let the image distance be v. the angular width of central maximum due to the effect of diffraction is ,

$$\theta = \frac{1.22\lambda}{D}$$

Width of central maximum

$$\theta v = \frac{1.22\lambda}{D} v$$

If image of two point like objects are at a separation les than vθ, then it will be seen as a mixed single object. It can be proved that a minimum distance (d_m) for which objects can be seen separately is given by

$$d_m = \theta \frac{v}{m}$$

Here m is magnification m=v/f substituting value of m we get d_m

$$d_m = \theta \frac{v}{v/f} = \theta f$$

Substituting value of θ we get

$$d_m = \frac{1.22\lambda}{D} f$$

From figure D/2 =f (tanβ)

D = 2f (tanβ) substituting value in above equation we get

$$d_m = \frac{1.22\lambda}{2f tan\beta} f = \frac{1.22\lambda}{2tan\beta}$$

For small angles tanβ = sinβ

Reciprocal of d_m known as Resolving Power(RP) of microscope

$$RP = \frac{1}{d_m} = \frac{2\sin\beta}{1.22\lambda}$$

If some medium with large refractive index (n) is used between object and objective resolving power of microscope increases n times

Formula for resolving power is given by

$$RP = \frac{2n\sin\beta}{1.22\lambda}$$

Here term nsinβ is known as 'Numerical Aperture". Resolving power is inversely proportional to wavelength.

Polarization

The phenomena of reflection, refraction, interference, diffraction are common to both transverse waves and longitudinal waves. But the transverse nature of light waves is demonstrated only by the phenomenon of polarization.

Unpolarized light

In an ordinary light source like bulb, there are large numbers of atomic emitters. They all emit electromagnetic waves with there Electrical vector E , vibrating randomly in all directions perpendicular to direction of propagation.

It means that vector E of one wave is not parallel to Vector E of another wave.

Wave emitted by different atom is of source propagate in same direction constitute beam of light.

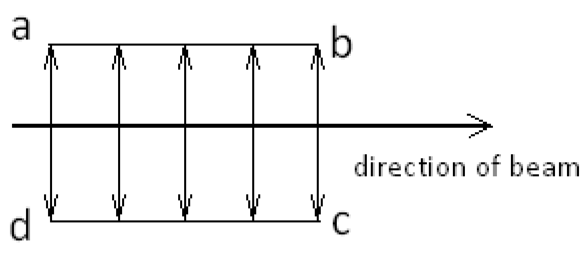
If such beam is assumed to be coming out of paper, light vectors (E) of its waves will be found in all random direction in a plane of paper. Such light is called Unpolarized light.

" In a beam of light, if the oscillations of E vectors are in all direction in a plane perpendicular to the direction of propagation, then the light is called unpolarized light"

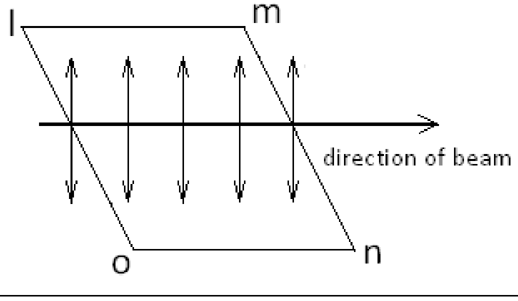
Polarized light

If in beam of light all electric vector (E) are coplanar and parallel to each other is plane polarized light

Process by which getting the plane polarized light from unpolarized light is called polarization



" The plane containing the direction of the beam and the direction of oscillation of E vectors is called the plane of oscillation . In figure abcd is the plane of oscillation



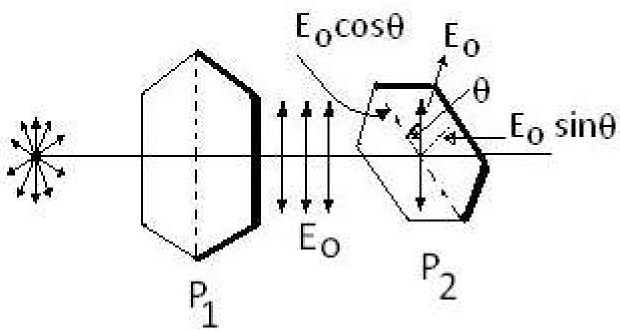
"A plane perpendicular to the plane of oscillation and passing through the beam of light is called the plane of polarization"

In figure lmno is the plane of polarization

When light passes through tourmaline crystal ,freely transmit the light components which are polarized to a definite direction. While crystal absorbs light strongly whose polarization is perpendicular to this definite direction. Thus emergent beam iof light only coplanar and parallel E vectors are found. This definite direction in a crystal is known as an *optic axis*

Malus' Law

If the light from an ordinary source (like a sodium lamp) passes through a polaroid sheet P_1, it is observed that its intensity is reduced by half. Rotating P_1 has no effect on the transmitted beam and transmitted intensity remains constant.
Now, let an identical piece of polaroid P_2 be placed before P_1. On rotating P_2 has a dramatic effect on the light coming from P_2.
In one position, the intensity transmitted by P_2 followed by P_1 is nearly zero. When turned by 90^o from this position, P_1 transmits nearly the full intensity emerging from P_2
An optic axis of plate P_2 makes an angle of θ with that of the plate P_1. In this situation vector E emerging from plate P_1 (E_0) makes angle θ with an optic axis of plate B. therefore we can resolve them into two components
91) $E_0\cos\theta$ parallel to the optic axis of plate P_2 and
(2) $E_0\sin\theta$ perpendicular to the optic axis of plate P_2



Thus, only $E_0\cos\theta$ components will emerge out of plate $P_{2'}$ while perpendicular components are absorbed. Since intensity is proportional to the square of amplitude, intensity of light incident on plate P_2 is

$$I \propto E_0^2 cos^2\theta$$

$$\therefore \frac{I}{I_0} = cos^2\theta$$

$$\therefore I = I_0 cos^2\theta$$

This equation is known as Malus Law. It is obvious from above equation that if plate P_2 is completely rotated, twice the intensity of emerging light is zero , corresponding to $\theta = \pi/2$ and $3\pi/2$ and twice it become maximum corresponding to $\theta = 0$ and $\theta = \pi$.
This procedure will help us to verify whether the given light is polarized or not. Since plate P_2 is used to analyze a state of polarization of incident light, it is known as Analyzer.

Solved numerical

Q) A ray of light travelling in water is incident on a glass plate immersed in it. What the angle of incident is 51° the reflected ray is totally plane polarized. Find the refractive index of glass. Refractive index of water is 1.33

Solution:

Angle of incidence $\theta_P = 51^o$

Since at this incidence angle, reflected ray is totally plane polarized, using Brewster'slaw refractive index of glass w.r.t. water is

$n' = \tan\theta_P = \tan 51 = 1.235$

But refractive index n' =

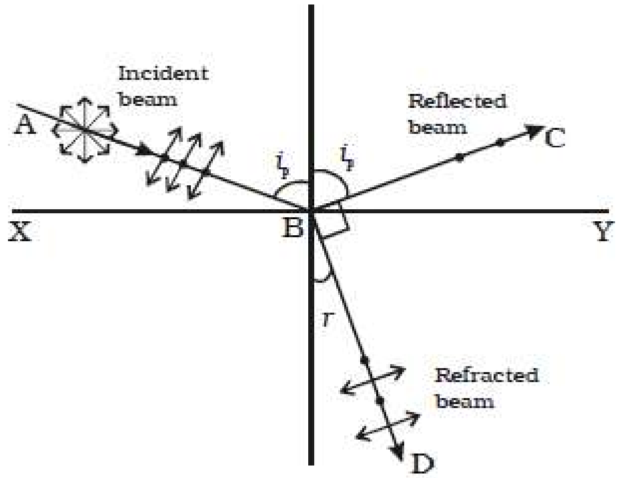
$$n' = \frac{RI.\,glass(n_g)}{R.\,i.\,of\ water(n_w)}$$

$n_g = n'\, n_w = 1.235 \times 1.33 = 1.64$

Polarisation by reflection

The simplest method of producing plane polarised light is by reflection. Malus, discovered that when a beam of ordinary light is reflected from the surface of transparent medium like glass or water, it gets polarised. The degree of polarisation varies with angle of incidence.

Consider a beam of unpolarised light AB, incident at any angle on the reflecting glass surface XY. Vibrations in AB which are parallel to the plane of the diagram are shown by arrows. The vibrations which are perpendicular to the plane of the diagram and parallel to the reflecting surface, shown by dots (Fig).



A part of the light is reflected along BC, and the rest is refracted along BD.
On examining the reflected beam with an analyzer, it is found that the ray is partially plane polarised. When the light is allowed to be incident at a particular angle, (for glass it is 57.5^o) the reflected beam is completely plane polarised. The angle of incidence at which the reflected beam is completely plane polarised is called the polarising angle (i_p).

Brewster's law

Sir David Brewster conducted a series of experiments with different reflectors and found a simple relation between the angle of polarization and the refractive index of the medium.
It has been observed experimentally that the reflected and refracted rays are at right angles to each other, when the light is incident at polarizing angle.
From Fig, ip +90^0 + r = 180^0
r = 900 – ip
From Snell's law,

$$\frac{sini_p}{sinr} = \mu$$

where μ is the refractive index of the medium (glass)
Substituting for *r*, we get

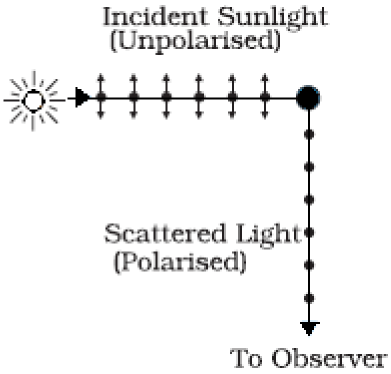
$$\frac{sini_p}{sin(90 - i_p)} = \frac{sini_p}{cosi_p} = \mu$$

$$\therefore \tan i_p = \mu$$

Tangent of polarizing angle is numerically equal to refractive index of medium

Polarisation by scattering

The light from a clear blue portion of the sky shows a rise and fall of intensity when viewed through a Polaroid which is rotated. This is nothing but sunlight, which has changed its direction (having been scattered) on encountering the molecules of the earth's atmosphere



As shown in figure, the incident sunlight is unpolarised. The dots stand for polarisation perpendicular to the plane of the figure. The double arrows show polarisation in the plane of the figure. There is no phase relation between these two in unpolarised light.
Under the influence of the electric field of the incident wave the electrons in the molecules acquire components of motion in both these directions.
We have drawn an observer looking at 90° to the direction of the sun. Clearly, charges accelerating parallel to the double arrows do not radiate energy towards this observer since their acceleration has no transverse component. The radiation scattered by the molecule is therefore represented by dots. It is polarized perpendicular to the plane of the figure. This explains the polarization of scattered light from the sky.

Fresnel distance, ray optics is a limiting case of wave optics

Fresnel distance is that distance from the slit at which the spreading of light due to diffraction becomes equal to the size of the slit. It is generally denoted by Z_F
We know that the first secondary minimum is formed at an angle θ_1 such that

$$\theta_1 = \frac{\lambda}{d}$$

After travelling a distance D, the width acquired by the beam due to diffraction is $D\lambda/d$
At Fresenel distance Z_F

$$\frac{Z_F\lambda}{d} = d$$

$$Z_F = \frac{d^2}{\lambda}$$

If the distance D between the slit and the screen is less than Fresnel distance Z_F then the diffraction effects may be regarded as absent. So, ray optics may be regarded as limiting case of wave optics

Solved Numerical

Light of wave length 600nm is incident on an aperture of size 2mm. calculate the distance upto which the ray of light can travel such that its spread is less than the size of the aperture
Solution

$$Z_F = \frac{d^2}{\lambda} = \frac{(2 \times 10^{-3})^2}{600 \times 10^{-9}} = 6.67\ m$$

Doppler effect for light

If there is no medium and the source moves away from the observer, then later wavefronts have to travel a greater distance to reach the observer and hence take a longer time. The time taken between the arrival of two successive wavefronts is hence longer at the observer than it is at the source.
Thus, when the source moves away from the observer the frequency as measured by the source will be smaller. This is known as the *Doppler effect*.
Astronomers call the increase in wavelength due to doppler effect as *red shift* since a wavelength in the middle of the visible region of the spectrum moves towards the red end of the spectrum.
When waves are received from a source moving towards the observer, there is an apparent decrease in wavelength, this is referred to as *blue shift.*

For velocities small compared to the speed of light, we can use the same formulae which we use for sound waves. The fractional change in frequency Δv/v is given by $–V_{radial}/c$, where V_{radial} is the component of the source velocity along the line joining the observer to the source relative to the observer; V_{radial} is considered positive when the source moves away from the observer. Thus, the Doppler shift can be expressed as:

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

Solved numerical

Q) Certain characteristic wavelengths of the light from a galaxy in the constellation Virgo are observed to be increased in wave length, as compaired with terrestrial sources, by 0.4%. What is the radial speed of this galaxy with respect to the earth? Is it approaching or receding?

<u>Solution</u>

From formula

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

We know that

$$\frac{\Delta v}{v} = -\frac{\Delta\lambda}{\lambda}$$

Thus

$$\frac{\Delta\lambda}{\lambda} = \frac{V_{radial}}{c}$$

Given : Δλ/λ = 0.004

$$V_{radial} = \frac{\Delta\lambda}{\lambda}c$$

$$V_{radial} = 0.004 \times 3 \times 10^8 = 1.2 \times 10^6\ ms^{-1}$$

Since v_{radial} is positive therefore galaxy is receding

Q) the red shift of radiation from a distant nebula consists of the light known to have a wavelength 4340×10^{-8} cm when observed in laboratory, appearing to have a wavelength of 4362×10^{-8} cm. What is the speed of the nebula in the line of sight relative to the earth ? Is it approaching or receding

Solution:

$\Delta\lambda = 4362\times10^{-8} - 4340\times10^{-8} = 22\times10^{-8}$ cm thus

$\Delta\lambda/\lambda = 22\times10^{-8}/4340\times10^{-8} = 0.0004$

Thus $\Delta\nu/\nu = -0.0004$

$$\frac{\Delta v}{v} = -\frac{V_{radial}}{c}$$

$$-0.0004 = -\frac{V_{radial}}{3\times10^{8}}$$

$V_{radial} = (0.0004)(3\times10^{8}) = 0.12852\times10^{8} = 1.2\times10^{5}$ m/s

Since V_{radial} is positive nebula is receding

----------------END--------------

COMMUNICATION SYSTEMS

The term 'communication' refers to sending, receiving and processing of information electronically. System is the setup to transmit information.

Basic terminology used in electronic communication system

(i) *Transducer:* Any device that converts one form of energy into another can be termed as a transducer. An electrical transducer may be defined as a device that converts some physical variable (pressure, displacement, force, temperature, etc) into corresponding variations in the electrical signal at its output.

(ii) *Signal:* Information converted in electrical form and suitable for transmission is called a signal. Signals can be either *analog or digital*.

(iii) *Noise:* Noise refers to the unwanted signals that tend to disturb the transmission and processing of message signals in a communication system. The source generating the noise may be located inside or outside the system.

(iv) *Transmitter:* A transmitter processes the incoming message signal so as to make it suitable for transmission through a channel and subsequent reception.

(v) *Receiver:* A receiver extracts the desired message signals from the received signals at the channel output.

(vi) *Attenuation:* The loss of strength of a signal while propagating through a medium is known as attenuation.

(vii) *Amplification*: It is the process of *increasing the amplitude* (and consequently the strength) of a signal using an electronic circuit called the amplifier. Amplification is necessary to compensate for the attenuation of the signal in communication systems. Amplification is done at a place between the source and the destination wherever signal strength becomes weaker than the required strength.

(viii) *Range:* It is the largest distance between a source and a destination up-to which the signal is received with sufficient strength.

(ix) *Band width :* Bandwidth refers to the frequency range over which an equipment operates or the portion of the spectrum occupied

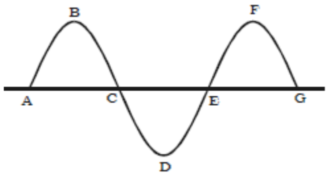
(x) *Modulation:* The original low frequency message/information signal cannot be transmitted to long distances, therefore information contained in low frequency signal is super imposed on high frequency wave which acts as a carrier of information. This process is called modulation

(xi) *Demodulation:* The process of retrieval of information from the carrier wave at the receiver is termed demodulation. This is the reverse process of modulation.

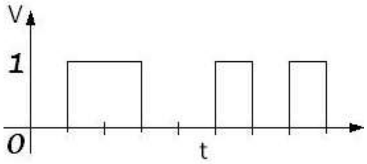
(xii) *Repeater:* A repeater is a combination of a receiver and a repeater picks up the signal from the transmitter change in carrier frequency. Repeaters are used to extend the range of a communication system. Communication satellite is essentially a repeater station in space.

Analog Signal and Digital signal:

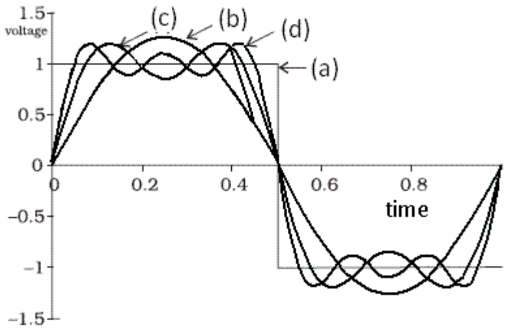
Analog signals are continuous variations of voltage or current. *They are essentially single-valued functions of time*.
Sine wave is a fundamental analog signal. Sound and picture signals in TV are analog in nature.



Digital signals are those which can take only discrete stepwise values. Binary system that is extensively used in digital electronics employs just two levels of a signal. ‘0’ corresponds to a low level and ‘1’ corresponds to a high level of voltage/ current.



One can show that this rectangular wave can be composed into a superposition of sinusoidal waves of frequencies ν_0, $2\nu_0$, $3\nu_0$, $4\nu_0$ ... $n\nu_0$ where n is an integer extending to infinity and $\nu_0 = 1/T_0$. The fundamental (ν_0), fundamental (ν_0) + second harmonic ($2\nu_0$), and fundamental (ν_0) + second harmonic ($2\nu_0$) + third harmonic ($3\nu_0$), are shown in the same figure to illustrate this fact.



(a) rectangular wave
(b) Fundamental ν_0
(c) Fundamental ν_0 + Second harmonic $2\nu_0$
(d) Fundamental ν_0 + Second harmonic $2\nu_0$ + Third harmonics 3 ν_0

It is clear that to reproduce the rectangular wave shape exactly we need to superimpose all the harmonics ν_0, $2\nu_0$, $3\nu_0$, $4\nu_0$..., which implies an infinite bandwidth.
However, for practical purposes, the contribution from higher harmonics can be neglected. This is so because the higher the harmonic, less is its contribution to the wave form.
There are several coding schemes useful for digital communication.

They employ suitable combinations of number systems such as the binary coded decimal (BCD).
American Standard Code for Information Interchange (ASCII) is a universally popular digital code to represent numbers, letters and certain characters.

Bandwidth of signals
In a communication system, the message signal can be voice, music, picture or computer data. Each of these signals has different ranges of frequencies.
The type of communication system needed for a given signal depends on the band of frequencies which is considered essential for the communication process. For speech signals, frequency range 300 Hz to 3100 Hz is considered adequate.
Therefore speech signal requires a bandwidth of 2800 Hz (3100 Hz– 300 Hz) for commercial telephonic communication.
To transmit music, an approximate bandwidth of 20 kHz is required because of the high frequencies produced by the musical instruments. The audible range of frequencies extends from 20 Hz to 20 kHz.
Video signals for transmission of pictures require about 4.2 MHz of bandwidth.
A TV signal contains both voice and picture and is usually allocated 6 MHz of bandwidth for transmission.

Bandwidth of transmission medium
Different types of transmission media offer different bandwidths. The commonly used transmission media are wire, free space and fiber optic cable. Coaxial cable is a widely used wire medium, which offers a bandwidth of approximately 750 MHz. Such cables are normally operated below 18 GHz.
Communication through free space using radio waves takes place over a very wide range of frequencies: from a few hundreds of kHz to a few GHz.
Optical communication using fibers is performed in the frequency range of 1 THz to 1000 THz (microwaves to ultraviolet). An optical fiber can offer a transmission bandwidth in excess of 100 GHz.
Spectrum allocations are arrived at by an international agreement. The International Telecommunication Union (ITU) administers the present system of frequency allocations

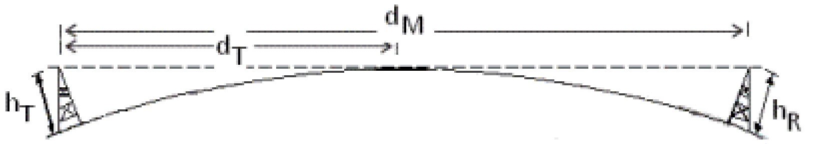
Propagation of electromagnetic wave
There are different ways which electromagnetic waves emitted by a transmitting antenna propagate through space and reach to the receiver.

Ground wave
To radiate signals with high efficiency, the antennas should have a size comparable to the wavelength λ of the signal (at least ~ λ/4).

At longer wavelengths (i.e., at lower frequencies), the antennas have large physical size and they are located on or very near to the ground. In standard AM broadcast, ground based vertical towers are generally used as transmitting antennas.
For such antennas, ground has a strong influence on the propagation of the signal. The mode of propagation is called surface wave propagation and the wave glides over the surface of the earth.
A wave induces current in the ground over which it passes and it is attenuated as a result of absorption of energy by the earth. The attenuation of surface waves increases very rapidly with increase in frequency.
The maximum range of coverage depends on the transmitted power and frequency (less than a few MHz).

Space wave propagation

Radio waves propagated through the troposphere of the Earth are known as space waves. Troposphere is the portion of the Earth's atmosphere which extends upto 15 km from the surface of the Earth. Space wave usually consists of two components as shown in Fig



(i) A component which travels straight from the transmitter to the receiver. line-of-sight (LOS) communication
(ii) A component which reaches the receiver after reflection from the surface of the Earth.
Space wave propagation is particularly suitable for the waves having frequency above 30MHz.

If the transmitting antenna is at a height h_T, then you can show that the distance to the horizon d_T is given as $dT = 2Rh_T$, where R is the radius of the earth (approximately 6400 km). d_T is also called the radio horizon of the transmitting antenna. With reference to Fig. the maximum line-of-sight distance d_M between the two antennas having heights h_T and h_R above the earth is given by

$$d_m = \sqrt{2Rh_T} + \sqrt{2Rh_R}$$

where h_R is the height of receiving antenna.
Area covered by transmission A = $\pi(d_T)^2 = \pi(2h_TR)$
Television broadcast, microwave links and satellite communication are some examples of communication systems that use space wave mode of propagation

Solved numerical

Q) What must be the height of the antenna of FM radio station so that people in a circular region of 3140 km^2 can enjoy the programme of an FM radio station? R = 6400 km]
Solution:
Area covered by transmission A = $\pi(2h_TR)$

$3140 = 3.14\times2\times h_T\times6400$
$h_T = 0.078125$ km = 78.125 m

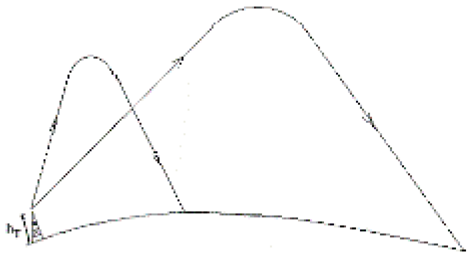
Q) A transmitting antenna at the top of a tower has a height of 50m and the height of the receiving antenna is 32m. What is the maximum distance between them for satisfactory communication is LOS mode? Give radius of earth R = 6400km

Solution:
h_R = 32m, h_T = 50 m,
From formula

$$d_m = \sqrt{2Rh_T} + \sqrt{2Rh_R}$$
$$d_m = \sqrt{2\times6400\times10^3\times50} + \sqrt{2\times6400\times10^3\times32}$$
$$d_m = 25.29\times10^3 + 20.23\times10^3 = 45.5\ km$$

Sky wave propagation

The propagation of radio waves (frequency 2MHz to 30 MHz) is due to sky waves. The electromagnetic waves emitted by the transmitter, return to the earth after getting reflected by the ionosphere at a height of about 80 -300km. A receiver at large distance can receive these reflected waves. The ionosphere behaves like a mirror for these radio waves.



Ionosphere of earth atmosphere contains electrons anions produced due to radiation from sun. There are different layers at various heights, depending on the density of gas, intensity of radiation and selective ionization of gases by various radiation. Electron density of all layers are different. So the radio waves with different frequencies get reflected from the ionosphere of diffident height. Due to total internal reflection phenomenon waves can be received at far distance from the transmitter on the earth Frequency from 2MHz to 30MHz are used above 30MHz frequency wave penetrate the ionosphere and cannot be reflected by ionosphere.

Modulation and its necessity

Most of the signals of information have low frequency and they are not able to travel long distance in free space. Because of flowing factors

(1)Length of antenna: A transmitter converts audio frequency electrical signal into electromagnetic radiation through an antenna and radiates in the space

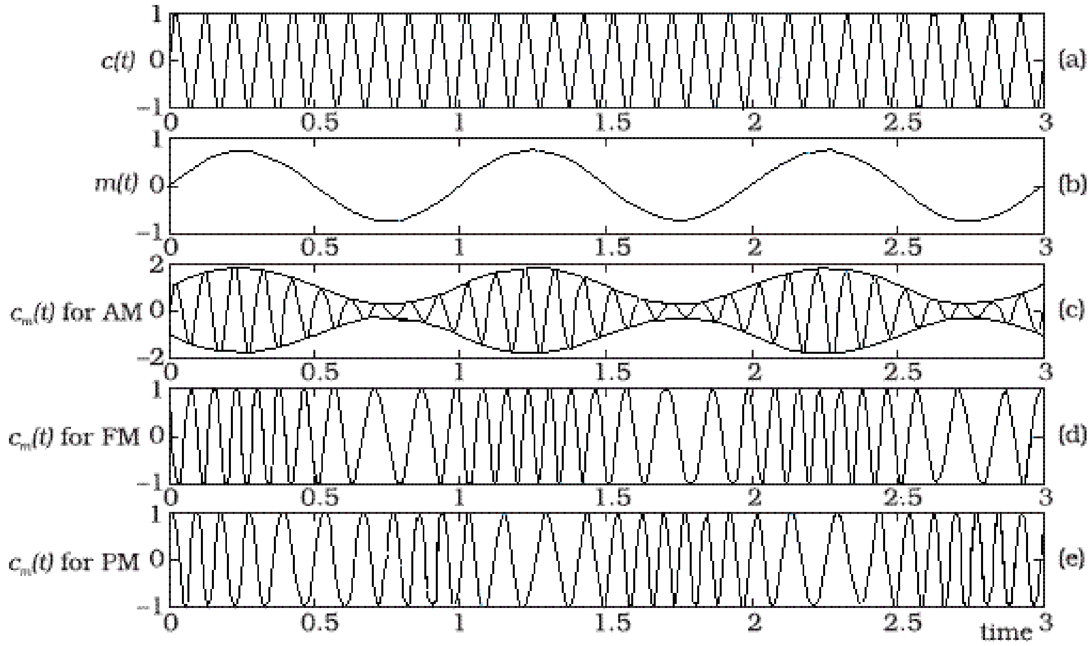
For effective transmission of electromagnetic radiation of audio signal, the minimum length of antenna must be λ/4. Where λ is the wavelength of the audio signal
If transmitted wavelength is 300 km whose frequency is about 1kHz then minimum length of antenna would be
300/4 = 75 km, which impractical as well as very costly
This shows that for effective transmission of high frequency signals, required antenna length is small and hence an antenna can be easily constructed.

(2)Power radiated from antenna: Transmitted power by an antenna of a given length is inversely proportional to the square of the wavelength λ i.e. $P \propto 1/\lambda^2$.
This indicates that an antenna can transmit short wavelength or high frequency radiation with more efficiency.

(3) Mixing up of signals from different transmitters: If there is more than one transmitter in a region and if these transmit the information using audio signals, then all such signals get mixed. It is not possible to separate information of one transmitter from the information of other transmitter. Such situation can be avoided if every transmitter is assigned different high frequencies for information.

Modulation

The process of superposing low frequency audio signals on waves with high frequency is called modulation.
Here, low frequency signal is called the modulating and the high frequency wave, since it carries the information is called a carrier wave.
A sinusoidal carrier wave can be represented as $c(t) = A_c \sin(\omega_c t + \phi)$
where $c(t)$ is the signal strength (voltage or current), A_c is the amplitude, ω_c ($= 2\pi\nu_c$) is the angular frequency and ϕ is the initial phase of the carrier wave. During the process of modulation, any of the three parameters, *viz* A_c, ω_c and ϕ, of the carrier wave can be controlled by the message or information signal.
This results in three types of modulation: (i) Amplitude modulation (AM), (ii) Frequency modulation (FM) and (iii) Phase modulation (PM) as shown in figure



Amplitude modulation

A modulation in which the amplitude of carrier wave C(t) is varied in accordance with the instantaneous value of the modulating wave is called amplitude modulation (AM). The frequency and initial phase remains constant

Let $c(t) = A_c \sin \omega_c t$ represent carrier wave and $m(t) = A_m \sin \omega_m t$ represent the message or the modulating signal where $\omega_m = 2\pi f_m$ is the angular frequency of the message signal.

Amplitude of carrier wave changes according to modulating signal but frequency and phase of the carrier wave remains constant thus

The modulated signal $c_m(t)$ can be written as

$c_m(t) = (A_c + A_m \sin \omega_m t) \sin \omega_c t$

$$C_m(t) = A_c\left(1 + \frac{A_m}{A_c} sin\omega_m t\right) sin\omega_c t$$

$C_m(t) = A_c(1 + \mu sin\omega_m t)sin\omega_c t \; ------eq(1)$

Above equation is a mathematical form of AM wave

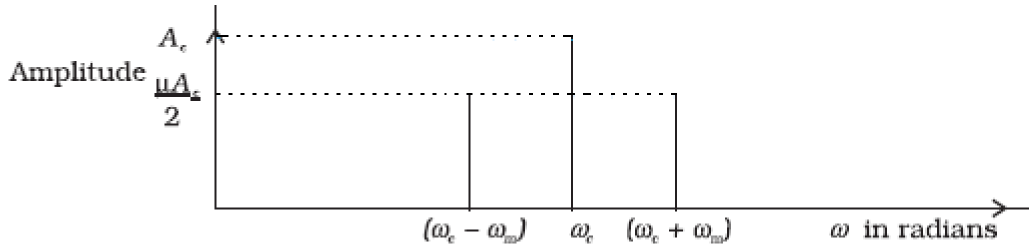
Here $\mu = A_m/A_c$ is the *modulation index*; in practice, μ is kept ≤ 1 to avoid distortion.

$$C_m(t) = A_c sin\omega_c t + A_c \mu sin\omega_m t sin\omega_c t$$

Using the trigonometric relation $\sin A \sin B = ½ (\cos(A - B) - \cos(A + B))$, above equation can be wrote as

$$C_m(t) = A_c sin\omega_c t + \frac{\mu A_c}{2} cos(\omega_c - \omega_m)t - \frac{\mu A_c}{2} cos(\omega_c + \omega_m)t$$

Here $\omega_c - \omega_m$ and $\omega_c + \omega_m$ are respectively called the lower side frequency band (LSB) and upper side frequency band (USB). The modulated signal now consists of the carrier wave of frequency ω_c plus two sinusoidal waves each with a frequency slightly different from, known as side bands. The frequency spectrum of the amplitude modulated signal is shown in Fig



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Amplitude of USB and LSB is $\mu A_c/2$
From eq(1) When sinωt = 1 Amplitude modulated wave have maximum amplitude
$A_{max} = A_C + A_m$
When sinωt = -1. Amplitude modulated wave have minimum amplitude
$A_{min} = A_c - A_m$
From above equations

$$A_c = \frac{A_{max} + A_{min}}{2} \ and \ A_m = \frac{A_{max} - A_{min}}{2}$$

According to definition of modulation index

$$\mu = \frac{A_m}{A_C} = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$$

$$\mu(\%) = \frac{A_{max} - A_{min}}{A_{max} + A_{min}} \times 100$$

Solved numerical

Q) A 10 MHz sinusoidal carrier wave of amplitude 10 mV is modulated by a 5 kHz sinusoidal audio signal wave of amplitude 6 mV. Find the frequency components of the resultant modulated wave and their amplitude.
Data: Frequency of the carrier = *fc* = 10 MHz
Frequency of the signal = *fs*= 5 kHz = 0.005 MHz
Amplitude of the carrier signal = *Ec* = 10 mV
Amplitude of the audio signal = *Es* = 6 mV
Frequency components of modulated wave = ?
Amplitude of the components in the modulated wave = ?
<u>Solution :</u>
The modulated carrier wave contains the following frequencies :
(i) Original carrier wave of frequency = *fc* = 10 MHz
(ii) Upper side band frequency, *fc + fs* = 10 + 0.005 = 10.005 MHz
(iii) Lower side band frequency *fc – fs* = 10 − 0.005 = 9.995 MHz
The modulation factor is,

$$\mu = \frac{A_m}{A_C} = \frac{6}{10} = 0.6$$

∴Amplitude of USB = Amplitude of LSB

$$\frac{\mu A_c}{2} = \frac{0.6 \times 10}{2} = 3mV$$

Q) The equation of AM wave is C = 100(1 +0.6sin6280t) sin2π×10^6 t. Calculate (i)Modulation Index (ii) Frequency of carrier wave (iii) frequency of modulating wave (iv) frequency of LSB and USB

Solution:

Comparing given equation with standard equation

$$C_m(t) = A_c(1 + \mu sin\omega_m t) sin\omega_c t$$

We get (i) modulation index μ = 0.6

(ii) Frequency of carrier wave $\omega_C = 2\pi\times10^6$

∴ $2\pi f_c = 2\pi\times10^6$

∴$f_c = 10^6$ Hz= 1MHz

(iii) Frequency of modulating wave

ω_m =6280

∴ $2\pi f_m = 6280$

∴$2\times3.14\times f_m = 6280$

∴f_m = 1000 Hz=1kHz

(iv) frequency of LSB

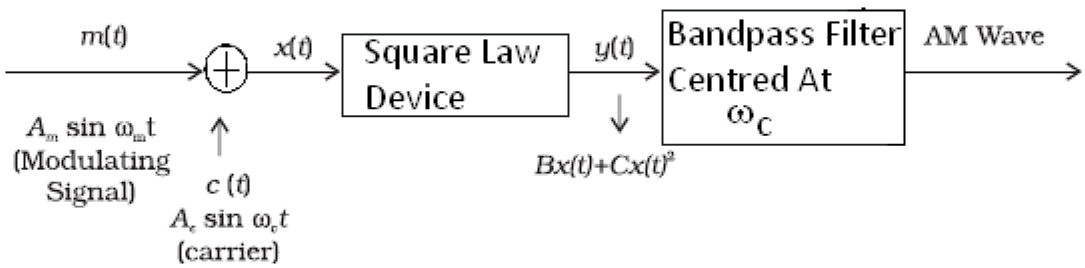
$f = f_c - f_m$ = 1MHz - 1kHz = 0.999 MHz

frequency USB

$f = f_c + f_m$ = 1MHz + 1kHz = 1.001 MHz

Production of amplitude modulated wave

Amplitude modulation can be produced by a variety of methods. A conceptually simple method is shown in the block diagram of Fig.



Here the modulating signal A_m sin $\omega_m t$ is added to the carrier signal Ac sin $\omega_c t$ to produce the signal $x(t)$. This signal $x(t) = A_m \sin\omega_m t + Ac \sin\omega_c t$ is passed through a square law device which is a non-linear device which produces an output

$y(t) = B x(t) + Cx^2(t)$

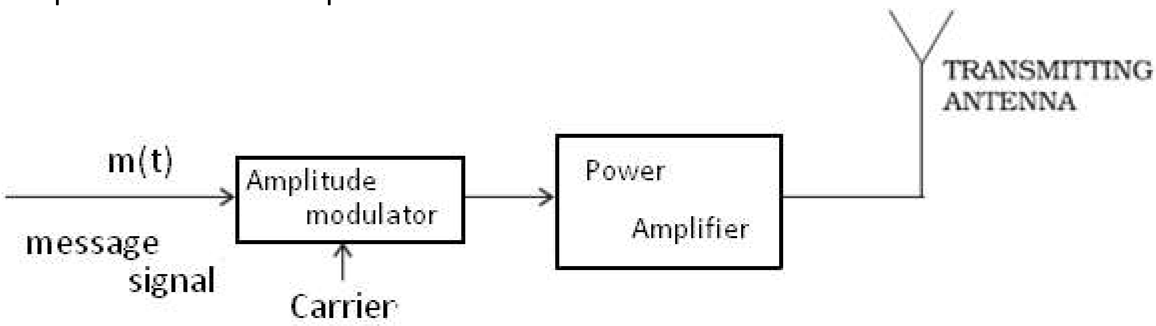
where B and C are constants. Thus,

$y(t) = BA_m \sin \omega_m t + BA_c \sin \omega_c t + C[A^2_m \sin^2_m \omega_m t + A^2_c \sin^2_c \omega_c t + 2A_m A_c \sin\omega_c t \sin\omega_c t]$

From trigonometry formula $\sin^2 A = (1-\cos2A)/2$ we get

$$y\,(t) = \text{BAm}\sin\omega\text{mt} + \text{BAc}\sin\omega\text{ct} + \frac{\text{CA}_\text{m}^2}{2} + \text{A}_\text{c}^2 - \frac{\text{CA}_\text{m}^2}{2}\cos 2\omega_\text{m}\text{t} - \frac{\text{CA}_\text{c}^2}{2}\cos 2\omega_\text{c}\text{t} + \text{CA}_\text{m}\text{A}_\text{C}\cos(\omega_\text{c} - \omega_\text{m})\text{t} - \text{CA}_\text{m}\text{A}_\text{C}\cos(\omega_\text{c} + \omega_\text{m})\text{t}$$

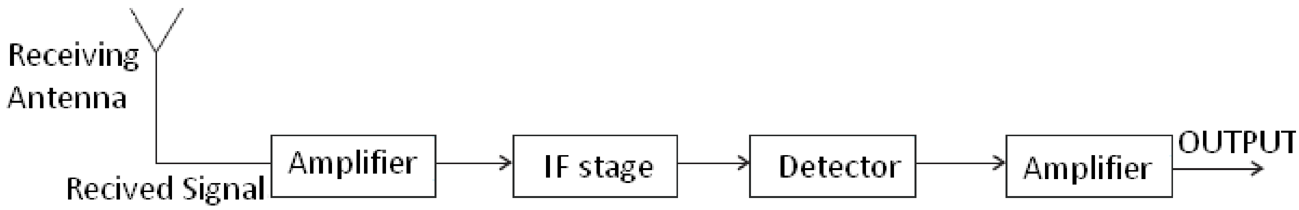
In above equation dc term is $C/2(A^2_m + A^2_c)$ and sinusoids of frequencies ω_m, $2\omega_m$, ω_c, $2\omega_C$, $\omega_c - \omega_m$, and $\omega_c + \omega_m$ this signal passes through a band filters which rejects dc and the frequencies of ω_m, $2\omega_m$, $2\omega_C$ and retains frequencies ω_c, $\omega_c - \omega_m$, and $\omega_c + \omega_m$. the out put therefore is amplitude modulated waves



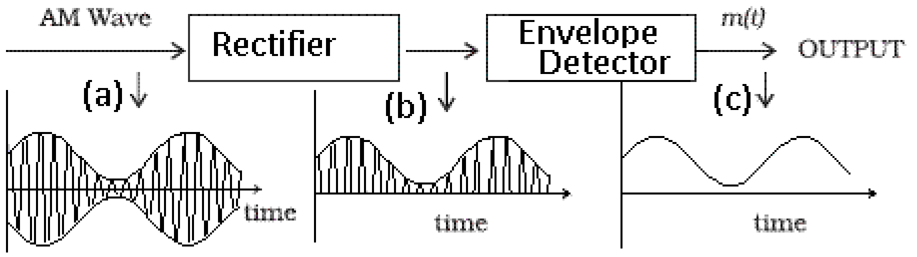
Modulated signals cannot be transmitted as such. The modulator is followed by a power amplifier which provides the necessary power and then modulated signals is forwarded to an antenna of appropriate size for radiation as shown in figure

Detection of amplitude modulated wave

The transmitted message gets attenuated in propagating through the channel. The receiving antenna is therefore to be followed by an amplifier and a detector.
In addition, to facilitate further processing, the carrier frequency is usually changed to a lower frequency by what is called an *intermediate frequency (IF) stage* preceding the detection.



The detected signal may not be strong enough to be made use of and hence is required to be amplified. A block diagram of a typical receiver is shown in Detection is the process of recovering the modulating signal from the modulated carrier wave.
We know that the modulated carrier wave contains the frequencies ω_c and $\omega_c \pm \omega_m$.
In order to obtain the original message signal $m(t)$ of angular frequency ω_m, a simple method is shown in the form of a block diagram in Fig.



AM input wave Rectified wave Output (without RF component)

The modulated signal of the form given in (a) of fig. is passed through a rectifier to produce the output shown in (b).
This envelope of signal (b) is the message signal. In order to retrieve $m(t)$, the signal is passed through an envelope detector (which may consist of a simple RC circuit).

SEMICONDUCTOR ELECTRONICS:

MATERIALS, DEVICES AND SIMPLE CIRCUITS

Semiconductors

It has been observed that certain materials like germanium, silicon etc. have resistivity between good conductors like copper and insulators like glass. These materials are known as semiconductors. A material which has resistivity between conductors and insulators is known as semiconductor. The resistivity of a semiconductor lie approximately between 10^{-2} and 10^{4} Ω m at room temperature. The resistance of a semiconductor decreases with increase in temperature over a particular temperature range. This behavior is contrary to that of a metallic conductor for which the resistance increases with increase in temperature.

The elements that are classified as semiconductors are Si, Ge, In, etc. Germanium and silicon are most widely used as semiconductors

Energy band in solids

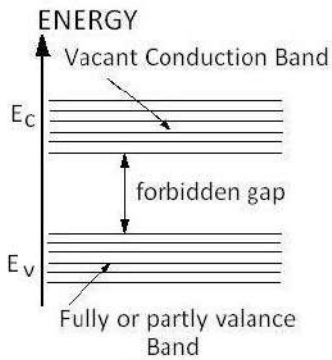
In the case of a single isolated atom, there are various discrete energy levels. In solids, the atoms are arranged in a systematic space lattice and each atom is influenced by neighboring atoms. The closeness of atoms results in the intermixing of electrons of neighboring atoms.

Inside the crystal each electron has a unique position and no two electrons see exactly the same pattern of surrounding charges. Because of this, each electron will have a different *energy level*.

These different energy levels with continuous energy variation form what are called *energy bands*. The energy band which includes the energy levels of the valence electrons is called the *valence band*. The energy band above the valence band is called the *conduction band*.

Energy difference between energy of conduction band and valance band is called band gap energy or forbidden energy gap

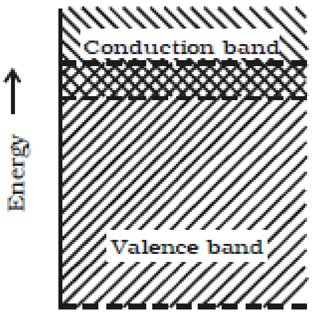
With no external energy, all the valence electrons will reside in the valence band. If the lowest level in the conduction band happens to be lower than the highest level of the valence band, the electrons from the valence band can move into the conduction band



Let us consider what happens in the case of Si or Ge crystal containing *N* atoms. For Si, the outermost orbit is the third orbit (*n* = 3), while for Ge it is the fourth orbit (*n* = 4). The number of electrons in the outermost orbit is 4 (3*s* and 3*p* electrons for Si). Hence, the total number of outer electrons in the crystal is 4*N*. The maximum possible number of electrons in the outer orbit is 8 (2*s* + 6*p* electrons).

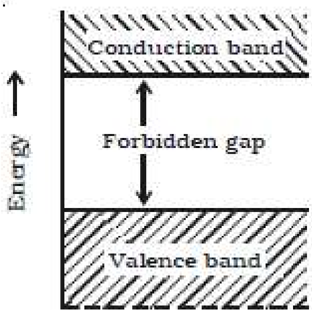
So, for the 4*N* valence electrons there are 8*N* available energy states. These 8*N* discrete energy levels can either form a continuous band or they may be grouped in different bands depending upon the distance between the atoms in the crystal

Conductors:



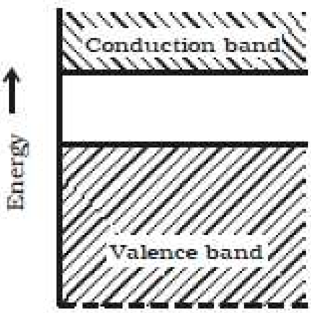
Normally the conduction band is empty. But when it overlaps on the valence band electrons can move freely into it. This is the case with metallic conductors.

Insulators



In an insulator, the forbidden energy gap is very large. In general, the forbidden energy gap is more than 3eV and almost no electrons are available for conduction. Therefore, a very large amount of energy must be supplied to a valence electron to enable it to move to the conduction band. In the case of materials like glass, the valence band is completely filled at 0 K. The energy gap between valence band and conduction band is of the order of 10 eV. Even in the presence of high electric field, the electrons cannot move from valence band to conduction band. If the electron is supplied with high energy, it can jump across the forbidden gap. When the temperature is increased, some electrons will move to the conduction band. This is the reason, why certain materials, which are insulators at room temperature become conductors at high temperature. The resistivity of insulator approximately lies between 10^{11} and 10^{16} Ω m

Semiconductors



In semiconductors (Fig), the forbidden gap is very small. Germanium and silicon are the best examples of semiconductors. The forbidden gap energy is of the order of 0.7eV for Ge and 1.1eV for Si.

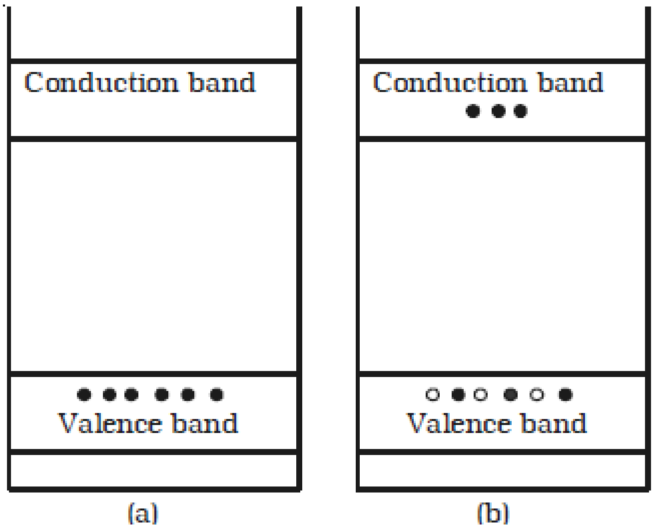
There are no electrons in the conduction band. The valence band is completely filled at 0 K. With a small amount of energy that is supplied, the electrons can easily jump from the valence band to the conduction band.

For example, if the temperature is raised, the forbidden gap is decreased and some electrons are liberated into the conduction band.

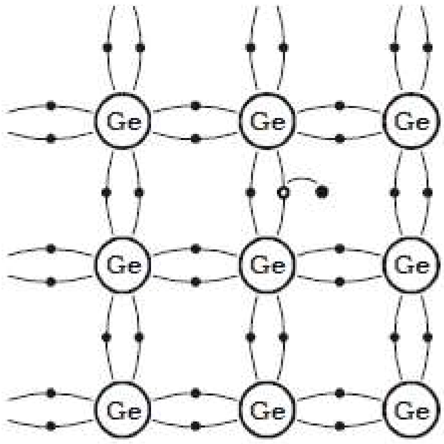
The conductivity of a semiconductor is of the order of 10^2 mho m^{-1}

INTRINSIC SEMICONDUCTOR

A semiconductor which is pure and contains no impurity is known as an intrinsic semiconductor. In an intrinsic semiconductor, the number of free electrons and holes are equal. Common examples of intrinsic semiconductors are pure germanium and silicon
Fig a and Fig b represent charge carriers at absolute zero temperature and at room temperature respectively.



The electrons in an intrinsic semiconductor, which move in to the conduction band at high temperatures are called as intrinsic carriers. In the valence band, a vacancy is created at the place where the electron was present, before it had moved in to the conduction band. This vacancy is called hole.
Fig c helps in understanding the creation of a hole. Consider the case of pure germanium crystal. It has four electrons in its outer or valence orbit. These electrons are known as valence electrons. When two atoms of germanium are brought close to each other, a covalent bond is formed between the atoms. If some additional energy is received, one of the electrons contributing to a covalent bond breaks and it is free to move in the crystal lattice



While coming out of the bond, a hole is said to be created at its place, which is usually represented by a open circle. The hole behaves as an *apparent free particle* with effective positive charge. An electron from the neighboring atom can break the covalent bond and

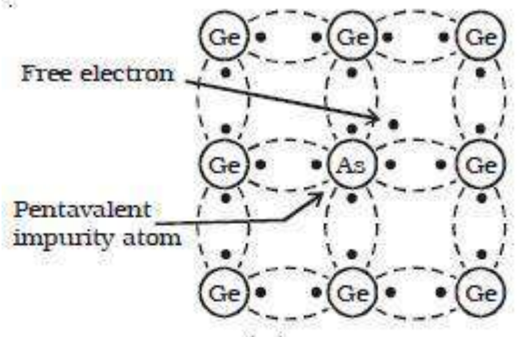
can occupy this hole, creating a hole at another place. Since an electron has a unit negative charge, the hole is associated with a unit positive charge. The importance of hole is that, it may serve as a carrier of electricity in the same manner as the free electron, but in the opposite direction.
In intrinsic semiconductors, the number of free electrons, n_e is equal to the number of holes, n_h. That is $n_e = n_h = n_i$
where n_i is called intrinsic carrier concentration
Under the action of an electric field, these holes move towards negative potential giving the hole current, I_h. The total current, I is thus the sum of the electron current I_e and the hole current I_h:
$I = I_e + I_h$
It may be noted that apart from the *process of generation* of conduction electrons and holes, a simultaneous *process of recombination* occurs in which the electrons *recombine* with the holes. At equilibrium, the rate of generation is equal to the rate of recombination of charge carriers. The recombination occurs due to an electron colliding with a hole.

EXTRINSIC SEMICONDUCTOR

Electrons and holes can be generated in a semiconductor crystal with heat energy or light energy. But in these cases, the conductivity remains very low. The efficient and convenient method of generating free electrons and holes is to add very small amount of selected impurity inside the crystal. The impurity to be added is of the order of 100 ppm (parts per million). The process of addition of a very small amount of impurity into an intrinsic semiconductor is called doping.
The impurity atoms are called dopants. The semiconductor containing impurity atoms is known as impure or doped or extrinsic semiconductor.
There are three different methods of doping a semiconductor.
(i) The impurity atoms are added to the semiconductor in its molten state.
(ii) The pure semiconductor is bombarded by ions of impurity atoms.
(iii) When the semiconductor crystal containing the impurity atoms is heated, the impurity atoms diffuse into the hot crystal.
Usually, the doping material is either pentavalent atoms (bismuth, antimony, phosphorous, arsenic which have five valence electrons) or trivalent atoms (aluminium, gallium, indium, boron which have three valence electrons).
The pentavalent doping atom is known as donor atom, since it donates one electron to the conduction band of pure semiconductor.
The trivalent atom is called an acceptor atom, because it accepts one electron from the pure semiconductor atom.
Depending upon the type of impurity atoms added, an extrinsic semiconductor can be classified as N-type or P-type.

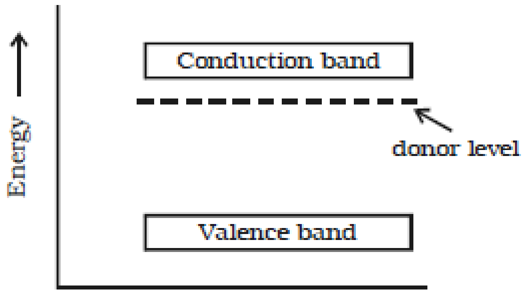
(A) N-TYPE SEMICONDUCTOR

When a small amount of pentavalent impurity such as arsenic is added to a pure germanium semiconductor crystal, the resulting crystal is called N-type semiconductor. Fig a shows the crystal structure obtained when pentavalent arsenic impurity is added with pure germanium crystal



(a)

The four valence electrons of arsenic atom form covalent bonds with electrons of neighboring four germanium atoms. The fifth electron of arsenic atom is loosely bound. This electron can move about almost as freely as an electron in a conductor and hence it will be the carrier of current. In the energy band picture, the energy state corresponding to the fifth valence electron is in the forbidden gap and lies slightly below the conduction band (Figb).



This level is known as the donor level. When the fifth valence electron is transferred to the conduction band, the arsenic atom becomes positively charged immobile ion. Each impurity atom donates one free electron to the semiconductor. These impurity atoms are called donors.

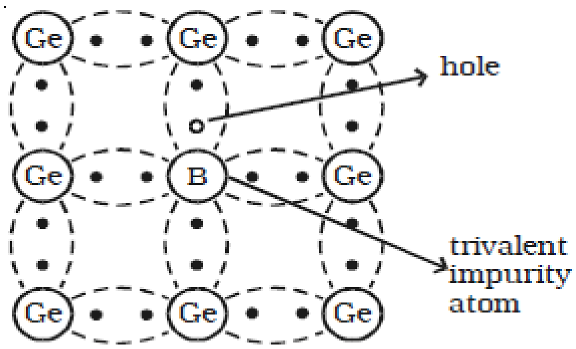
In N-type semiconductor material, the number of electrons increases, compared to the available number of charge carriers in the intrinsic semiconductor. This is because, the available larger number of electrons increases the rate of recombination of electrons with holes. Hence, in N-type semiconductor, free electrons are the majority charge carriers and holes are the minority charge carriers in an extrinsic

therefore, known as n-*type semiconductors*. For n-type semiconductors,

we have, $n_e >> n_h$

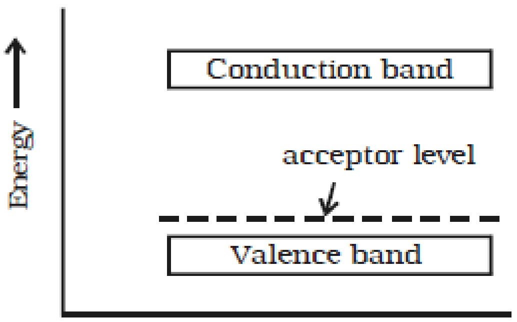
(b) P-type semiconductor

When a small amount of trivalent impurity (such as indium, boron or gallium) is added to a pure semiconductor crystal, the resulting semiconductor crystal is called P-type semiconductor. (Fig a)shows the crystal structure obtained, when trivalent boron impurity is added with pure germanium crystal.



The three valence electrons of the boron atom form covalent bonds with valence electrons of three neighborhood germanium atoms. In the fourth covalent bond, only one valence electron is available from germanium atom and there is deficiency of one electron which is called as a hole.

Hence for each boron atom added, one hole is created. Since the holes can accept electrons from neighborhood, the impurity is called acceptor. The hole, may be filled by the electron from a neighboring atom, creating a hole in that position from where the electron moves. This process continues and the hole moves about in a random manner due to thermal effects. Since the hole is associated with a positive charge moving from one position to another, this is called as P-type semiconductor. In the P-type semiconductor, the acceptor impurity produces an energy level just above the valence band. (Fig b).



Since, the energy difference between acceptor energy level and the valence band is much smaller, electrons from the valence band can easily jump into the acceptor level by thermal agitation. In P-type semiconductors, holes are the majority charge carriers and free electrons are the minority charge carriers.

Therefore, extrinsic semiconductors doped with trivalent impurity are called p-*type semiconductors*. For p-type semiconductors, the recombination process will reduce the number (*ni*)of intrinsically generated electrons to n_e. We have, for p-type semiconductors $n_h >> n_e$

Note that the crystal maintains an overall charge neutrality as the charge of additional charge carriers is just equal and opposite to that of the ionised cores in the lattice.

Conduction in p-type and n-type semiconductors

The semiconductor's energy band structure is affected by doping. In the case of extrinsic semiconductors, additional energy states due to donor impurities and acceptor impurities also exist.

In the energy band diagram of n-type Si semiconductor, the donor energy level is slightly below the bottom of the conduction band and electrons from this level move into the conduction band with very small supply of energy. At room temperature, most of the donor atoms get ionised but very few (~10–12) atoms of Si get ionised. So the conduction band will have most electrons coming from the donor impurities,

Similarly for p-type semiconductor, the acceptor energy level is slightly above the top of the valence band. With very small supply of energy an electron from the valence band can jump to the level and ionise the acceptor negatively. (Alternately, we can also say that with very small supply of energy the hole from level sinks down into the valence band. Electrons rise up and holes fall down when they gain external energy.) At room temperature, most of the acceptor atoms get ionised leaving holes in the valence band. Thus at room temperature the density of holes in the valence band is predominantly due to impurity in the extrinsic semiconductor. The electron and hole concentration in a semiconductor *in thermal equilibrium* is given by $n_e n_h = n_i^2$

Solved Numerical

Q) Suppose a pure Si crystal has 5 × 1028 atoms m^{-3}. It is doped by 1 ppm concentration of pentavalent As. Calculate the number of electrons and holes. Given that $n_i = 1.5 \times 10^{16}\ m^{-3.}$

Solution:

Note that thermally generated electrons ($n_i \sim 10^{16}\ m^{-3}$) are negligibly small as compared to those produced by doping.

Therefore, $n_e \approx N_D$.

Since $n_e n_h = n_i^2$

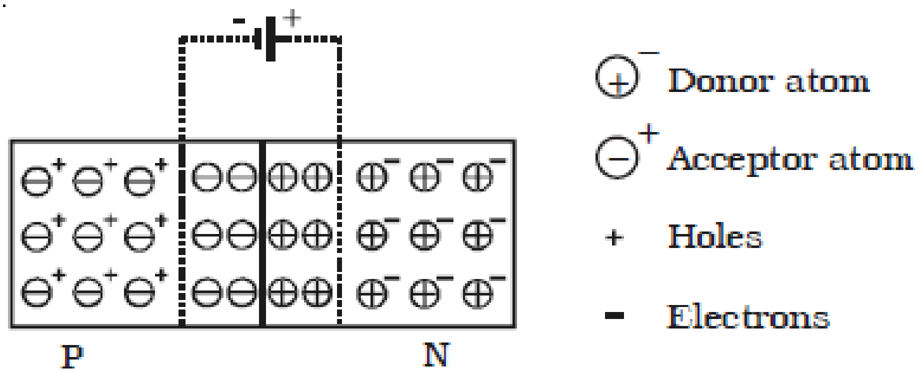
The number of holes

$n_h = (2.25 \times 10^{32})/(5 \times 10^{22})$

$n_h \sim 4.5 \times 10^9\ m^{-3}$

PN Junction diode

If one side of a single crystal of pure semiconductor (Ge or Si) is doped with acceptor impurity atoms and the other side is doped with donor impurity atoms, a PN junction is formed as shown in Fig



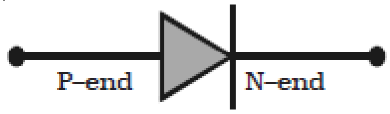
P region has a high concentration of holes and N region contains a large number of electrons. As soon as the junction is formed, free electrons and holes cross through the junction by the process of diffusion. During this process, the electrons crossing the junction from N-region into the P region, recombine with holes in the P-region very close to the junction.

Similarly holes crossing the junction from the P-region into the N-region, recombine with electrons in the N-region very close to the junction. Thus a region is formed, which does not have any mobile charges very close to the junction. This region is called depletion region. In this region, on the left side of the junction, the acceptor atoms become negative ions and on the right side of the junction, the donor atoms become positive ions

An electric field is set up, between the donor and acceptor ions in the depletion region. The potential at the N-side is higher than the potential at P-side. Therefore electrons in the N-side are prevented to go to the lower potential of P-side. Similarly, holes in the P-side find themselves at a lower potential and are prevented to cross to the N-side. Thus, there is a barrier at the junction which opposes the movement of the majority charge carriers. The difference of potential from one side of the barrier to the other side is called potential barrier. The potential barrier is approximately 0.7V for a silicon PN junction and 0.3V for a germanium PN junction. The distance from one side of the barrier to the other side is called the width of the barrier, which depends upon the nature of the material.

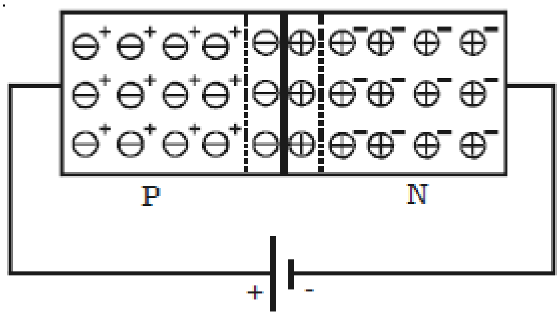
Symbol for a semiconductor diode

The diode symbol is shown in Fig The P-type and N-type regions are referred to as P–end and N–end respectively. The arrow on the diode points the direction of



Forward biased PN junction diode

When the positive terminal of the battery is connected to P-side and negative terminal to the N-side, so that the electric field across diode due to battery is in opposite direction to the electric field of barrier, then the PN junction diode is said to be forward biased.

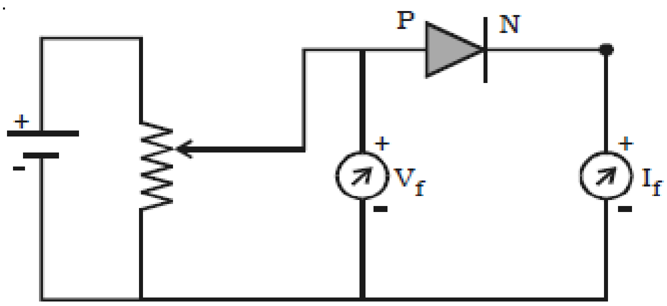


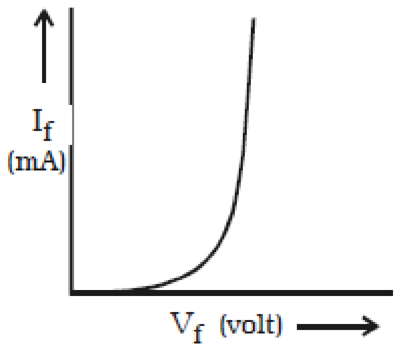
When the PN junction is forward biased (Fig), the applied positive potential repels the holes in the P-region, and the applied negative potential repels the electrons in the N-region, so the charges move towards the junction.

If the applied potential difference is more than the potential barrier, some holes and free electrons enter the depletion region.

Hence, the potential barrier as well as the width of the depletion region are reduced. The positive donor ions and negative acceptor ions within the depletion region regain electrons and holes respectively. As a result of this, the depletion region disappears and the potential barrier also disappears. Hence, under the action of the forward potential difference, the majority charge carriers flow across the junction in opposite direction and constitute current flow in the forward direction.

Forward bias characteristics





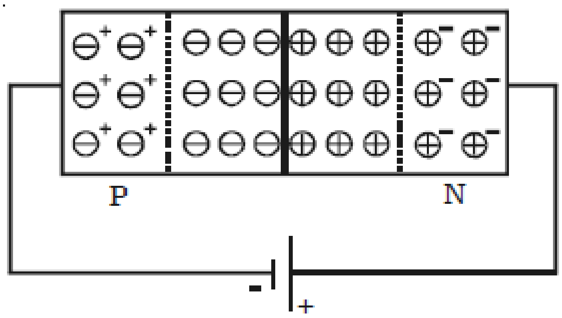
The circuit for the study of forward bias characteristics of PN junction diode is shown in Fig The voltage between P–end and N–end is increased from zero in suitable equal steps and the corresponding currents are noted down. (Fig b) shows the forward bias characteristic curve of the diode. Voltage is the independent variable. Therefore, it is plotted along X–axis. Since, current is the dependent variable, it is plotted against Y–axis. From the characteristic curve, the following conclusions can be made.

(i) The forward characteristic is not a straight line. Hence the ratio V/I is not a constant (i.e) the diode does not obey Ohm's law. This implies that the semiconductor diode is a non-linear conductor of electricity.

(ii) It can be seen from the characteristic curve that initially, the current is very small. This is because, the diode will start conducting, only when the external voltage overcomes the barrier potential (0.7V for silicon diode). As the voltage is increased to 0.7 V, large number of free electrons and holes start crossing the junction. Above 0.7V, the current increases rapidly. The voltage at which the current starts to increase rapidly is known as cut-in voltage or knee voltage of the diode

Reverse biased PN junction diode

When the positive terminal of the battery is connected to the difference is in the same direction as that of barrier potential, the junction is said to be reverse biased.



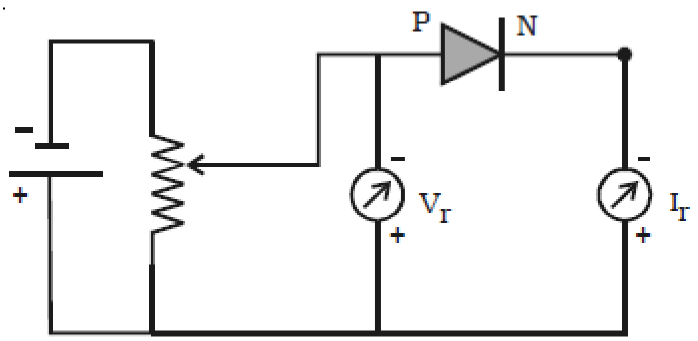
When the PN junction is reverse biased (Fig), electrons in the N region and holes in the P-region are attracted away from the junction N-side and negative terminal to the P-side, so that the applied potential Because of this, the number of negative ions in the P-region and positive ions in the N-region increases. Hence the depletion region becomes wider and the potential barrier is increased.

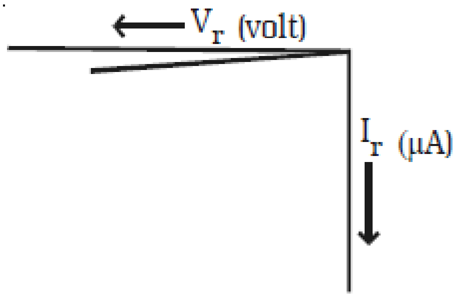
Since the depletion region does not contain majority charge carriers, it acts like an insulator. Therefore, no current should flow in the external circuit. But, in practice, a very small current of the order of few microamperes flows in the reverse direction. This is due to the minority carriers flowing in the opposite direction. This reverse current is small, because the number of minority carriers in both regions is very small. Since the major source of minority carriers is, thermally broken covalent bonds, the reverse current mainly depends on the junction temperature.

Reverse bias characteristics

The circuit for reverse bias characteristics of PN junction diode is shown(Fig.)

The voltage is increased from zero in suitable steps. For each voltage, the corresponding current readings are noted down. Fig b shows the reverse bias characteristic curve of the diode. From the characteristic curve, it can be concluded that, as voltage is increased from zero, reverse current (in the order of microamperes) increases and reaches the maximum value at a small value of the reverse voltage. When the voltage is further increased, the current is almost independent of the reverse voltage upto a certain critical value.
This reverse current is known as the reverse saturation current or leakage current. This current is due to the minority charge carriers, which depends on junction temperature.



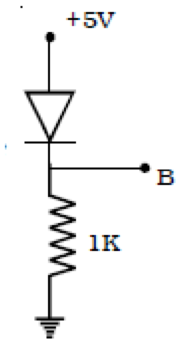


Avalanche breakdown :

When both sides of the PN junction are lightly doped and the depletion layer becomes large, avalanche breakdown takes place. In this case, the electric field across the depletion layer is not so strong. The minority carriers accelerated by the field, collide with the semiconductor atoms in the crystal.
Because of this collision with valence electrons, covalent bonds are broken and electron hole pairs are generated. These charge carriers, so produced acquire energy from the applied potential and in turn produce more and more carriers. This cumulative process is called avalanche multiplication and the breakdown is called avalanche breakdown.

Solved Numerical

Q) Find the voltage at the point B in the figure (Silicon diode is used).

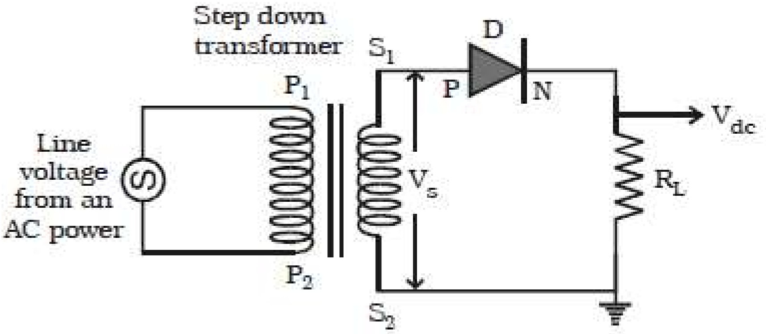


Solution:
The potential drop across the diode is equal to the knee voltage when diode is in forward biases. This voltage for Si diode is 0.7V

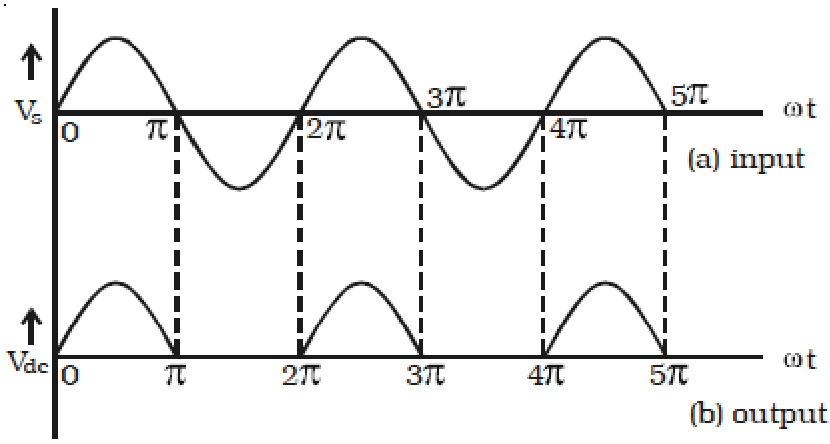
Now by applying Kirchhoff law
$5 = 0.7 + V_R$
$V_R = 4.3$ V
Now V_R = potential at B
Thus potential at B is 4.3V

Half wave rectifier

A circuit which rectifies half of the a.c wave is called half wave rectifier. Fig shows the circuit for half wave rectification. The a.c. voltage (Vs) to be rectified is obtained across the secondary ends S_1 S_2 of the transformer.

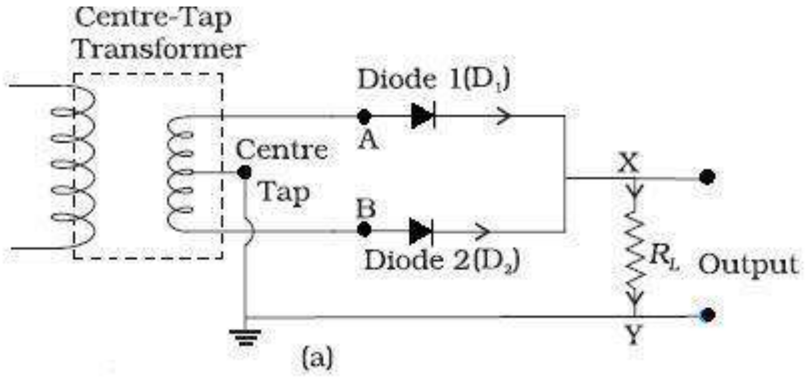


The P-end of the diode D is connected to S_1 of the secondary coil of the transformer. The N-end of the diode is connected to the other end S_2 of the secondary coil of the transformer, through a load resistance R_L.
The rectified output voltage V_{dc} appears across the load resistance R_L. During the positive half cycle of the input a.c. voltage Vs, S1 will be positive and the diode is forward biased and hence it conducts.
Therefore, current flows through the circuit and there is a voltage drop across RL. This gives the output voltage as shown in Fig. During the negative half cycle of the input a.c. voltage (Vs), S_1 will be negative and the diode D is reverse biased. Hence the diode does not conduct. No current flows through the circuit and the voltage drop across R_L will be zero. Hence no output voltage is obtained.
Thus corresponding to an alternating input signal, unidirectional pulsating output is obtained. The ratio of d.c. power output to the a.c. power input is known as rectifier efficiency. The efficiency of half wave rectifier is approximately 40.6%.

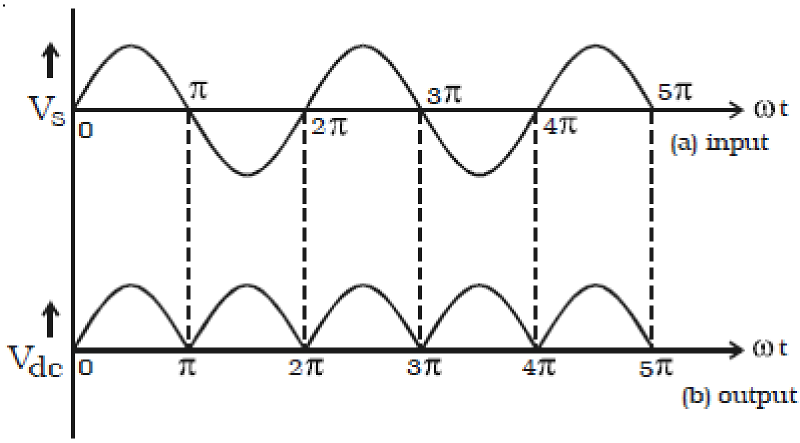


Full-wave rectifier:

The circuit using two diodes, shown in Fig. (a), gives output rectified voltage corresponding to both the positive as well as negative half of the ac cycle. Hence, it is known as *full-wave rectifier*.
Here the p-side of the two diodes are connected to the ends of the secondary of the transformer. The n-side of the diodes are connected together and the output is taken between this common point of diodes and the midpoint of the secondary of the transformer. So for a full-wave rectifier the secondary of the transformer is provided with a centre tapping and so it is called *centre-tap transformer*.
Suppose the input voltage to A with respect to the centre tap at any instant is positive. It is clear that, at that instant, voltage at B being out of phase will be negative as shown in So, diode D_1 gets forward biased and conducts (while D2 being reverse biased is not conducting). Current flows through path AD1XY to central tapping.
Hence, during this positive half cycle we get an output current (and a output voltage across the load resistor R_L) as shown in (Fig.b).
In the course of the ac cycle , the voltage at B would be positive. In this part of the cycle diode D_1 would not conduct but diode D_2 would, giving an output current path of current will be BD_2 XY to central tapping and output voltage (across R_L) during the negative half cycle of the input ac.
Thus, we get output voltage during both the positive as well as the negative half of the cycle. This is a more efficient circuit for getting rectified voltage or current than the halfwave rectifier
The rectified voltage is in the form of pulses of the shape of half sinusoids. Though it is unidirectional it does not have a steady value.



To get steady dc output from the pulsating voltage normally a capacitor is connected across the output terminals (parallel to the load R_L). One can also use an inductor in series with *RL* for the same purpose. Since these additional circuits appear to *filter* out the *ac ripple* and give a *pure dc* voltage, so they are called filters.



Filter circuits and regulation property of the power supply

The rectified voltage is in the form of pulses of the shape of half sinusoids. Though it is unidirectional it does not have a steady value.

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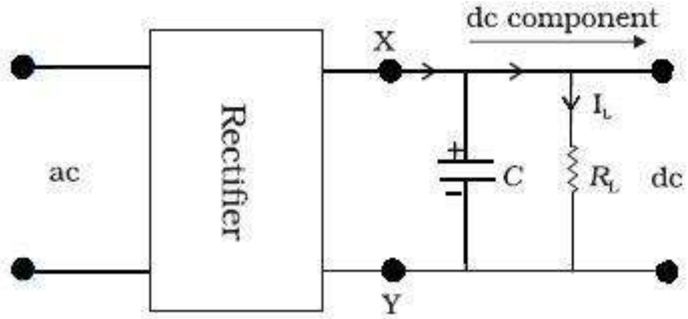
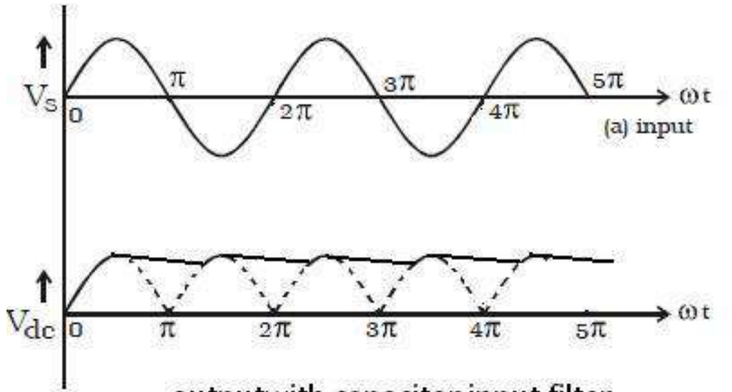


When the voltage across the capacitor is rising, it gets charged. If there is no external load, it remains charged to the peak voltage of the rectified output.

When there is a load, it gets discharged through the load and the voltage across it begins to fall. In the next half-cycle of rectified output it again gets charged to the peak value (Fig.).

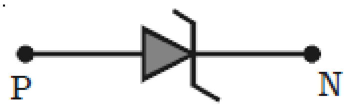


outputwith capacitor input filter

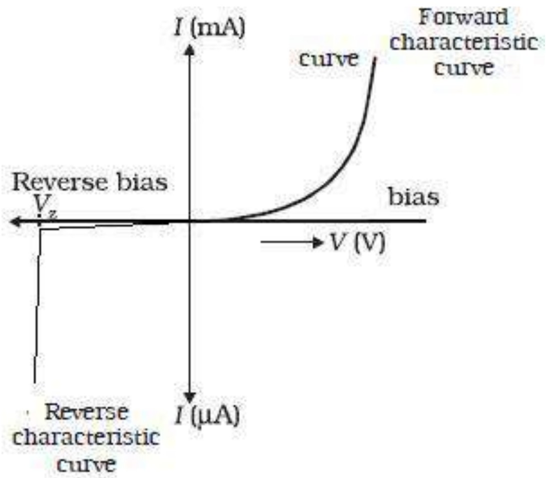
The rate of fall of the voltage across the capacitor depends upon the inverse product of capacitor *C* and the effective resistance R_L used in the circuit and is called the *time constant*.
To make the time constant large value of *C* should be large. So capacitor input filters use large capacitors. The *output voltage* obtained by using capacitor input filter is nearer to the *peak voltage* of the rectified voltage. This type of filter is most widely used in power supplies.

Zener diode

It is a special purpose semiconductor diode, named after its inventor C. Zener.
It is designed to operate under reverse bias in the breakdown region and used as a voltage regulator. The symbol for Zener diode is shown in (a).



Zener diode is fabricated by heavily doping both p-, and n- sides of the junction.
Due to this, depletion region formed is very thin ($<10^{-6}$ m) and the electric field of the junction is extremely high (~5×106 V/m) even for a small reverse bias voltage of about 5V. The I-V characteristics of a Zener diode is shown in Fig.



It is seen that when the applied reverse bias voltage(V) reaches the breakdown voltage (V_z) of the Zener diode, there is a large change in the current. Note that after the breakdown voltage V_z, a large change in the current can be produced by almost insignificant change in the reverse bias voltage. In other words, Zener voltage remains constant, even though current through the Zener diode varies over a wide range. This property of the Zener diode is used for regulating supply voltages so that they are constant.

Zener breakdown :

We know that reverse current is due to the flow of electrons (minority carriers) from p → n and holes from n → p.

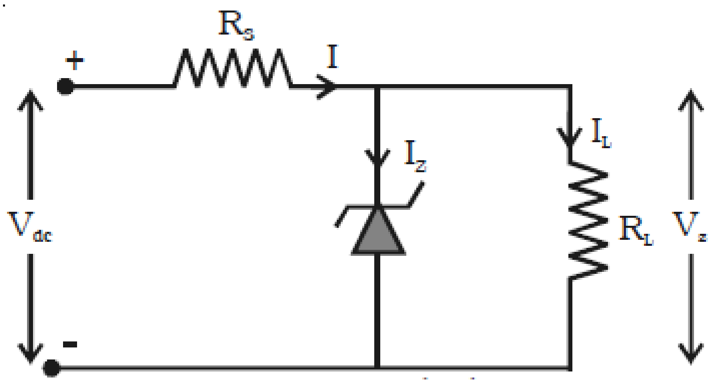
When both sides of the PN junction are heavily doped, consequently the depletion layer is narrow. Zener breakdown takes place in such a thin narrow junction

As the reverse bias voltage is increased, the electric field at the junction becomes significant. When the reverse bias voltage V = V_z, then the electric field strength is high enough to pull valence electrons from the host atoms on the p-side which are accelerated These electrons account for high current observed at the breakdown. The emission of electrons from the host atoms due to the high electric field is known as internal field emission or field ionisation. The electric field required for field ionisation is of the order of 10^6 V/m.

Zener diode as voltage regulator:

To maintain a constant voltage across the load, even if the input voltage or load current varies, voltage regulation is to be made.

A Zener diode working in the break down region can act as voltage regulator. The circuit in which a Zener diode is used for maintaining a constant voltage across the load R_L is shown in Fig

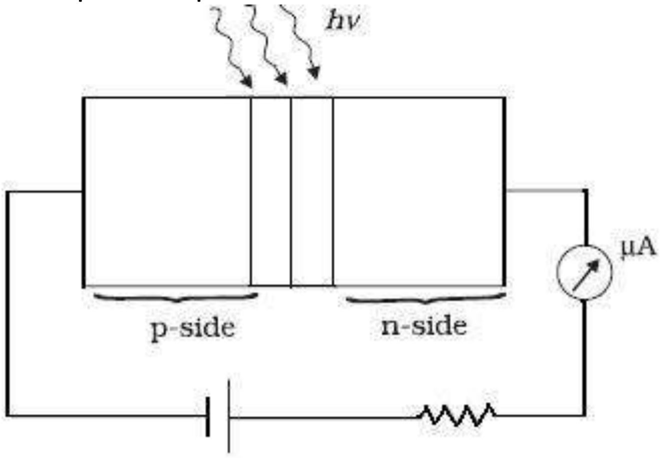


The Zener diode in reverse biased condition is connected in parallel with the load R_L. Let V_{dc} be the unregulated dc voltage and V_z be Zener voltage (regulated output voltage).
R_s is the current limiting resistor. It is chosen in such a way that the diode operates in the breakdown region. Inspite of changes in the load current or in the input voltage, the Zener diode maintains a constant voltage across the load. The action of the circuit can be explained as given below.
(i) load current varies, input voltage is constant : Let us consider that the load current increases. Zener current hence decreases, and the current through the resistance Rs is a constant.
The output voltage is $V_z = V_{dc} - IR_s$, since the total current I remains constant, output voltage remains constant.
(ii) input voltage varies :
If input voltage increases, In the breakdown region, Zener voltage remains constant even though the current through the Zener diode changes.
Similarly, if the input voltage decreases, the current through *Rs* and Zener diode also decreases. The voltage drop across *Rs* decreases without any change in the voltage across the Zener diode.
Thus any increase/decrease in the input voltage results in, increase/decrease of the voltage drop across *Rs* without any change in voltage across the Zener diode. Thus the Zener diode acts as a voltage regulator.
We have to select the Zener diode according to the required output voltage and accordingly the series resistance *Rs*.

(i) *Photodiode*

A Photodiode is again a special purpose p-n junction diode fabricated with a transparent window to allow light to fall on the diode.
It is operated under reverse bias.
Reverse saturation current flows through the PN junction diode on connecting it in a reverse bias mode.
The reverse saturation current can be increased by making more light incident on it

When the photodiode is illuminated with light (photons) with energy ($h\nu$) greater than the energy gap (E_g) of the semiconductor, then electron-hole pairs are generated due to the absorption of photons.



(a)

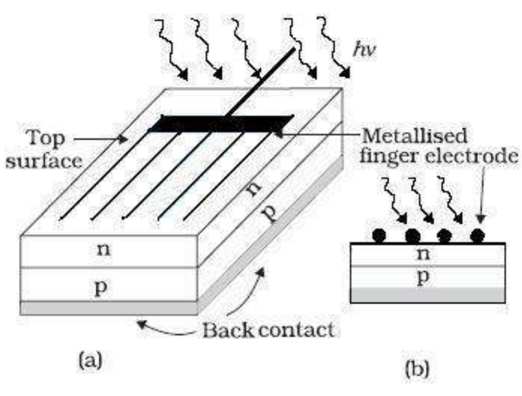
The diode is fabricated such that the generation of *e-h* pairs takes place in or near the depletion region of the diode.
Due to electric field of the junction, electrons and holes are separated before they recombine. The direction of the electric field is such that electrons reach n-side and holes reach p-side.
When an external load is connected ,electrons are collected on n-side and holes are collected on p-side giving rise to reverse saturation current.
The magnitude of the photocurrent depends on the intensity of incident light photocurrent is proportional to incident light intensity).
Thus photodiode can be used as a photo detector to detect optical signals.

Light emitting diode

It is a heavily doped p-n junction which under forward bias emits spontaneous radiation.
The diode is encapsulated with a transparent cover so that emitted light can come out.
When the diode is forward biased, electrons are sent from n → p and holes are sent from p → n.
At the junction boundary the concentration of minority carriers increases compared to the equilibrium concentration (i.e., when there is no bias).
Thus at the junction boundary on either side of the junction, excess minority carriers are there which recombine with majority carriers near the junction.
On recombination, the energy is released in the form of photons. Photons with energy equal to or slightly less than the band gap are emitted.
When the forward current of the diode is small, the intensity of light emitted is small. As the forward current increases, intensity of light increases and reaches a maximum. Further increase in the forward current results in decrease of light intensity.
LEDs are biased such that the light emitting efficiency is maximum.

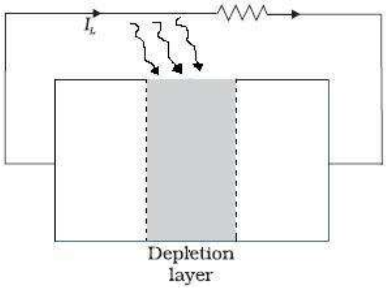
The *V-I* characteristics of a LED is similar to that of a Si junction diode. But the threshold voltages are much higher and slightly different for each colour.
The reverse breakdown voltages of LEDs are very low, typically around 5V. So care should be taken that high reverse voltages do not appear across them.
LEDs that can emit red, yellow, orange, green and blue light are commercially available.
The semiconductor used for fabrication of visible LEDs must at least have a band gap of 1.8 eV (spectral range of visible light is from about 0.4 μm to 0.7 μm, i.e., from about 3 eV to 1.8 eV).
The compound semiconductor Gallium Arsenide – Phosphide ($GaAs_{1-x}P_x$) is used for making LEDs of different colours.
$GaAs_{0.6}P_{0.4}$ (E_g ~ 1.9 eV) is used for red LED.
GaAs (E_g ~ 1.4 eV) is used for making infrared LED.
These LEDs find extensive use in remote controls, burglar alarm systems, optical communication, etc. Extensive research is being done for developing white LEDs which can replace incandescent lamps.
LEDs have the following advantages over conventional incandescent low power lamps:
(i) Low operational voltage and less power.
(ii) Fast action and no warm-up time required.
(iii) The bandwidth of emitted light is 100 Å to 500 Å or in other words it is nearly (but not exactly) monochromatic.
(iv) Long life and ruggedness.
(v) Fast on-off switching capability.

Solar cell

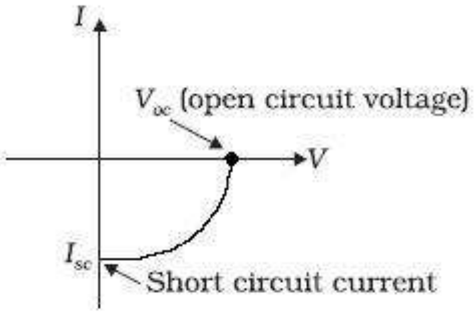


A solar cell is basically a p-n junction which generates emf when solar radiation falls on the p-n junction. It works on the same principle (photovoltaic effect) as the photodiode, except that no external bias is applied and the junction area is kept much larger for solar radiation to be incident because we are interested in more power. A simple p-n junction solar cell is shown in figure
A p-Si wafer of about 300 μm is taken over which a thin layer (~0.3 μm) of n-Si is grown on

one-side by diffusion process. The other side of p-Si is coated with a metal (back contact). On the top of n-Si layer, metal finger electrode (or metallic grid) is deposited. This acts as a front contact. The metallic grid occupies only a very small fraction of the cell area (<15%) so that light can be incident on the cell from the top.
The generation of emf by a solar cell, when light falls on, it is due to the following three basic processes: generation, separation and collection—



(i) generation of e-h pairs due to light (with $h\nu > E_g$) close to the junction;
(ii) separation of electrons and holes due to electric field of the depletion region. Electrons are swept to n-side and holes to p-side;
(iii) the electrons reaching the n-side are collected by the front contact and holes reaching p-side are collected by the back contact. Thus p-side becomes positive and n-side becomes negative giving rise to *photovoltage*.
When an external load is connected as shown in the Fig. a photocurrent I_L flows through the load. A typical *I-V* characteristics of a solar cell is shown in the Fig.b



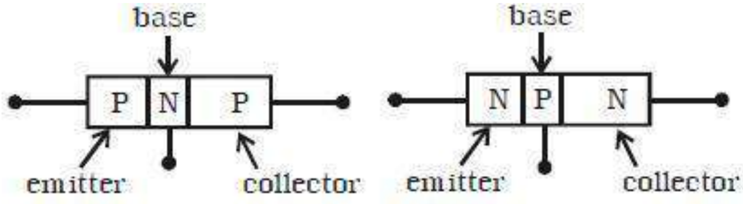
(b)

Note that the *I* – *V* characteristics of solar cell is drawn in the fourth quadrant of the coordinate axes. This is because a solar cell does not draw current but supplies the same to the load.
Semiconductors with band gap close to 1.5 eV are ideal materials for solar cell fabrication. Solar cells are made with semiconductors like
Si (E_g = 1.1 eV),
GaAs (E_g = 1.43 eV),
CdTe (E_g = 1.45 eV),
$CuInSe_2$ (E_g = 1.04eV), etc.
The important criteria for the selection of a material for solar cell fabrication are
(i) Band gap (~1.0 to 1.8 eV),
(ii) High optical absorption (~104 cm–1), electrical conductivity,
(iv) Availability of the raw material, and
(v) Cost. Note that sunlight is not always required for a solar cell.
Any light with photon energies greater than the band gap will do. Solar cells are used

to power electronic devices in satellites and space vehicles and also as power supply to some calculators.

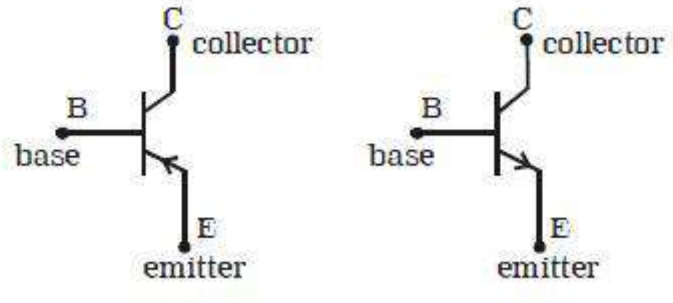
Junction transistor

A junction transistor is a solid state device. It consists of silicon or germanium crystal containing two PN junctions.
The two PN junctions are formed between the three layers. These are called base, emitter and collector.
(i) Base (B) layer : It is a very thin layer, the thickness is about 25 microns. It is the central region of the transistor.
(ii) Emitter (E) and Collector (C) layers : The two layers on the opposite sides of B layer are emitter and collector layers. They are of the same type of the semiconductor.
An ohmic contact is made to each of these layers. The junction between emitter and base is called emitter junction. The junction between collector and base is called collector junction.
In a transistor, the emitter region is heavily doped, since emitter has to supply majority carriers. The base is lightly doped. The collector region is lightly doped. Since it has to accept majority charge carriers, it is physically larger in size. Hence, emitter and collector cannot be interchanged.
The construction of PNP and NPN transistors are shown in Fig a and Fig b respectively.



(a) *Construction of PNP transistor* (b) *Construction of NPN transistor*

For a transistor to work, the biasing to be given are as follows :
(i) The emitter-base junction is forward biased, so that majority charge carriers are repelled from the emitter and the junction offers very low resistance to the current.
(ii) The collector-base junction is reverse biased, so that it attracts majority charge carriers and this junction offers a high resistance to the current.
Transistor circuit symbols
The circuit symbols for a PNP and NPN transistors are shown in Fig

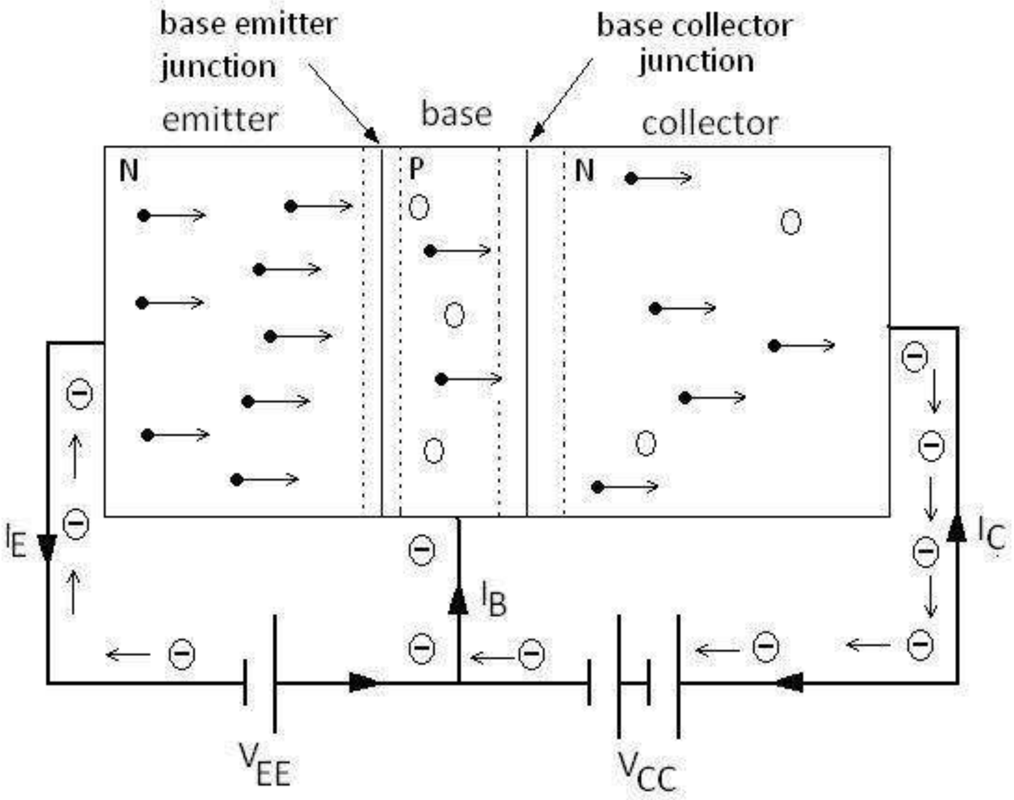


(a) PNP (b) NPN

The arrow on the emitter lead pointing towards the base represents a PNP transistor
When the emitter-base junction of a PNP transistor is forward biased, the direction of the conventional current flow is from emitter to base.
NPN transistor is represented by arrow on the emitter lead pointing away from the base.
When the emitter base junction of a NPN transistor is forward biased, the direction of the conventional current is from base to emitter.

Working of a NPN transistor

A NPN transistor is like two PN junction diodes, which are placed back-to-back. At each junction, there is a depletion region which gives rise to a potential barrier. The external biasing of the junction is provided by the batteries V_{EE} and V_{CC} as shown in Fig.
The emitter base junction is forward biased and the collector base junction is reverse biased.
Since the emitter-base junction is forward biased, a large number of electrons cross the junction and enters the base constitutes current I_E.
The base width is small and has fewer concentration of impurity as a result only 5% of electrons entering base recombine with holes.
The rest of the electrons enters collector region due to influence of the battery V_{CC}.
For every electron entering the collector one electron flows in the external circuit and constitutes the collector current I_C.
Similarly for every electron combing with the hole in the base section, there is one electron which gets attracted by V_{EE} and flows as base current I_B in external circuit
Applying Kirchoff's current law to the circuit, the emitter current is the sum of collector current and base current. $I_E = I_C + I_B$



This equation is the fundamental relation between the currents in a transistor circuit. This equation is true regardless of transistor type or transistor configuration. The action of PNP transistor is similar to that of NPN transistor.

Solved Numerical

Q) In NPN transistor about 10^{10} electrons enter the emitter in 1μs when it is connected to a battery. About 2% electrons recombine with the holes in the base. Calculate the values of I_E , I_B, I_C, α_{dc}, and β_{dc} (e = 1.6 ×10^{-19} C)

Solution:

As per the definition of current

$$I_E = \frac{Q}{t} = \frac{ne}{t}$$

$$I_E = \frac{10^{10} \times 1.6 \times 10^{-19}}{10^{-6}} = 1600\mu A$$

2% of the total current entering the base from the emitter recombine with the holes which constitutes the base current I_B. The rest of the 98% electrons reaches the collector and constitutes the collector current

I_B = 0.021, I_E = 0.02 × 1600 = 32 μA

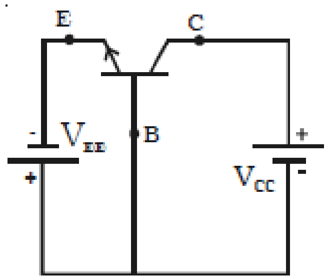
I_C = 0.98, I_E = 0.98 × 1600 =1568 μA

$$\alpha_{dc} = \frac{I_C}{I_E} = \frac{1568 \times 10^{-6}}{1600 \times 10^{-6}} = 0.98$$

$$\beta_{dc} = \frac{I_C}{I_B} = \frac{1568 \times 10^{-6}}{32 \times 10^{-6}} = 49$$

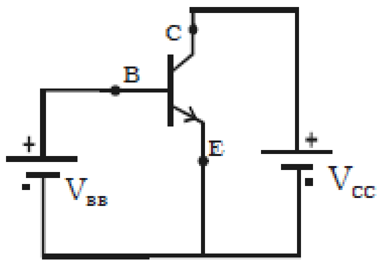
Transistor circuit configurations

There are three types of circuit connections (called configurations or modes) for operating a transistor. They are (i) common base (CB) mode



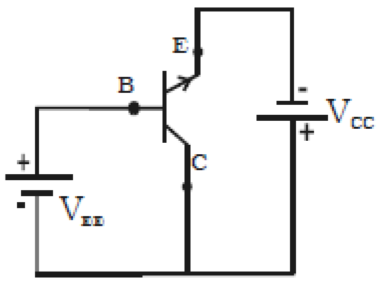
(a) CB mode

(ii)common emitter (CE) mode



(b) CE mode

(iii)common collector (CC) mode.



(c) CC mode

In a similar way, three configurations can be drawn for PNP transistor.

Current amplification factors α and β and the relation between them

The current amplification factor or current gain of a transistor is the ratio of output current to the input current.

If the transistor is connected in common base mode, the current gain α

$$\alpha = \frac{I_C}{I_E}$$

and if the transistor is connected in common emitter mode, the current gain β

$$\beta = \frac{I_C}{I_B}$$

Since 95% of the injected electrons reach the collector, the collector current is almost equal to the emitter current. Almost all transistors have α , in the range 0.95 to 0.99
We know that

$$\alpha = \frac{I_C}{I_E}$$

And $I_E = I_C + I_B$ Thus

$$\alpha = \frac{I_C}{I_C + I_B}$$

$$\frac{1}{\alpha} = \frac{I_C + I_B}{I_C} = 1 + \frac{I_B}{I_C}$$

$$\frac{1}{\alpha} - 1 = \frac{I_B}{I_C}$$

$$\frac{1}{\alpha} - 1 = \frac{1}{\beta}$$

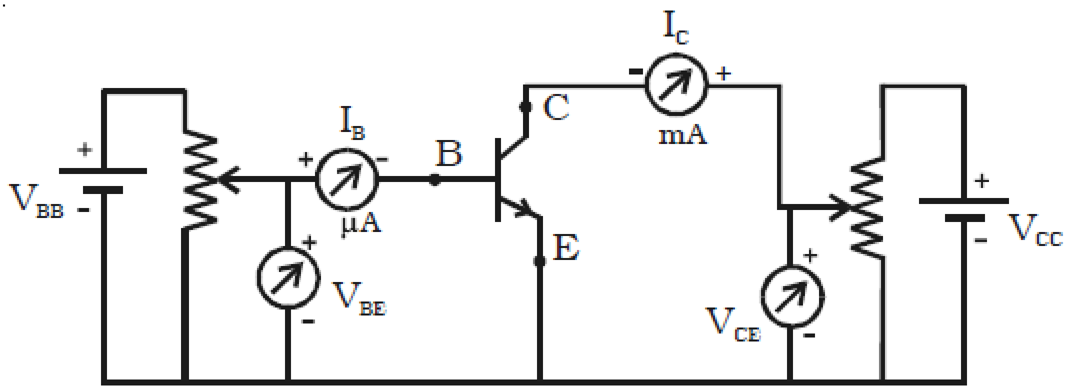
$$\beta = \frac{\alpha}{1-\alpha}$$

Usually β lies between 50 and 300. Some transistors have β as high as 1000.
Similarly we can prove

$$\alpha = \frac{\beta}{1+\beta}$$

Characteristics of an NPN transistor in common emitter configuration

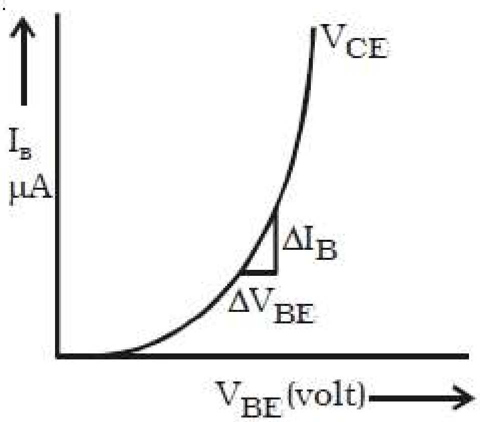
The three important characteristics of a transistor in any mode are (i) input characteristics (ii) output characteristics and (iii) transfer characteristics.
The circuit to study the characteristic curves of NPN transistor in common emitter mode is as shown in Fig



Input characteristics

Input characteristic curve is drawn between the base current (I_B) and voltage between base and emitter (V_{BE}), when the voltage between collector and emitter (V_{CE}) is kept constant at a particular value. V_{BE} is increased in suitable equal steps and corresponding base current is noted. The procedure is repeated for different values of V_{CE}.

I_B values are plotted against V_{BE} for constant V_{CE}. The input characteristic thus obtained is shown in Fig



The input impedance of the transistor is defined as the ratio of small change in base – emitter voltage to the corresponding change in base current at a given V_{CE}.

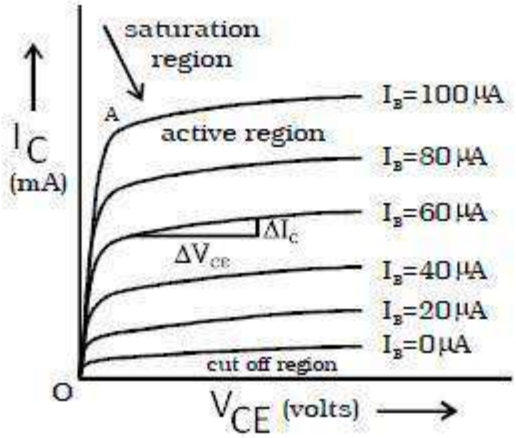
Input impedance, r_i

$$r_i = \left(\frac{\Delta V_{BE}}{\Delta I_B}\right)_{V_{CE}}$$

The input impedance of the transistor in CE mode is very high.

Output characteristics

Output characteristic curves are drawn between I_C and V_{CE}, when I_B is kept constant at a particular value. The base current I_B is kept at a constant value, by adjusting the base emitter voltage V_{BE}. V_{CE} is increased in suitable equal steps and the corresponding collector current is noted. The procedure is repeated for different values of I_B. Now, Ic versus V_{CE} curves are drawn for different values of I_B. The output characteristics thus obtained are represented in Fig.



The three regions of the characteristics can be discussed as follows :
Saturation region *:*
The initial part of the curve (ohmic region, OA) is called saturation region. (i.e) The region in between the origin and knee point. (Knee point is the point, where I_c is about to become a constant). In this region both base-emitter region and base-collector region are forward bias.

Cut off region :

There is very small collector current in the transistor, even when the base current is zero (I_B = 0). In the output characteristics, the region below the curve for I_B = 0 is called cut off region. Below the cut off region, the transistor does not function. In this region both base-emitter region and base-collector region are reverse biased.

Active region :

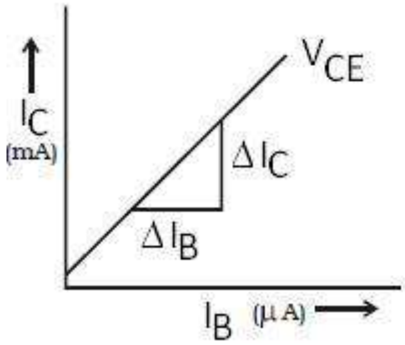
The central region of the curves is called active region. In the active region, the curves are uniform. In this region, E-B junction is forward biased and C-B junction is reverse biased.
The output impedance r_o is defined as the ratio of variation in the collector emitter voltage to the corresponding variation in the collector current at a constant base current in the active region of the transistor characteristic curves.
output impedance, r_o

$$\boldsymbol{r_o = \left(\frac{\Delta V_{CE}}{\Delta I_C}\right)_{I_B}}$$

The output impedance of a transistor in CE mode is low.
Its value can be found out from the input characteristic curve. Normally its value is found between 50kΩ to 100kΩ

Transfer characteristics

The transfer characteristic curve is drawn between I_C and I_B, when V_{CE} is kept constant at a particular value. The base current I_B is increased in suitable steps and the collector current I_C is noted down for each value of I_B. The transfer characteristic curve is shown in Fig.



The current gain is defined as the ratio of a small change in the collector current to the corresponding change in the base current at a constant V_{CE}.
current gain, β

$$\beta = \left(\frac{\Delta I_C}{\Delta I_B}\right)_{V_{CE}}$$

The common emitter configuration has high input impedance, low output impedance and higher current gain when compared with common base configuration.
Taking the ratio of β and r_i for ac circuit

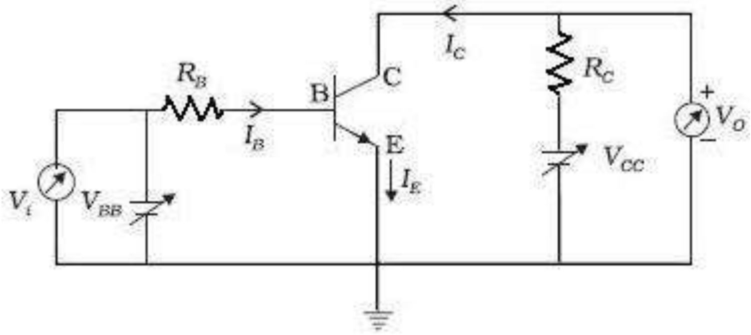
$$\frac{\beta}{r_i} = \frac{\Delta I_C/_{\Delta I_B}}{\Delta V_{BE}/_{\Delta I_B}} = \frac{\Delta I_C}{\Delta V_{BE}}$$

Ratio of the change in the current in the output circuit (ΔI_C) to the change in the input voltage (ΔV_{BE}) is known as the trans-conductance g_m its unit is *mho*

$$g_m = \frac{\Delta I_C}{\Delta V_{BE}} = \frac{\beta}{r_i}$$

Transistor as switch

In an ideal ON/OFF switch, when it is OFF the current is not flowing in the circuit because switch offers infinite resistance.
When switch is in On condition, maximum current flows because its resistance is zero.
We can prepare such an electronic switch by using the resistor.
The operating point switch from cutoff to saturation along the load line.



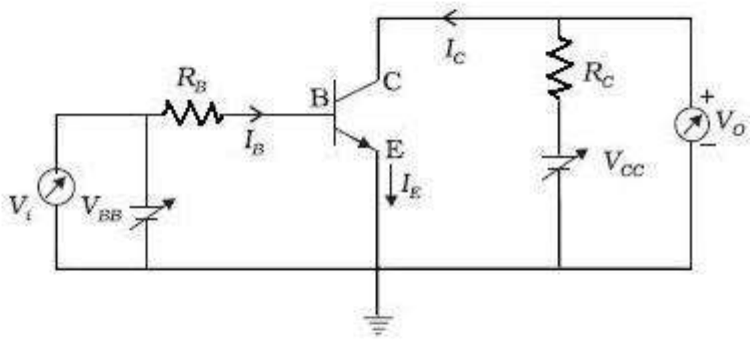
We shall try to understand the operation of the transistor as a switch by analyzing the behavior of the base-biased transistor in CE configuration as shown in Fig.
Applying Kirchhoff's voltage rule to the input and output sides of this circuit, we get
$V_{BB} = I_BR_B + V_{BE}$ and $V_{CE} = V_{CC} – I_CR_C$.
We shall treat V_{BB} as the dc input voltage V_i and V_{CE} as the dc output voltage V_O.
So, we have
$V_i = I_BR_B + V_{BE}$ ----eq(1)and
$V_o = V_{CC} – I_CR_C$. ----eq(2)
Let us see how V_o changes as V_i increases from zero onwards.

(i) When input voltage V_i is zero or less than 0.6V for Si transistor (transistor cut in voltage), the base current I_B will be zero. Hence the collector current will also zero $I_C = 0$
From eq(2) $V_O = V_{CC}$
In this situation resistance of output circuit is very large. Hence the current is not flowing through it. This is the 'OFF' or "cut off" condition of the transistor.
(ii) When the input voltage will be $V_i = V_{CC}$, the base current is maximum, hence the collector current is maximum. The voltage drop (I_CR_L) across the load resistance R_L will be approximately V_{CC} . According to eq(2)
$V_O = 0$
In this condition resistance of the output circuit of the transistor is very small to the effect that maximum current is flowing through it. This is called the "ON" condition or saturation condition of the transistor.

Alternatively, we can say that a *low* input to the transistor gives a *high* output and a *high* input gives a *low* output. The switching circuits are designed in such a way that the transistor does not remain in active state. This circuit is used to make 'NOT' gate in the digital electronics

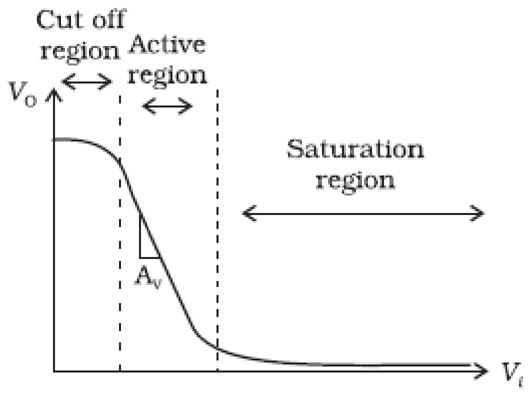
Transistor as an Amplifier

The important function of a transistor is the amplification. An amplifier is a circuit capable of magnifying the amplitude of weak signals. The important parameters of an amplifier are input impedance, output impedance, current gain and voltage gain. A good design of an amplifier circuit must possess high input impedance, low output impedance and high current gain.
Transistor act as an amplifier when operated in active region.
In this region base-emitter region is forward biased and base-collector region is reverse biased.



The A.C. signal (V_i) causes the change in the base emitter voltage V_{BE}. This result in the change in the base current I_B. the change in the base current is of the order of microampere. This results in the change in the collector current equal to βI_B, which is of the order of milliampere.

For using the transistor as an amplifier we will use the active region of the V_o versus V_i curve.



The slope of the linear part of the curve represents the rate of change of the output with the input. It is negative because the output is $V_O = V_{CC} - I_C R_C$ and not $I_C R_C$. That is why as input voltage of the C_E amplifier increases its output voltage decreases and the output is said to be out of phase with the input.

If we consider ΔV_o and ΔV_i as small changes in the output and input voltages then $\Delta V_o/\Delta V_i$ is called the small signal voltage gain A_V of the amplifier.

If the V_{BB} voltage has a fixed value corresponding to the midpoint of the active region, the circuit will behave as a CE amplifier with voltage gain $\Delta V_o/\Delta V_i$. We can express the voltage gain A_V in terms of the resistors in the circuit

Working of the circuit

(1) Input Circuit

In absence of input signal $V_i = 0$ as per the Kirchhoff's second law for input circuit

$V_{BB} = V_{BE}$

In presence of signal V_I, the change in the base emitter voltage is ΔV_{BE}

$\therefore V_{BB} + V_i = V_{BE} + \Delta V_{BE}$

From above equations

$V_i = \Delta V_{BE}$

Change in base current ΔI_B is due to the voltage change ΔV_{BE}. As per definition of the input resistance r_i we have

$$r_i = \frac{\Delta V_{BE}}{\Delta I_{BE}}$$

Or $\Delta V_{BE} = V_i = r_i \Delta I_B$

(2) Output circuit

The collector current increases by an amount ΔI_C due to the change in the base circuit ΔI_B. As a result the voltage change by an equal amount $R_L I_C$ across resistor R_L

As per the kirchhoff's law

$V_{CC} = I_C R_C + V_{CE}$

$\therefore \Delta V_{CC} = R_C \Delta I_C + \Delta V_{CE}$

As the battery V_{CC} remains constant $\Delta V_{CC} = 0$

Thus $\Delta V_{CE} = -R_C \Delta I_C$

Here ΔV_{CE} is obtained across the two ends of load resistor and is known as the output voltage V_O

$\therefore V_O = -R_C \Delta I_C$

Negative sign shows input and output voltages are out of phase by 180^O

Whenever the input voltage increases output voltage decrease and vice versa

Voltage gain(A_V):

As per the definition of voltage gain

$$A_V = \frac{output\ voltage}{Input\ voltage} = \frac{V_O}{V_i}$$

Since $V_O = -R_C \Delta I_C$ and $V_i = r_i \Delta I_B$

$$A_V = -\frac{R_C \Delta I_C}{r_i \Delta I_B}$$

As current gain $\beta = \Delta I_C / \Delta I_B$

$$A_V = -\beta \frac{R_C}{r_i}$$

Since trans-conductance $g_m = \beta / r_i$

$$\therefore A_V = -g_m R_C$$

Power gain (A_P): AS per definition of the gain A_P

$$A_P = \frac{Ouput\ AC\ power}{Input\ AC\ power}$$

$$A_P = \frac{\Delta V_{CE} \Delta I_C}{\Delta V_{BE} \Delta I_B}$$

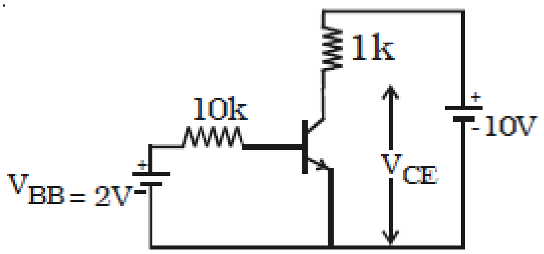
$$A_P = A_V A_i$$

$$A_P = \left(-\beta \frac{R_C}{r_i}\right)(\beta)$$

$$|A_P| = \beta^2 \frac{R_C}{r_i}$$

Solved Numerical

Q) The current gain β of the silicon transistor used in the circuit as shown in figure is 50. (Barrier potential for silicon is 0.69 V)



Find: (i) I_B (ii) I_E (iii) I_C and (iv) V_{CE}

Given:

V_{BB} = 2 V, V_{CC} = 10 V; β = 50 ;R_B = 10 k Ω ; R_C = 1 k Ω The barrier potential for silicon transistor V_{BE} = 0.69 V

Solution :

From input circuit

$V_{BB} = I_B R_B + V_{BE}$

$$I_B = \frac{V_{BB} - V_{BE}}{R_B} = \frac{2.0 - 0.69}{10000} = 131\mu A$$

Current gain β

$$\beta = \frac{I_C}{I_B}$$
$$I_C = \beta I_B$$
$$I_C = 50 \times 131 \times 10^{-6} = 6.5mA$$

Emitter current $I_E = I_C + I_B$

I_E = 6.5 mA + 131 μA

I_E = 6.5 mA + 0.131 mA

I_E = 6.631 mA

$V_{CC} = V_{CE} + I_C R_C$

$V_{CE} = V_{CC} - I_C R_C$

$V_{CE} = 10 - (6.5 \times 10^{-3} \times 1 \times 10^{3})$

V_{CE} = 3.5 V

Q) A transistor is connected in CE configuration. The voltage drop across the load resistance (R_C) 3 k Ω is 6 V. Find the base current. The current gain α of the transistor is 0.97

Given : Voltage across the collector load resistance (R_C) = 6 V α = 0.97; R_C = 3 k Ω

Solution : The voltage across the

collector resistance is, $R_C = I_C R_C$ = 6V

Hence, I_C

$$I_c = \frac{6}{R_C} = \frac{6}{3 \times 10^3} = 2\ mA$$

Current gain β

$$\beta = \frac{\alpha}{1-\alpha} = \frac{0.97}{1-0.97} = 32.33$$

$$I_B = \frac{I_C}{\beta} = \frac{2 \times 10^{-3}}{32.33} = 61.86\ \mu A$$

Q) A change of 0.02V takes place between the base and emitter when an input signal is connected to CE transistor amplifier. As a result, 20μA change takes place in the base current and change of 2mA takes place in the collector current. Calculate the following quantities

(1) Input resistance (2) A.C. Current gain (3) Trans-conductance (4) If the load resistance is 5kΩ, what will be the voltage gain and power gain

Solution:

Here ΔI_B = 20μA = 20×10^{-6} A ; ΔV_{BE} = 0.02 V ; ΔI_C = 20 mA = 20×10^{-3} A, R_L = 5 kΩ = 5 $\times10^3$ Ω

(1) Input resistance

$$r_i = \frac{\Delta V_{BE}}{\Delta I_B} = \frac{0.02}{20 \times 10^{-6}} = 1k\Omega$$

(2) AC current gain

$$A_i = \beta = \frac{\Delta I_C}{\Delta I_B} = \frac{2 \times 10^{-3}}{20 \times 10^{-6}} = 100$$

(3) Trans-conductance

$$g_m = \frac{\beta}{r_i} = \frac{100}{1000} = 0.1\ mho$$

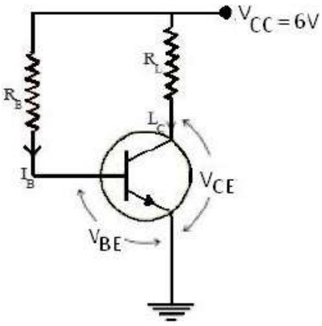
(4) Voltage gain

$$|A_V| = g_m R_L$$
$$|A_V| = (0.1)(5000) = 500$$

(5) Power gain

$$A_P = A_V A_i$$
$$A_P = (500)(100) = 5 \times 10^4$$

Q) For the circuit shown in figure I_B = 5μA. R_B = 1 MΩ., R_L = 1.1 kΩ, I_C = 5 mA and V_{CC} = 6V. Calculate the values of V_{BE} and V_{CE}



Solution:
For input circuit $V_{CC} = I_B R_B + V_{BE}$
$V_{BE} = V_{CC} - I_B R_B$
$V_{BE} = 6 - (5\times10^{-6})(1\times10^{6}) = 1$ V
From output circuit $V_{CC} = I_C R_C + V_{CE}$
$V_{CE} = V_{CC} - I_C R_C$
$V_{CE} = 6 - (5\times10^{-3})(1.1\times10^{3}) = 0.5$ V

Q) The A.C current gain of a PNP common emitter circuit is 100. The value of the input resistance is 1kΩ. What should be the value of the load resistor R_L in order to obtain power gain of 2000?
Solution:
Power gain

$$|A_P| = \beta^2 \frac{R_L}{r_i}$$

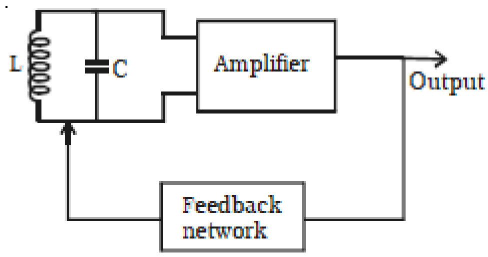
$$2000 = (100^2)\frac{R_L}{1 \times 10^3}$$

R_L = 200 Ω

Transistor oscillators

An oscillator may be defined as an electronic circuit which converts energy from a d.c. source into a periodically varying output.
Oscillators are classified according to the output voltage, into two types viz. sinusoidal and non-sinusoidal oscillators.
If the output voltage is a sine wave function of time, the oscillator is said to be sinusoidal oscillator. If the oscillator generates non-sinusoidal waveform, such as square, rectangular waves, then it is called as non-sinusoidal oscillator (multivibrator).
The oscillators can be classified according to the range of frequency as audio-frequency (AF) and radio-frequency (RF) oscillators.
Sinusoidal oscillators may be any one of the following three types:
(i) LC oscillators

(ii) RC oscillators
(iii) Crystal oscillators
Essentials of LC oscillator:
Fig shows the block diagram of an oscillator.

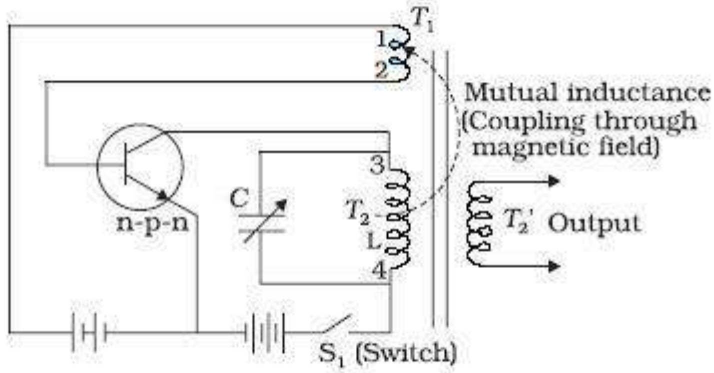


It's essential components are (i) tank circuit, ii) amplifier and (iii) feedback circuit.

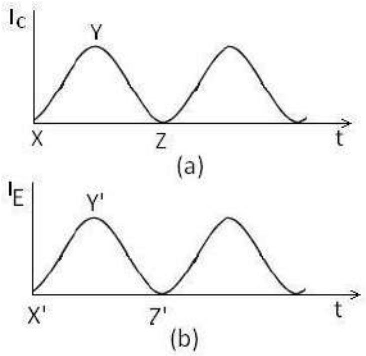
(i) Tank circuit : It consists of inductance coil (L) connected in parallel with capacitor (C). The frequency of oscillations in the circuit depends upon the values of inductance coil and capacitance of the capacitor.

(ii) Amplifier : The transistor amplifier receives d.c. power from the battery and changes it into a.c. power for supplying to the tank circuit.

(iii) Feedback circuit : It provides positive feedback (i.e.) this circuit transfers a part of output energy to LC circuit in proper phase, to maintain the oscillations



Suppose switch S_1 is put *on* to apply proper bias for the first time. Obviously, a *surge* of collector current flows in the transistor. This current flows through the coil T_2 where terminals are numbered 3 and 4 [Fig.]. This current does not reach full amplitude instantaneously but increases from X to Y, as shown in [Fig.a]
The inductive coupling between coil T_2 and coil T_1 now causes a current to flow in the emitter circuit (note that this actually is the 'feedback' from input to output).
As a result of this positive feedback, this current (in T_1; emitter current) also increases from X′ to Y′ [Fig.b]



The current in T_2 (collector current) connected in the collector circuit acquires the value Y when the transistor becomes *saturated*.
This means that maximum collector current is flowing and can increase no further. Since there is no further change in collector current, the magnetic field around T_2 ceases to grow. As soon as the field becomes static, there will be no further feedback from T_2 to T_1. Without continued feedback, the emitter current begins to fall.
Consequently, collector current decreases from Y towards Z [Fig. a]. However, a decrease of collector current causes the magnetic field to decay around the coil T_2. Thus, T_1 is now seeing a decaying field in T_2 (opposite from what it saw when the field was growing at the initial *start* operation). This causes a further decrease in the emitter current till it reaches Z' when the transistor is *cut-off*.
This means that both I_E and I_C cease to flow. Therefore, the transistor has reverted back to its original state (when the power was first switched on).
The whole process now repeats itself. That is, the transistor is driven to saturation, then to cut-off, and then back to saturation. The time for change from saturation to cut-off and back is determined by the constants of the tank circuit or tuned circuit (inductance *L* of coil T_2 and *C* connected in parallel to it). The resonance frequency (ν) of this tuned circuit determines the frequency at which the oscillator will oscillate.

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

In the circuit of the tank or tuned circuit is connected in the collector side. Hence, it is known as *tuned collector oscillator*.
If the tuned circuit is on the base side, it will be known as *tuned base oscillator*.
There are many other types of tank circuits (say *RC*) or feedback circuits giving different types of giving different types of oscillators like Colpitt's oscillator, Hartley oscillator, *RC*-oscillator

Solved Numerical

Q) In transistor oscillator circuit an output signal of 1MHz frequency is obtained. The value of capacitance C = 100pF. What should be the value of the capacitor is a signal of 2MHz frequency is to be obtained
Solution:
$C_1 = 100\text{pF} = 10 \times 10^{-13}$ F, $f_1 = 1\text{MHz} = 10^6$ Hz

f_2 = 2 MHz = 2×10^6 Hz, C_2 = ?

$$f_1 = \frac{1}{2\pi\sqrt{LC_1}}$$

And

$$f_2 = \frac{1}{2\pi\sqrt{LC_2}}$$

$$\frac{f_1}{f_2} = \sqrt{\frac{C_2}{C_1}}$$

$$C_2 = \left(\frac{f_1}{f_2}\right)^2 \times C_1$$

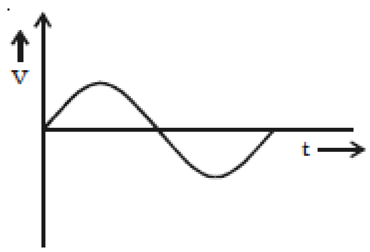
$$C_2 = \left(\frac{1}{2}\right)^2 \times 100 \times 10^{-12} = 25pF$$

Digital electronics

Digital Electronics involves circuits and systems in which there are only two possible states which are represented by voltage levels. Other circuit conditions such as current levels open or closed switches can also represent the two states.

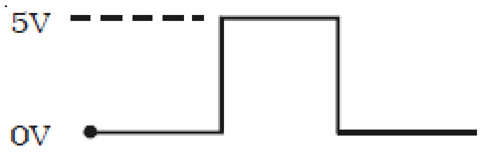
Analog signal

The signal current or voltage is in the form of continuous, time varying voltage or current (sinusoidal). Such signals are called continuous or analog signals. A typical analog signal is shown in Fig



Digital signal and logic levels

A digital signal (pulse) is shown in Fig. It has two discrete levels, 'High' and 'Low'. In most cases, the more positive of the two levels is called HIGH and is also referred to as logic 1. The other level becomes low and also called logic 0. This method of using more positive voltage level as logic 1 is called a positive logic system. A voltage 5V refers to logic 1 and 0 V refers to logic 0. On the other hand, in a negative logic system, the more negative of the two discrete levels is taken as logic 1 and the other level as logic 0. Both positive and negative logic are used in digital systems. But, positive logic is more common of logic gates. Hence we consider only positive logic for studying the operation of logic gates.



Logic gates

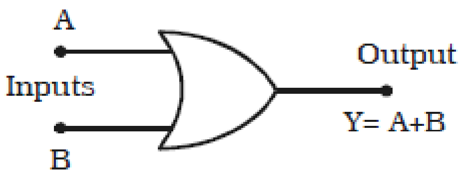
Circuits which are used to process digital signals are called logic gates. They are binary in nature. Gate is a digital circuit with one or more inputs but with only one output. The output appears only for certain combination of input logic levels. Logic gates are the basic building blocks from which most of the digital systems are built up. The numbers 0 and 1 represent the two possible states of a logic circuit. The two states can also be referred to as ‘ON and OFF’ or ‘HIGH and LOW’ or ‘TRUE and FALSE’.

Basic logic gates using discrete components

The basic elements that make up a digital system are ‘OR’, ‘AND’ and ‘NOT’ gates. These three gates are called basic logic gates. All the possible inputs and outputs of a logic circuit are represented in a table called TRUTH TABLE. The function of the basic gates are explained below with circuits and truth tables.

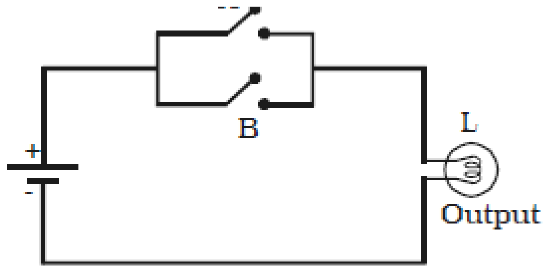
(i) OR gate

An OR gate has two or more inputs but only one output. It is known as OR gate, because the output is high if any one or all of the inputs are high. The logic symbol of a two input OR gate is shown in Fig



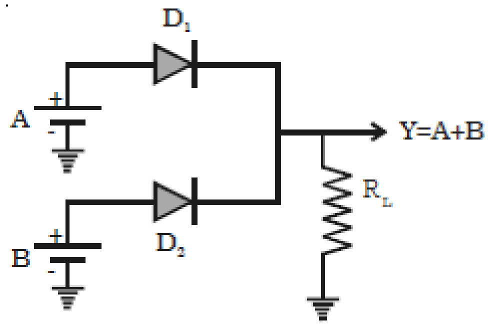
The Boolean expression to represent OR gate is given by Y= A+B (+ symbol should be read as OR)

The OR gate can be thought of like an electrical circuit shown in Fig, in which switches are connected in parallel with each other. The lamp will glow if both the inputs are closed or any one of them is closed.



Diode OR gate

Fig shows a simple circuit using diodes to build a two input OR gate. The working of this circuit can be explained as follows.



Case (i) A = 0 and B = 0

When both A and B are at zero level, (i.e.) low, the output voltage will be low, because the diodes are non-conducting.

Case (ii) A = 0 and B = 1

When A is low and B is high, diode D_2 is forward biased so that current flows through RL and output is high.

Case (iii) A = 1 and B = 0

When A is high and B is low, diode D_1 conducts and the output is high.

Case (iv) A = 1 and B = 1

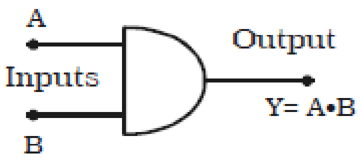
When A and B both are high, both diodes D_1 and D_2 are conducting and the output is high. Therefore Y is high

Truth table of OR gate:

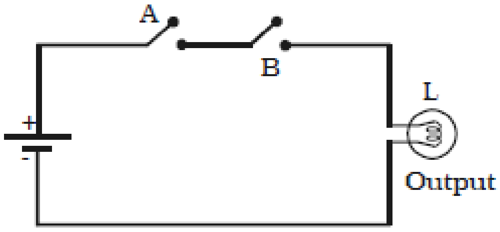
Inputs		Output
A	B	Y = A + B
0	0	0
0	1	1
1	0	1
1	1	1

(ii) AND gate

An AND gate has two or more inputs but only one output. It is known as AND gate because the output is high only when all the inputs are high. The logic symbol of a two input AND gate is shown in Fig.



The Boolean expression to represent AND gate is given by Y = A·B (· should be read as AND) AND gate may be thought of an electrical circuit as shown in Fig,



in which the switches are connected in series. Only if A and B are closed, the lamp will glow, and the output is high.

Diode AND gate

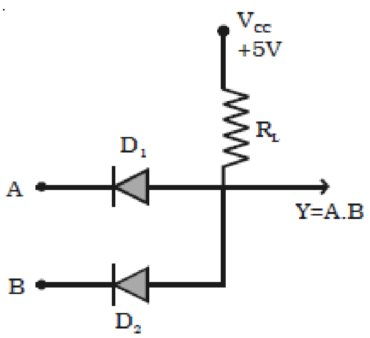


Fig shows a simple circuit using diodes to build a two-input AND gate.

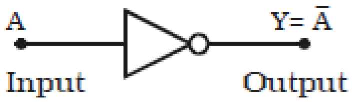
The working of the circuit can be explained as follows :
Case (i) A = 0 and B = 0
When A and B are zero, both diodes are in forward bias condition and they conduct and hence the output will be zero, because the supply voltage V_{CC} will be dropped across RL only. Therefore Y = 0.
Case (ii) A = 0 and B = 1
When A = 0 and B is high, diode D_1 is forward biased and diode D_2 is reverse biased. The diode D_1 will now conduct due to forward biasing. Therefore, output Y = 0.
Case (iii) A = 1 and B = 0
In this case, diode D_2 will be conducting and hence the output Y = 0.
Case (iv) A = 1 and B = 1
In this case, both the diodes are not conducting. Since D1 and D2 are in OFF condition, no current flows through RL. The output is equal to the supply voltage. Therefore Y = 1.
Thus the output will be high only when the inputs A and B are high.
Truth table of AND gate:

Inputs		Output
A	B	$Y = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

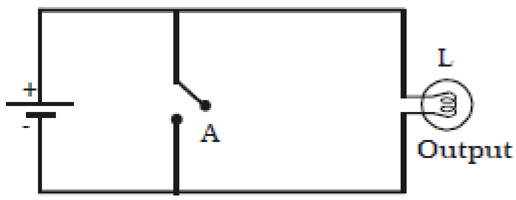
.

(iii) NOT gate (Inverter)

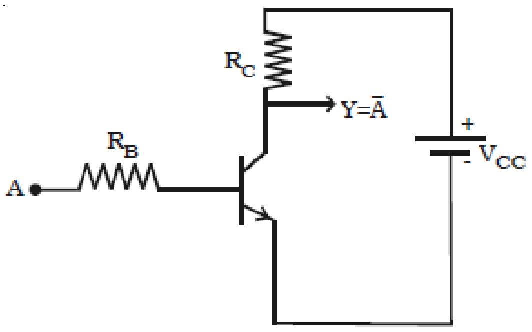
The NOT gate is a gate with only one input and one output. It is so called, because its output is complement to the input. It is also known as inverter. Fig shows the logic symbol for NOT gate.



The Boolean expression to represent NOT operation is $Y = A$.
The NOT gate can be thought of like an electrical circuit as shown in Fig.



When switch A is closed, input is high and the bulb will not glow (i.e) the output is low and vice versa.
When the input A is high, the transistor is driven into saturation and hence the output Y is low. If A is low, the transistor is in cutoff and hence the output Y is high. Hence, it is seen that whenever input is high, the output is low and vice versa.

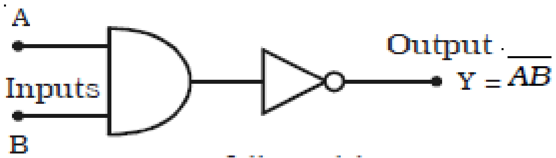


Input	Output
A	$Y = \bar{A}$
0	1
1	0

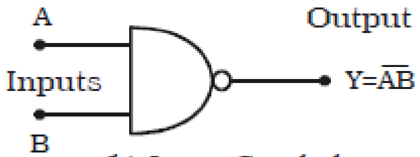
Truth table of NOT gate:

(iv) NAND gate

This is a NOT–AND gate. It can be obtained by connecting a NOT gate at the output of an AND gate (Fig).



The logic symbol for NAND gate is shown in Fig 9.53b.



(b) Logic Symbol

The Boolean expression to represent NAND Operation is Y = AB

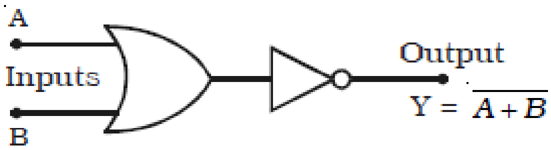
NAND gate function is reverse of AND gate function. A NAND gate will have an output, only if both inputs are not 1. In other words, it gives an output 1, if either A or B or both are 0.

Truth table of NAND gate:

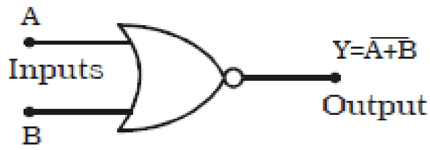
Inputs		Output
A	B	Y = AB
0	0	1
0	1	1
1	0	1
1	1	0

(V)NOR gate

This is a NOT–OR gate. It can be made out of an OR gate by connecting an inverter at its output (Fig).



The logic symbol for NOR gate is given in Fig.



The Boolean expression to represent NOR gate is Y = $A + B$

The NOR gate function is the reverse of OR gate function. A NOR gate will have an output, only when all inputs are 0. In a NOR gate, output is high, only when all inputs are low.

Truth table of NOR gate:

Inputs		Output
A	B	Y = $\overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

De-Morgan's theorems

First theorem

"The complement of a sum is equal to the product of the complements."

If A and B are the inputs, then

$$\overline{\boldsymbol{A}+\boldsymbol{B}} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}}$$

Second theorem

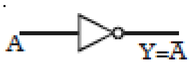
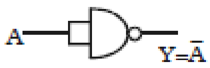
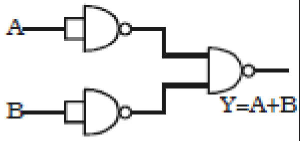
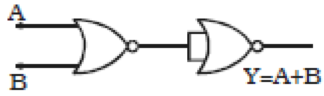
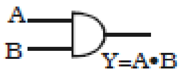
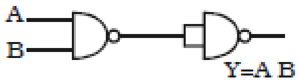
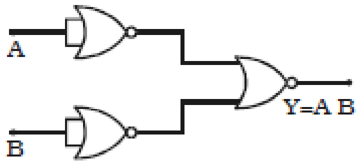
"The complement of a product is equal to the sum of the complements." If A and B are the inputs, then $\overline{A\,.B} = \bar{A} + \bar{B}$.

The theorems can be proved, first by considering the two variable cases and then extending this result as shown in Table

A	B	$\overline{A}$	$\overline{B}$	$\overline{A\ .\ B}$	$\overline{A}+\overline{B}$	$\overline{A+B}$	$\overline{A}\cdot\overline{B}$
0	0	1	1	1	1	1	1
0	1	1	0	1	1	0	0
1	0	0	1	1	1	0	0
1	1	0	0	0	0	0	0

NAND and NOR as Universal gates

NAND and NOR gates are called Universal gates because they can perform all the three basic logic functions. Table gives the construction of basic logic gates NOT, OR and AND using NAND and NOR gates

Logic function	Symbol	Circuits using NAND gates only	Circuits using NOR gates only
NOT			
OR			
AND			

Boolean algebra

Boolean algebra, named after a mathematician George Boole is the algebra of logic, which is applied to the operation of computer devices. The rules of this algebra is simple, speed and accurate. This algebra is helpful in simplifying the complicated logical expression.

Laws and theorems of Boolean algebra

The fundamental laws of Boolean algebra are given below which are necessary for manipulating different Boolean expressions.

Basic laws :

Commutative laws : A + B = B + A ; AB = BA

Associative Laws: A + (B + C) = (A + B) + C ; A (BC) = (AB) C

Distributive law: A (B+C) = AB + AC

Special theorems :

A + AB = A

(A + B) (A + C) = A + BC

A (A + B) = A

A + *A* B = A + B

A (*A* + B) = AB

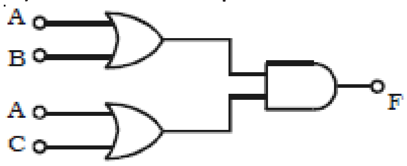
(A + B) (*A* + C) = AC + *A* B

AB + *A* C = (A + C) (*A* + B)

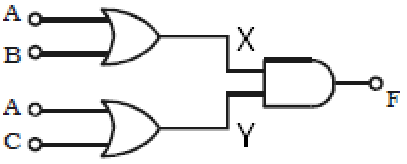
Theorems involving a single variable can be proved by considering every possible value of the variable.

Solved Numerical

Q) Find the output F of the logic circuit given below:



Solution :



Let X and Y be the output of two OR gates

Thus X = A + B

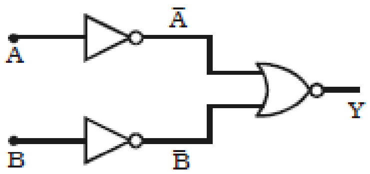
And Y = A + C

Output X and Y acts as input for AND gate

Thus F = X · Y

F = (A+B)(A+C) = AC + AB (from special theorems)

Q) The outputs of two NOT gates are in put for NOR, as shown in figure. What is this combination equivalent to?



Solution:

From the logic circuit it follows that the output

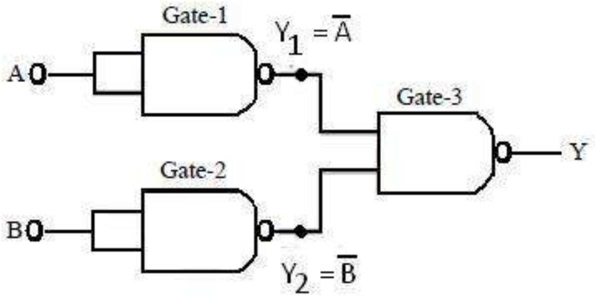
$$y = \overline{\bar{A} + \bar{B}}$$

Applying DeMorgan's first theorem,

we get, $y = \bar{\bar{A}} \cdot \bar{\bar{B}} = AB$

Hence given logic circuit is AND operation.

Q) Show that the circuit drawn in figure comprising of three NAND gates behave as an OR gate



Solution:
Gate 1 and Gate 2 have identical inputs. Hence both behaves as NOT gate
Hence $y_1 = \bar{A}$ and $y_2 = \bar{B}$
Gate 3 is NAND hence output

$$y = \overline{y_1 y_2}$$
$$y = \overline{\bar{A}\bar{B}}$$

By DeMorgan's theorem y = AB
Hence the above circuit behaves as OR gate and can be verified using truth table

----------------END--------------